When does the first derivative exceed the geometric mean of a function and its second derivative?

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Given a real function $u = u(x)$ defined in some region of real x, the quantity $w = {u'}^2 - uu''$ is "often" (but by no means always) positive throughout that region. The purpose of this note is to suggest some restrictions on u that are sufficient to guarantee that $w > 0$. [Strictly speaking, the verbalised title of the note is relevant only to the special case when u, u' and u'' are all separately positive, in which case $w > 0$ does indeed imply that $u' > \sqrt{uu''}$].

Since w/u^2 is the derivative of $-u'/u$, the condition $w > 0$ is equivalent to u'/u being a decreasing function, or to $-\log u$ being a convex function. There is a relationship between a decreasing property of u'/u and reality of the zeros of u (see e.g. [1], p. 128), and most functions u of interest here possess many real zeros.

Examples on the whole real line

The example $u = \sin x$, which is such that $w = 1$ for all x, is perhaps too simple, though emphatic! The slightly more general example $u = k + \sin x$ for real constant k, which is such that $w = 1 + k \sin x$, is instructive in that $w > 0$ for all x if and only if $|k| < 1$. Tellingly, if $|k| < 1$ then u has infinitely many real zeros but no complex zeros, whereas if $|k| > 1$ then u has only complex zeros with non-zero imaginary part.

Simple polynomials like $u = x^2 - k$ with $w = 2(x^2 + k)$ again illustrate $w > 0$ for functions u having only simple real zeros ($k > 0$), and w changing sign if u has complex zeros ($k < 0$).

On the other hand, real zeros are not essential for $w > 0$, as the example $u = \exp(-x^2)$ with $w = 2\exp(-2x^2)$ shows; but it is of interest that this function u has two real inflexion points.

These examples and some others are considered further later.

Preliminary results

(1) Directly from its definition, $w = {u'}^2 - uu'' > 0$ in any region where $uu'' < 0$.

(2) Since $w' = u'u'' - uu'''$, similarly $w' > 0$ in any region where $u'u'' > 0$ and $uu''' < 0$. If this is so in $x > a$, then also $w > 0$ in $x > a$, providing $w(a) \geq 0$.

(3) Since $w'' = {u''}^2 - uu''''$, then $w'' > 0$ in any region where $uu'''' < 0$. If this is so in $x > a$, then also $w' > 0$ in $x > a$ providing $w'(a) \geq 0$, and then also $w > 0$ providing $w(a) \geq 0$.

(4) Also directly from the definition, in order that $w > 0$, any turning point ($u' = 0$) must have $uu'' < 0$, so it must be either a positive ($u > 0$) maximum ($u'' < 0$), or a negative ($u < 0$) minimum ($u'' > 0$).

It is straightforward using these four results to analyse fully the region between any two consecutive simple zeros, paying particular attention to turning points and inflexion points. Then, by considering a general segment as the union of such regions and zero-free end regions, the following can be shown.

Sufficient conditions

Let $u = u(x)$ be a smooth real function defined in some segment (or all) of the real x-axis. Then ${u'}^2 - uu'' > 0$ if all of the following are true:

– any zeros of u are simple, i.e. u' does not vanish where u vanishes;

– any turning points of u are either positive maxima or negative minima, i.e. $uu'' < 0$ where u' vanishes;

– there is exactly one turning point and at most two inflexion points between any two consecutive zeros, and if there are two such inflexion points, the turning point lies between them;

– either $uu'' < 0$, or else (in a region with $uu'' > 0$ between a zero and an inflexion point) u''' changes sign at most once and $uu'''' < 0$ in any part of that region where $u'u''' > 0$.

Remarks

The conditions stated are sufficient, but it is not clear how many of them are necessary. In particular, the involvement of the fourth derivative in the last condition seems at first sight a little unlikely; nevertheless it may well be necessary. I note via direct computation that the real-valued Riemann function $u = \xi(1/2 + ix)$ on its critical line ([1], p. 16) satisfies all of the above, at least for $6 < x < 10^8$. In particular, it always has $uu'''' < 0$ in the relatively uncommon regions where both $uu'' > 0$ and $u'u''' > 0$.

A good example of the necessity of some of the other conditions is $u(x) = (4-x^2)(2+x^2)$ on $[-2, 2]$, which has three turning points, and suffers $w(0) = -32$. Similarly, $u(x) = (4 - x^2)(3 + 2x + x^2)$ has only one turning point, at $x = 1$, but two inflexion points in $(-2, 1)$, and suffers $w(-1/2) = -99/16$. On the other hand $u(x) = (4 - x^2)(4 + 2x + x^2)$ succeeds in having $w(x) > 0$ in all of $[-2, 2]$, in spite of the fact that its two inflexion points are also both to the left of its (single) turning point.

The above sufficient conditions are mostly of interest for functions with many real zeros. In particular they say little about the everywhere-positive example $u = \exp(-x^2)$, except for showing that $w > 0$ between the inflexion points at $x^2 = 1/2$, although we actually know that $w > 0$ everywhere for this function.

The "too-simple" example $u = \sin x$ with infinitely many zeros has the property that its inflexion points and zeros coincide. Functions like $u = \xi(1/2 + ix)$ with many more-or-less evenly spaced zeros often have inflexion points that are in a sense "close" to their zeros. That is, there is one (and only one) turning point about midway between every pair of zeros, and then one (and only one) inflexion point about midway between every pair of turning points. Only in very special cases like $u = \sin x$ does this rough idea become exact enough to cause the inflexion points to coincide with the zeros, but in many other cases (e.g. $u = k + \sin x$ with $|k| < 1$) there is not more than one inflexion point between a zero and the nearest turning point, which is consistent with the above sufficient conditions.

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References

[1] H.M. Edwards, *Riemann's Zeta Function* (Academic Press 1974).

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