ON PENALTY APPROACHES FOR NAVIER-SLIP AND OTHER BOUNDARY CONDITIONS IN VISCOUS FLOW

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Summary With the advent of microscale and nanoscale devices, the Navier-slip boundary condition as a macroscale model of fluid behaviour at a solid wall has seen renewed interest. The penalty concept and variational formulation are extended here to treat partial slip and related boundary conditions in viscous flow simulation. An analysis of the penalty partial-slip formulation permits us to relate it to the classical Navier slip condition and easily embed this class of boundary conditions in existing finite element flow software.

INTRODUCTION

While the most common macroscale model of the boundary condition for a viscous fluid at a solid wall is "no slip" with "no penetration", the validity of the no-slip condition has long been debated [6] and there are flows for which it must be relaxed to allow some slip. Moving contact line problems are examples that have received much attention over a long period of time [7]. Flows in micro and nanofluidic devices are more recent examples that have renewed interest in slip boundary conditions [2]. The most common alternative to the no-slip condition is the slip condition proposed by Navier (see [8]), that at a solid wall the component v_t of the fluid velocity in the direction of the unit tangent $\mathbf{t}$ is proportional to the rate of strain in that direction, the constant of proportionality (λ) being the 'slip length' which is determined empirically. Equivalently,

$$v_t = -\frac{\lambda}{\mu} T_t, \tag{1}$$

where μ is the fluid viscosity and $T_t = \mathbf{T} \cdot \mathbf{t}$ is the tangential stress, defined in terms of the stress vector $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$, with $\boldsymbol{\sigma}$ denoting the standard stress tensor and $\mathbf{n}$ denoting the unit outward normal to the wall. Thus, a zero slip length ($\lambda = 0$) corresponds to zero tangential velocity (no-slip), while full slip or zero tangential stress is obtained as $\lambda \to \infty$. Note that in three dimensions there will be two conditions of the form (1), in the directions of the tangent and binormal vectors, respectively. An expression similar to (1), relating the normal velocity and stress components, is appropriate for a porous wall ('partial penetration').

We here consider penalty methods for implementing such boundary conditions. A variational finite element method is the rational choice for investigating and interpreting penalty approaches. We note that other implementations of partial slip/penetration boundary conditions have presented significant computational difficulties [5], and particular challenges when inclined or curved walls are involved [1, 4]. The penalty method here described has no such computational difficulties, places no limitations on boundary shape and, additionally, is simple to implement.

PENALTY FORMULATION AND ANALYSIS

To introduce the main penalty concepts, we consider the stationary incompressible Stokes-flow problem

$$-\mu\nabla^2\mathbf{v} + \boldsymbol{\nabla} p - \mathbf{f} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{2}$$

for the velocity $\mathbf{v}$ and pressure p in domain Ω with boundary Γ on which $\mathbf{v} = \mathbf{0}$, where μ is the viscosity and $\mathbf{f}$ is the body force. For clarity, we first pose the associated variational problem on the space of divergence free velocity fields $H(\text{div})$ in domain Ω, using the standard penalty approach for including the no-slip condition on Γ and repeated indices to denote summation, which is: find $\mathbf{v} \in H(\text{div})$ that minimizes the functional [3]

$$J(\mathbf{v}) = \int_\Omega \left(\frac{\mu}{2} \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} - f_i v_i \right) dV + \frac{1}{2\epsilon} \oint_\Gamma v_i v_i \, ds. \tag{3}$$

The boundary integral term is a constraint with ϵ a small positive penalty parameter. As ϵ tends to zero, the constraint is more strongly enforced and the solution to the modified minimization problem approaches the solution to the original problem. On the other hand, if the penalty parameter is allowed to increase, the constraint will be enforced less strongly and the no-slip constraint will be progressively weakened allowing both slip and penetration.

A simple extension follows from the above form on introducing tangential-normal coordinates at the boundary and enforcing the tangential boundary condition via a larger scaled penalty coefficient ϵ_t to permit slip while enforcing the no-normal flow condition by the previous small penalty parameter. More specifically, the boundary penalty functional in (3) becomes

$$P = \frac{1}{2\epsilon_t} \oint_\Gamma (v_t^2 + v_b^2) \, ds + \frac{1}{2\epsilon_n} \oint_\Gamma v_n^2 \, ds, \tag{4}$$

where we have separated penalty contributions associated with respective tangential (v_t), binormal (v_b) and normal (v_n) boundary velocity components. If we set both penalty parameters (ϵ_n, ϵ_t) to the same small value then this reduces to the previous penalty form for the no-slip condition. If, as advocated here, certain penalty parameters are scaled relative to others, then we can imbed the Navier slip or other similar conditions in this simple penalty structure.

To analyse this using natural boundary conditions, the divergence condition is not satisfied a priori but instead is enforced in the variational problem through the use of a Lagrange multiplier which, interpreted physically, is the pressure [3]. The corresponding Lagrangian for this saddle-point problem, with the added penalty terms , is

$$\begin{aligned} L_\epsilon(\mathbf{v}, p) \quad = \quad & \int_\Omega \left(\frac{\mu}{2} \frac{\partial v_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - f_i v_i - p \frac{\partial v_j}{\partial x_j} \right) dV \\ & + \frac{1}{2\epsilon_t} \oint_\Gamma (v_t^2 + v_b^2)\, ds + \frac{1}{2\epsilon_n} \oint_\Gamma v_n^2 \, ds, \end{aligned} \tag{5}$$

where we have completed the deformation gradient tensor in the viscous term so as to recover the associated stress boundary term in the resulting natural boundary condition. With penalty parameters equal and small, we have a penalty stick boundary condition, while a penalty slip type of boundary condition in the tangential directions is obtained with ϵ_n small and ϵ_t finite.

To see this, let us examine the associated natural boundary condition for this latter form. On taking variations of the Lagrangian and applying divergence manipulations to the viscous and pressure terms the resulting boundary integral contributions involving $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ can be paired with the penalty boundary integral contributions to yield:

$$\oint_\Gamma \left[\left(T_t + \frac{1}{\epsilon_t} v_t \right) \delta v_t + \left(T_b + \frac{1}{\epsilon_t} v_b \right) \delta v_b + \left(T_n + \frac{1}{\epsilon_n} v_n \right) \delta v_n \right] ds, \tag{6}$$

and the respective mixed type natural boundary conditions follow as $T_n + v_n/\epsilon_n = 0,\ T_t + v_t/\epsilon_t = 0,\ T_b + v_b/\epsilon_t = 0.$ For the case of Navier slip with no penetration the normal penalty parameter is chosen very small so the first of the natural boundary conditions approximates the no normal flow boundary condition and converges to this in the limit as ϵ_n approaches zero, while the choice $\epsilon_t = \lambda/\mu$ yields the Navier slip boundary condition in equation (1). Conversely, for a given choice of ϵ_t we have $\lambda = \mu\, \epsilon_t$ as a compatible Navier slip length associated with the penalty solution. Similarly, for finite ϵ_n we have partial penetration or fluid leakage through the boundary.

CONCLUDING REMARKS

Since penalty methods are now a standard practical strategy for enforcing essential boundary conditions (like the no-slip condition) in finite element computations, this implies that partial slip and penetration conditions can be trivially included in finite element codes by appropriate setting of the desired penalty parameters. In particular we see that the partial slip condition obtained above by this means is related to the classical Navier slip condition, with the tangential penalty parameter dependent on both the Navier slip length parameter and the fluid viscosity. Finally, one can treat free surface flow boundary conditions and contact interactions with a free surface by appropriate penalty treatments. For example, choosing both penalty parameters to be large in the above penalty formulation approximates a stress-free free surface. Dividing a flow domain boundary into multiple segments, each with its own penalty parameters, enables penalty treatments of a variety of boundary conditions in a single flow problem. In addition to fixed boundaries on which no-slip, slip, partial penetration and stress-free conditions apply, by using an algorithm that updates the flow geometry over time, moving contact lines can also be handled. This also allows the penalty parameters on a boundary segment to be varied in time, which is useful for implementing time-varying boundary conditions or for accelerating convergence of algorithms. These penalty strategies have been implemented recently in numerical simulations of problems with partial slip, moving boundary, free surface contact and similar features [9].

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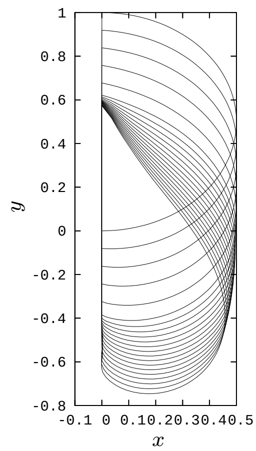

An initially semicircular 2D drop sliding down a vertical plane with Navier-slip and free surface BCs specified using penalty methods.