# Riemann's Hypothesis

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**Abstract**

Riemann's hypothesis (that all non-trivial zeros of the zeta function have real part one-half) is the most famous currently unproved conjecture in mathematics, and a $1M prize awaits its proof. The mathematical statement of this problem is only at about middle undergraduate level; after all, the zeta function is much like the trigonometric sine function, and every (?) student who has completed a first course in calculus knows that all zeros of the sine function are (real) integer multiples of $\pi$. Many of the steps needed to make progress on the proof are also not much more complicated than that. Some of these elementary steps, together with numerical explorations, will be described here. Nevertheless the Riemann hypothesis has defied proof so far, and very complicated and advanced abstract mathematics (that will NOT be described here) is often brought to bear on it. Does it need abstract mathematics, or just a flash of elementary inspiration?

# Preamble

There is little that is new in this colloquium – the new things are mainly a few computations, and some suggestions for changes in emphasis. However, in preparing to introduce this colloquium it seemed to me that almost everything I wanted to say was accessible to anyone who has completed a first course in calculus. The Riemann hypothesis is certainly *describable* at that level, and it might also be *proveable* at that level. In contrast to some who have commented on this matter (see [8], p. 234), I would be *delighted* if that were to be the case. Those of us on the more-applied side of mathematics use special functions routinely as part of our kit of tools, and in principle there is nothing magic about the zeta function compared to the many other special functions in the handbook [1]. Such an assertion might be denied by the pure mathematicians, and its "magic" then leads into their abstract world. But what if all that was needed was a simple identity or inequality? It will have to be a very obscure identity, or else it would have surely been found in the 150 years since Riemann. But perhaps the wrong sort of mathematicians have been looking, and all I really want to do here is to encourage a wider range of mathematically-literate people to try to solve this problem.

# Text for speaking script plus slides

The Riemann zeta function is defined ([1] p.807) by the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

which converges for $\Re s > 1$ if $s$ is a complex number. [Recall for example that the case $s = 1$ gives the *divergent* harmonic series of reciprocal integers, whereas the case $s = 2$ is a *convergent* series of reciprocal squares.]

The zeta function is in some ways similar to the trigonometric sine function, which can be defined ([1] p.74) by the series

$$\sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{6} + \frac{t^5}{120} - \dots$$

which converges for all real or complex values of $t$. (The reason for using $s$ for the argument of one function and $t$ for the other will become clear shortly!).

Now Riemann's hypothesis is simply that "if $\zeta(s) = 0$ then $\Re s = 0.5$". There are a couple of things that need qualifying about that statement in quotes, notably the fact that if $\Re s = 0.5 < 1$, the above series definition of $\zeta(s)$ doesn't converge, but let us briefly put that difficulty aside, and make a comparison with the sine function.

Namely, the equivalent statement for the sine function is that "if $\sin(t) = 0$ then $t$ is real". In fact, everyone knows (see e.g. Figure 1) that these zeros of $\sin(t)$ are integer multiples of $\pi$, but the equivalent of the Riemann hypothesis only requires that they are all real. The proof of reality of the zeros of $\sin(t)$ is a high-school trigonometry exercise as follows.

If $t = x + iy$ is complex, then

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = 0$$

only if both $\sin(x)\cosh(y) = 0$ and $\cos(x)\sinh(y) = 0$. Since $\cosh(y) > 0$ for all $y$, this means that $\sin(x) = 0$. But because $\cos^2(x) + \sin^2(x) = 1$, if $\sin(x) = 0$ then $\cos(x)$ can not be zero, so necessarily $\sinh(y) = 0$ which holds only if $y = 0$, so $t = x =$real, as required.

Incidentally although I have said that everyone knows that (for example) $\sin(\pi) = 0$, what if the only thing we know about the sine function is its series, i.e. what if we don't know any trigonometry? Can we use the series alone to prove $\sin(\pi) = 0$, and eventually to prove that all zeros of the sine function are real? If we can answer "Yes" to this question (and we can for the sine function), maybe we can prove Riemann's hypothesis by using the same methods on the zeta function. Sadly, this is a forlorn hope!

Let us now return to the difficulty that the series for $\zeta(s)$ doesn't converge if $\Re s = 0.5$. At first sight that looks rather fatal, since lack of convergence means that we can't compute the zeta function via its series! However, we can make further progress by analytically continuing the function $\zeta(s)$ into the region $\Re(s) < 1$.

There are lots of ways to do this. I quote first one simple way, though I won't use it in this talk. Namely ([9], p.23),

$$\zeta(s) = \frac{1}{s-1} + \Gamma(s)^{-1} \int_0^\infty x^{s-1} e^{-x} \left[ \frac{1}{1 - e^{-x}} - \frac{1}{x} \right] dx$$

where

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

is the gamma function ([1], p.255), taught to our first year students. It is easy to prove (by expanding $1/(1-e^{-x})$ in a geometric series of powers of $e^{-x}$) that this integral formula for $\zeta(s)$ equals the series when $\Re s > 1$, but in fact this integral converges for $\Re s > 0$, so can be used to define $\zeta(s)$ when $0 < \Re s < 1$.

So anyway, now we at least know how to compute $\zeta(s)$ when $\Re s = 0.5$, and it is possible by numerical integration to show that $\zeta(s) = 0$ when $s = 0.5 + i14.135\ldots$ and $s = 0.5 + i21.022\ldots$ and $s = 0.5 + i25.011\ldots$ and etc. In fact, there are infinitely many such zeros with $\Re s = 0.5$, a result proved in 1914 by G.H. Hardy ([5], [9] p.256). Riemann's hypothesis is (almost) that there are no others, i.e. that $\zeta(s)$ is not zero anywhere else in the complex $s$-plane.

There is that nasty word "almost" still to explain! I won't do it it here, but we can analytically continue even into the negative half-plane $\Re s < 0$ and if we do, we can easily show that $\zeta(-2) = \zeta(-4) = \zeta(-6) = \ldots = 0$. These are the "trivial" zeros, and we won't worry further about these.

Just a bit of history here before we go on with the maths. Bernhard Riemann conjectured his hypothesis in a remarkable short paper presented to the Berlin Academy in 1859, an English translation of which is an Appendix to Edwards' book [4]. Actually he didn't really have a great interest in this hypothesis per se at the time. His paper was entitled "On the number of prime numbers less than a given quantity", and he just mentioned in passing regarding the zeros or roots of a function (see below) related to the zeta function that "... it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation." Later it became clear that this hypothesis was very important in number theory, with implications for prime numbers that I will not discuss here, though I shall mention primes again briefly before the talk ends.

Since 1859, there have been many attempts to prove Riemann's hypothesis, so far without success. The Clay Mathematics Institute of New York has offered prizes of US$1M each for solution of several unsolved problems in mathematics, one of which is the Riemann hypothesis, see *www.claymath.org/prizeproblems/riemann.htm*. Supposedly some powerful and respected mathematicians are very close to a proof, notably Louis de Branges of Purdue University, who proved in 1984 a related result, the Bieberbach conjecture. A summary of de Branges' ideas for proof of the Riemann hypothesis, at least as they stood in 2002, is an Appendix to the popular book by Sabbagh [8]. The fact that it has so far defied proof in spite of massive efforts by good mathematicians is interesting in itself; why should this be so for a function that is so like the simple trigonometric sine?

Let us return to the mathematics, and another analytic continuation ([9], p.22) which emphasises the similarity to the trigonometric sine function. Let us define a new function

$$u(t) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

where $s = 0.5 + it$. Riemann himself showed in his 1859 paper that this function is real when $t$ is real, i.e. when $\Re s = 0.5$, and that it has a Fourier-cosine representation

$$u(t) = \int_0^\infty U(x) \cos xt \; dx$$

where

$$U(x) = -2e^{-x/2} + 4e^{x/2} \sum_{n=1}^{\infty} e^{-n^2 \pi e^{2x}}$$

This looks a bit messy, but it is actually quite straightforward to use to compute $u(t)$ by numerical integration, the series of "exponentials of exponentials" in the definition of the Fourier transform $U(x)$ being very rapidly convergent. Although we are mostly interested in real $t$ or $\Re s = 0.5$, the Fourier representation converges for $0 < \Re s < 1$, so it is as useful an analytic continuation as the previous integral.

Incidentally, an important element in Riemann's proof of the above analytic continuation is the fact that $U(x)$ is an *even* function of $x$ when considered on the whole real $x$-axis. In effect this relates a sum of exponentials with argument proportional to $e^{2x}$ to a sum of exponentials involving its *reciprocal* $e^{-2x}$. This is a rather surprising result, which was actually proved by Poisson as early as 1823, according to Whittaker and Watson [14], p. 124; see also [10], p.444.

Now since $\zeta(s) = 0$ if $u(t) = 0$, Riemann's hypothesis simply states that all zeros of $u(t)$ are real (it has no trivial zeros), so the new function $u(t)$ is quite analogous to the trigonometric function $\sin(t)$. Actually it is more analogous to a "damped" sine, since it tends to zero exponentially for large $t$. Hence in order to compute and plot graphs for large real $t$, it is often convenient to first re-scale to remove this decay (and some other unimportant factors), writing

$$u(t) = -2^{-1/4}\pi^{1/4}t^{7/4}e^{-\pi t/4}\ \hat{u}(t)$$

where the scaled function $\hat{u}(t)$ takes (more-or-less) order one values for large $t$.

When $t$ is large, there is a simple asymptotic expansion of this (scaled) function $\hat{u}(t)$ in terms of a finite sum of cosines, namely

$$\hat{u}(t) = 2\sum_{k=1}^{m} k^{-1/2}\cos\left[t\log\frac{\sqrt{t/2\pi}}{k} - \frac{1}{2}t - \frac{\pi}{8}\right] + O(t^{-1/4})$$

where $m$ is the largest integer less than $\sqrt{t/2\pi}$. [As a historical aside ([3], p.256), Riemann himself obtained this expansion, including an explicit estimate of the $\mathrm{O}(t^{-1/4})$ error term, but did not publish it. Carl Siegel in 1932 found Riemann's handwritten notes and polished them for publication, so the above expression is called the Riemann-Siegel sum.]

The scaled function $\hat{u}(t)$ is usually called the Hardy function and is written $Z(t)$. However, here I will retain the notation $u(t)$, ignoring the "hat" from now on, even when the scaling has been done.

Anyway, we now have various ways to compute $u(t)$ numerically. I have programmed three such methods (including another method of my own [12]), and they give results in numerical agreement with each other in appropriate ranges of validity. But our greatest interest is in large $t$, where the Riemann-Siegel sum comes into its own, whereas numerical integration loses accuracy if $t$ is large.

Figure 2 is a graph of (scaled) $u(t)$ for $0 < t < 100$. It oscillates, with infinitely many zeros, like the sine function. However, its oscillations are neither periodic, nor of constant magnitude. Much effort (e.g. [9], Ch.7) has gone into trying to estimate various statistical properties of this noise-like function.

But still, you may say that if we really want to prove the Riemann hypothesis, what is the use of computing $u(t)$ for real $t$? We have to show not just that $u(t)$ has lots of real zeros, but also that it has no zeros that are *not* real. So in principle it appears that we have to look at complex $t$ as well.

Well, actually this may not be so. Here I have to stray into conjecture-land a bit, or at least into things I don't quite understand. Let me first state a result due to Jeffrey Lagarias [6], who proved in 1997 (in effect) that the Riemann hypothesis is true if and only if

$$\Im\left[\frac{u'(t)}{u(t)}\right] \lesseqgtr 0 \quad \text{when} \quad \Im t \gtreqless 0\,. \tag{1}$$

I have not yet been able to follow Lagarias's proof myself, but there are several reasons why we might believe the result. First there is the simple residue theorem result

$$\int_C \frac{u'(t)}{u(t)}\,dt = 2\pi i N$$

for any analytic function $u(t)$ with no poles and $N$ simple zeros inside a closed curve $C$. After all, locally near each simple zero of $u(t)$ the integrand has a simple pole with residue 1. If we now choose $C$ to be a rectangle (just) in $\Im t > 0$, and do some approximate estimates of the integral, the condition $N = 0$ that there be no such zeros is likely to reduce to the above condition on its integrand (I haven't checked that though). The result also generalises in an obvious way to functions with multiple zeros and poles.

The second reason is based on observing that the Riemann hypothesis must be true if the magnitude of $u(t)$ (and hence the logarithm of that magnitude) always increases as we move off the real $t$-axis in a direction perpendicular to it. (Otherwise how can it ever reduce to zero?) Thus if $t = x + iy$, then

$$\frac{\partial}{\partial y}\log|u(t)| = \frac{\partial}{\partial y}\Re\log u(t)$$

must be positive for $y > 0$ and negative for $y < 0$. But

$$\frac{\partial}{\partial y}\Re\log u(t) = -\Im\frac{d}{dt}\log u(t) = -\Im\frac{u'(t)}{u(t)}$$

Hence the Lagarias equivalence follows.

A third reason takes the above in a geometrical direction, and involves properties of level curves of the real and imaginary parts of $u(t)$. Thus if

$$u(x + iy) = u_1(x, y) + iu_2(x, y) = Re^{i\theta}$$

where $\tan\theta = u_2/u_1$, then also

$$\Im\frac{u'(t)}{u(t)} = \frac{\partial\theta}{\partial x}$$

Now consider level curves $u_1 = 0$ and $u_2 = 0$ in the $(x, y)$ plane near the real axis $y = 0$. The former (say $U$) must cross the real axis at right angles at each real zero of $u_1$. The latter (say $V$) must also cross the real axis at right angles, at turning points of $u_1$, where $u_{1x} = u_{2y} = 0$, and this must happen at least once between any two zeros of $u_1$, by Rolle's theorem. Suppose the turning point is a maximum, and let $U$ start from the left of $V$. Now Riemann's hypothesis is true if and only if $U$ and $V$ do not intersect, since if they did, $u_1 = u_2 = 0$, and such an intersection point would be a zero of $u(t)$. But then in $y > 0$, $\theta = +\pi/2$ on $U$ and $\theta = 0$ on $V$; thus $\theta$ decreases as we move to the right, or $\partial\theta/\partial x < 0$ in $y > 0$, as required. Similar arguments apply in $y < 0$, and for $V$ curves that begin at a minimum.

There is an interesting immediate consequence of the Lagarias equivalence. Clearly on the real axis $y = 0$ itself the quantity $\Im u'(t)/u(t)$ vanishes, but if we expand it in a Taylor series for small $y$, we find that

$$\Im \left[ \frac{u'(t)}{u(t)} \right] = y \left[ \frac{u_1 u_{1xx} - u_{1x}^2}{u_1^2} \right] + O(y^3)$$

where all quantities involving $u_1$ are evaluated on $y = 0$. Now if the Riemann hypothesis is true, the Lagarias equivalence asserts that the numerator is negative, i.e. for real $t$ we have $w(t) > 0$ where

$$w(t) = u'(t)^2 - u(t)u''(t) ,$$

since on the real axis $x = t$ and $u_1 = u$.

This is true if the Riemann hypothesis is true; I believe that the converse may also hold, i.e. that the Riemann hypothesis may be true if $w(t) > 0$. If so, then we need only concern ourself with real values of $t$. Even if this converse fails for some functions, I am confident that it will not fail for the Riemann function. That is, if we could prove that $w(t) > 0$, so that the Lagarias equivalence is satisfied on $y = 0$, I am confident that this will imply its satisfaction for small non-zero $y$.

A further analogy [11] with the trigonometric sine function holds here. Thus if we replace $u(t)$ by $\sin(t)$ we get $w(t) = 1$ which is certainly positive. On the other hand, if we replace $u(t)$ by $\sin(t) + k$ for any real positive constant $k$, we get $w(t) = 1 + k\sin(t)$ and it is interesting that if $k > 1$ there are complex zeros of this replacement $u(t)$, while this $w(t)$ takes some negative values!

When $u(t)$ is a Riemann function, there is a small extra feature though, namely that we can exclude any finite range of $t$; after all, exhaustive searches for zeros of $u(t)$ have failed to find any that are not real in circles about the origin of any finite radius, even up to many billions. That is, I conjecture that the Riemann hypothesis is true if there is a $t_0 > 0$ such that $w(t) > 0$ for all real $t > t_0$.

In fact, my numerical computations [11] strongly suggest that this is indeed true, with $t_0 \approx 5.901$. Figure 3 is a plot for $0 < t < 100$; note that $w(t)$ is (like $u(t)$) scaled to remove the exponential decay factor. But numerical computations can't prove $u(t) > 0$ for *all* $t > 5.901$; I have only computed $w(t)$ up to $t = 12,000,000$ so far [now $30,000,000$, see below]. While that seems to be a big number, we need a proof that $w > 0$ no matter how high is $t$.

Meanwhile, there are some pretty small positive values of $w$. The smallest I have found is (scaled) $w(4,926,490.101792) \approx 0.0011416$ [now lower, see below]. The behaviour of $w(t)$ near this point is interesting, as shown in Figure 4, together with corresponding plot of (scaled) $u(t)$. That value of $t$ is *very* close to (but not exactly at) a maximum of $u(t)$, with $u'(4,926,490.101794) = 0$. This maximum lies almost half-way between two very close zeros of $u(t)$, namely $u(4,926,490.10030) = 0$ and $u(4,926,490.10330) = 0$ . My computations of the locations of these two zeros have been confirmed to at least 11 figures, by comparison with a program available on the internet from G.R. Pugh [7].

As an side issue, perhaps of importance, I have also compared $w(t)$ to the non-negative quantity $u'(t)^2$. Because $u'(t)$ is usually very small when $w(t)$ is very small (as we found near $t = 4,926,490.1$ above), the ratio $K(t) = w(t)/u'(t)^2$ is not necessarily small, and indeed I have never found a value of this ratio less than 0.351 [now 0.34961]. Here [now replaced below by a larger table] are the lowest 12 values of $K(t)$ in the range $10 < t < 12,000,000$.

I think that some attention could be paid to this ratio. Like most functions in this game, it oscillates as if it included random noise, but in this case appears only to have local minima, all of which lie between 0.35 and 1.00. There are no finite maxima, but $K(t)$ tends to positive infinity at all turning points where $u'(t) = 0$. Figure 5 is a graph near the lowest value of this ratio that I have found, namely $K(2,276,676.34) = 0.35122$.

[Added 5 October 2007: I now have extended the computation to $t = 30,000,000$, and here in increasing size are the 30 values of $K(t) < 0.370$. The lowest $K$ of all is now slightly lower at 0.34961, and also I observed a new lowest value of $w = 0.000270$ at $t = 28,355,719.2$.]

$$
\begin{aligned}
K(24,476,747.18) &= 0.34961 \\
K(22,641,125.64) &= 0.34983 \\
K(2,276,676.34) &= 0.35122 \\
K(9,338,720.85) &= 0.35218 \\
K(13,449,079.34) &= 0.35549 \\
K(22,723,124.58) &= 0.35565 \\
K(24,782,255.51) &= 0.35661 \\
K(14,253,736.74) &= 0.35748 \\
K(11,151,603.67) &= 0.35752 \\
K(6,820,051.73) &= 0.36064 \\
K(17,095,484.13) &= 0.36106 \\
K(18,481,859.38) &= 0.36135 \\
K(28,655,617.95) &= 0.36277 \\
K(4,998,855.58) &= 0.36297 \\
K(22,965,339.64) &= 0.36321 \\
K(10,385,792.57) &= 0.36349 \\
K(10,775,253.85) &= 0.36392 \\
K(22,383,080.74) &= 0.36402 \\
K(20,672,095.87) &= 0.36426 \\
K(3,745,331.66) &= 0.36501 \\
K(5,393,528.30) &= 0.36514 \\
K(17,185,753.77) &= 0.36617 \\
K(29,477,788.25) &= 0.36668 \\
K(18,046,379.01) &= 0.36789 \\
K(19,642,897.25) &= 0.36867 \\
K(7,723,713.96) &= 0.36875 \\
K(11,155,215.54) &= 0.36905 \\
K(8,694,161.99) &= 0.36915 \\
K(14,147,334.67) &= 0.36938 \\
K(26,998,373.84) &= 0.36982
\end{aligned}
$$

But we still haven't proved $w(t) > 0$, so even if my conjecture is right, we haven't proved the Riemann hypothesis. I have pursued quite a few avenues toward a proof that $w(t) > 0$, with no success so far. Perhaps there is an identity, expressing $w(t)$ as a sum of squares, yet to be discovered.

One avenue toward a proof of $w(t) > 0$ is to make use of the fact that we have a Fourier-cosine representation for $u(t)$, to express the quadratic combination $w(t)$ as a similar Fourier integral [13] by the convolution theorem. Then one might be able to use certain known results for positivity of Fourier integrals; no luck so far! In this context it is important that, as I have observed, the Fourier-cosine transform $U(x)$ of $u(t)$ is an *even* function on the whole real axis, and hence so is the Fourier-cosine transform of $w(t)$. Figure 6 shows the latter function, which is very smooth and bell-shaped. It is quite difficult [13] to prove positivity of functions whose Fourier-cosine transform is of that character.

I close by reminding you that prime numbers have something to do with this story! They certainly are very important in *applications* of the Riemann hypothesis; are they also important in its *proof*? One of the most fundamental results involving the Riemann zeta function ([1] p.807) is the Euler product

$$\zeta(s) = \prod_{p=2}^{\infty} (1 - p^{-s})^{-1}$$

over the primes $p$, which implies by logarithmic differentiation that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{p=2}^{\infty} \frac{\log p}{1 - p^s}$$

Like the original series definition, these Euler products and sums converge only if $\Re s > 1$, and nobody has so far succeeded in analytically continuing them into the neighborhood of $\Re s = 0.5$, as would be required to use them to prove the Riemann hypothesis. Why would we want to, given that we already have other analytic continuations?

On the other hand, an interesting observation of Titchmarsh ([9] p. 45) is that "...examples are known of functions that are extremely like the zeta function in ... (other representations) ..., but which have no Euler product, and for which the analogue of the Riemann hypothesis is false." So it is possible that *only* by making some use of the Euler product can we prove the Riemann hypothesis. In that context, it is at least interesting that the ratio $\zeta'(s)/\zeta(s)$, related to $u'(t)/u(t)$, plays a role, and we have already seen the importance of the latter.

The Riemann hypothesis is remarkably different from other famous mathematical problems, especially some that have been solved recently like the 4-colour problem and Fermat's last theorem. The latter problems are easy to describe to a non-mathematician, but their proofs require abstract mathematics of the highest order of difficulty, and a lot of it.

In contrast, Riemann's hypothesis is somewhat harder to describe to a total non-mathematician, though I hope you will agree after hearing this talk that it is not at all hard to describe to anyone who has completed an undergraduate calculus course. It also "looks" like it should be possible to prove it in a few lines of undergraduate-level mathematics. All we need is an appropriate identity, e.g. one that shows that the quadratic expression $w(t)$ defined here is positive. However, that may just be an illusion – surely such an identity would have been found after 148 years of effort, including work by geniuses like Ramanujan who specialised in such identities. So maybe it needs some big guns after all. Personally I hope not, and I would love to see an "elementary" proof some day.

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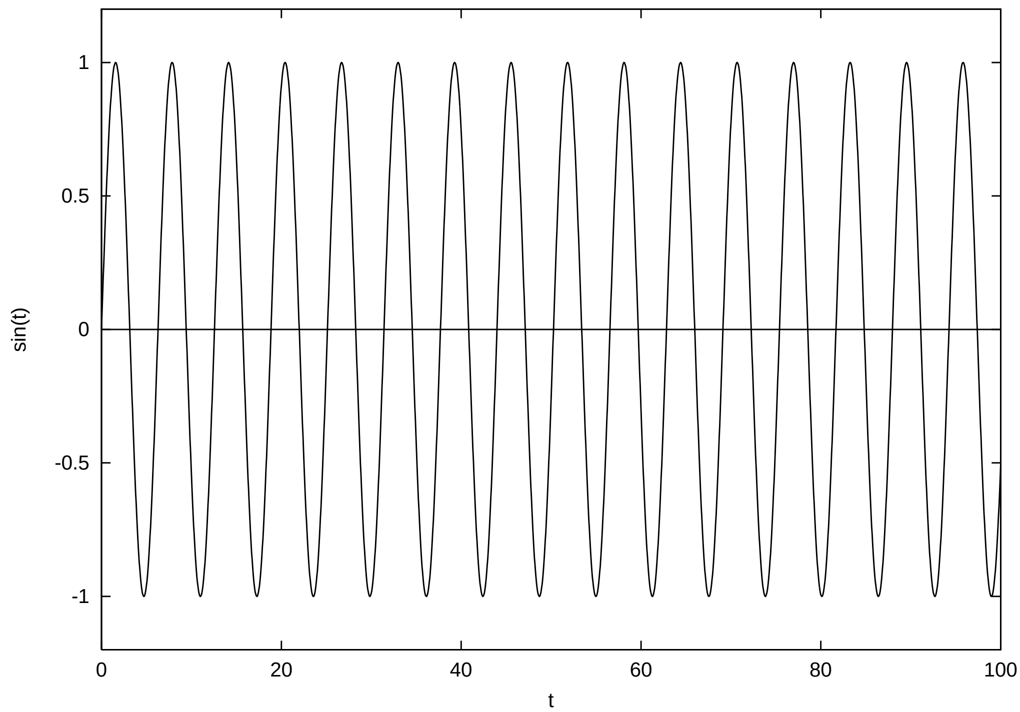


Figure 1: Plot of $\sin(t)$, for $0 < t < 100$

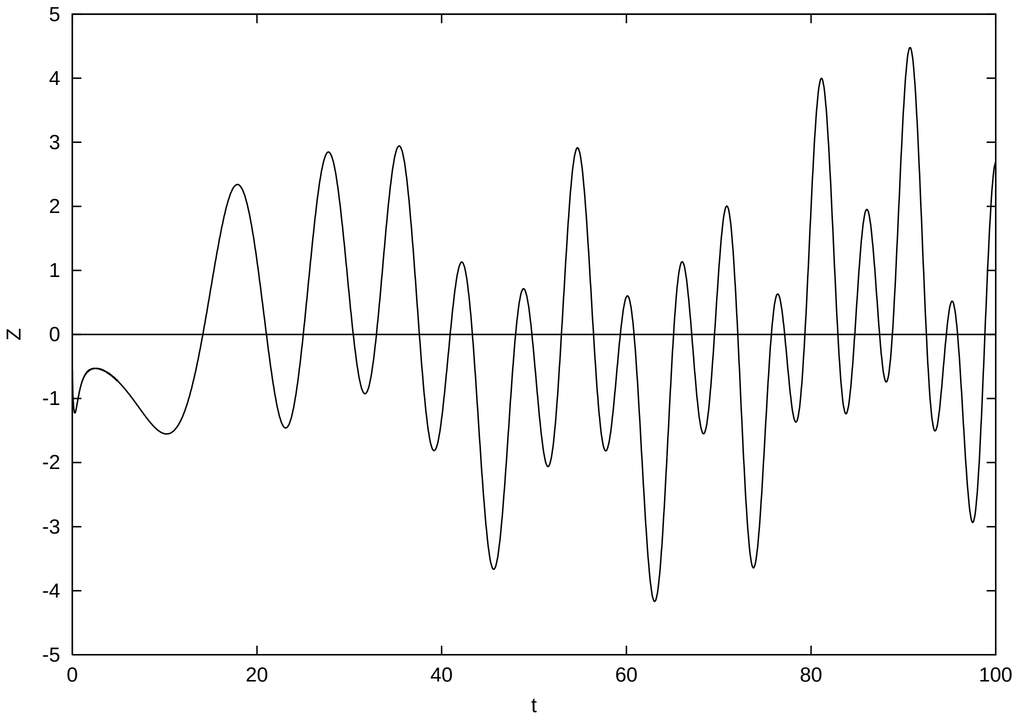


Figure 2: Plot of (scaled) $u(t)$, for $0 < t < 100$

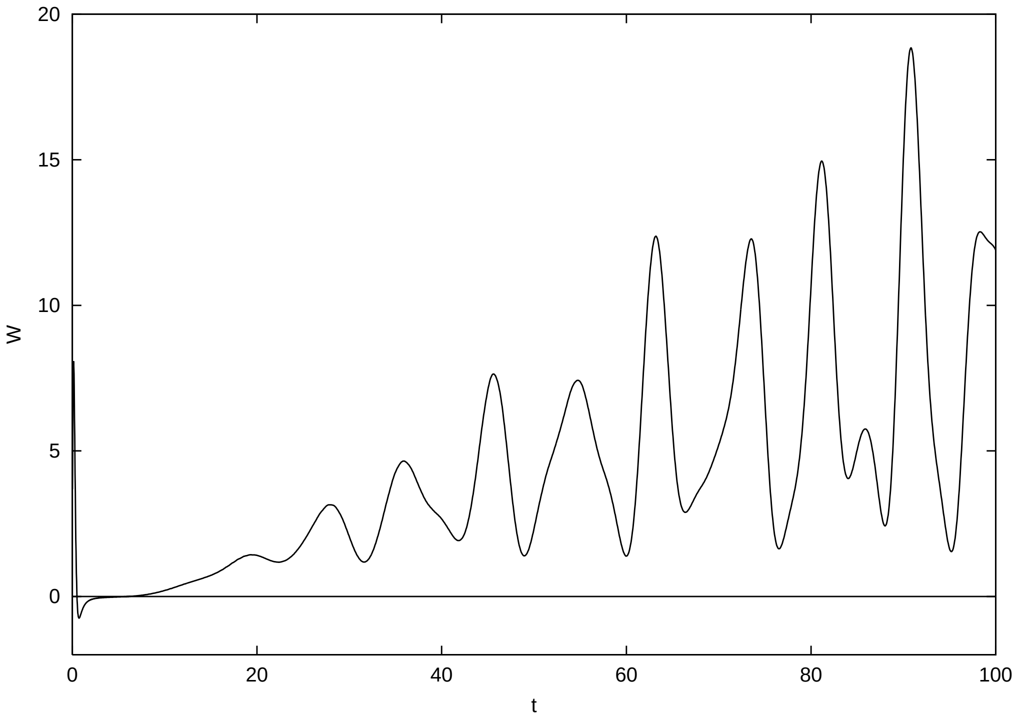


Figure 3: Plot of (scaled) $w(t)$, for $0 < t < 100$

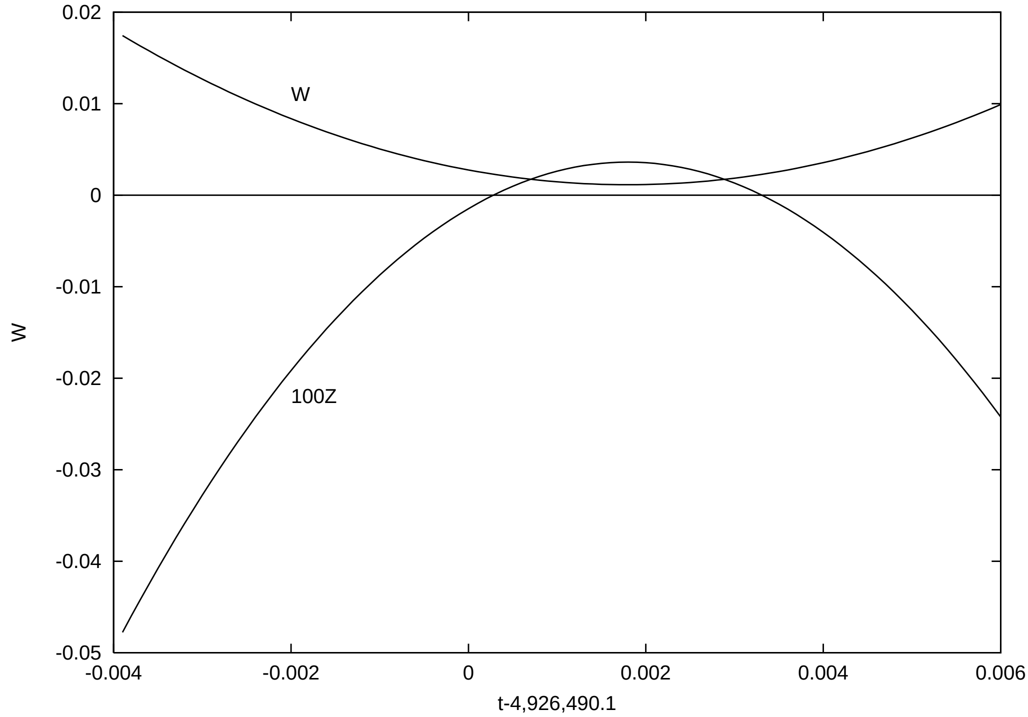


Figure 4: Plot of (scaled) $w(t)$ and $u(t)$ near $t = 4,926,490.1$

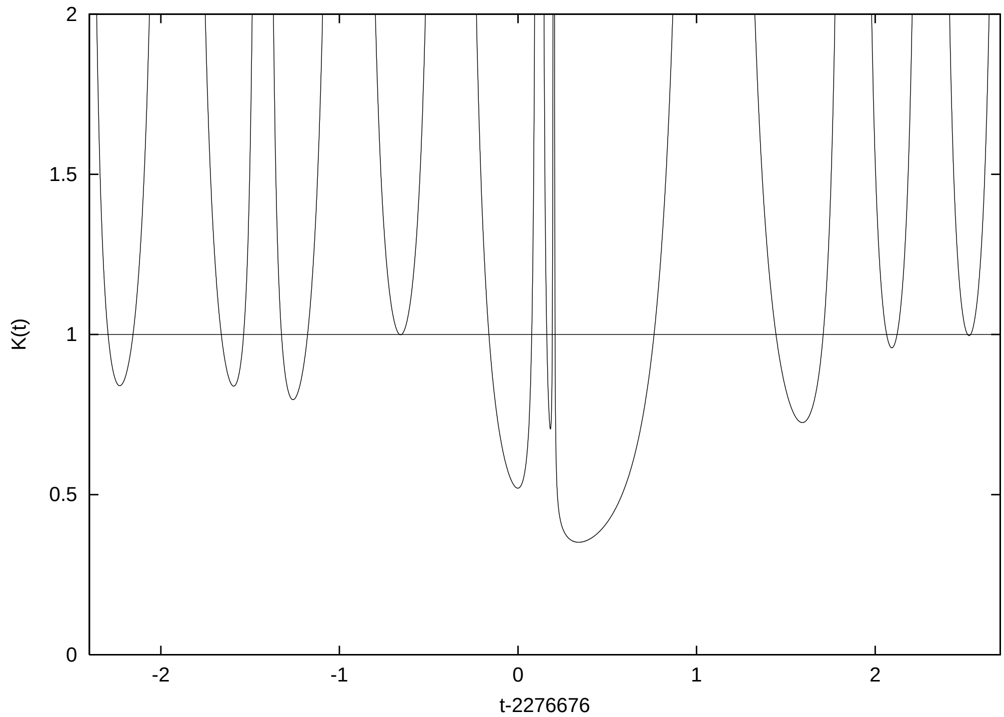


Figure 5: $K(t)$ near its lowest local minimum at $t = 2,276,676$.

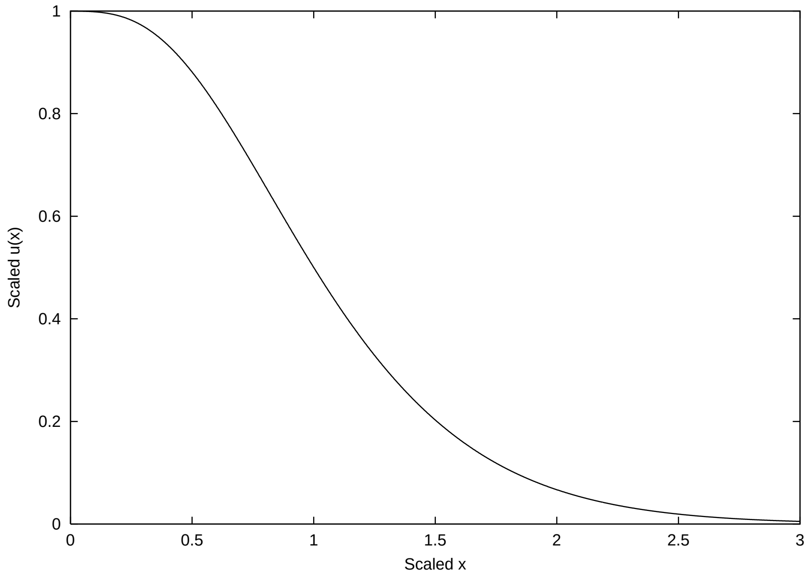


Figure 6: Fourier-cosine transform of $w(t) = u'(t)^2 - u(t)u''(t)$.