Smallest minima of $1 - Z(t)Z''(t)/Z'(t)^2$

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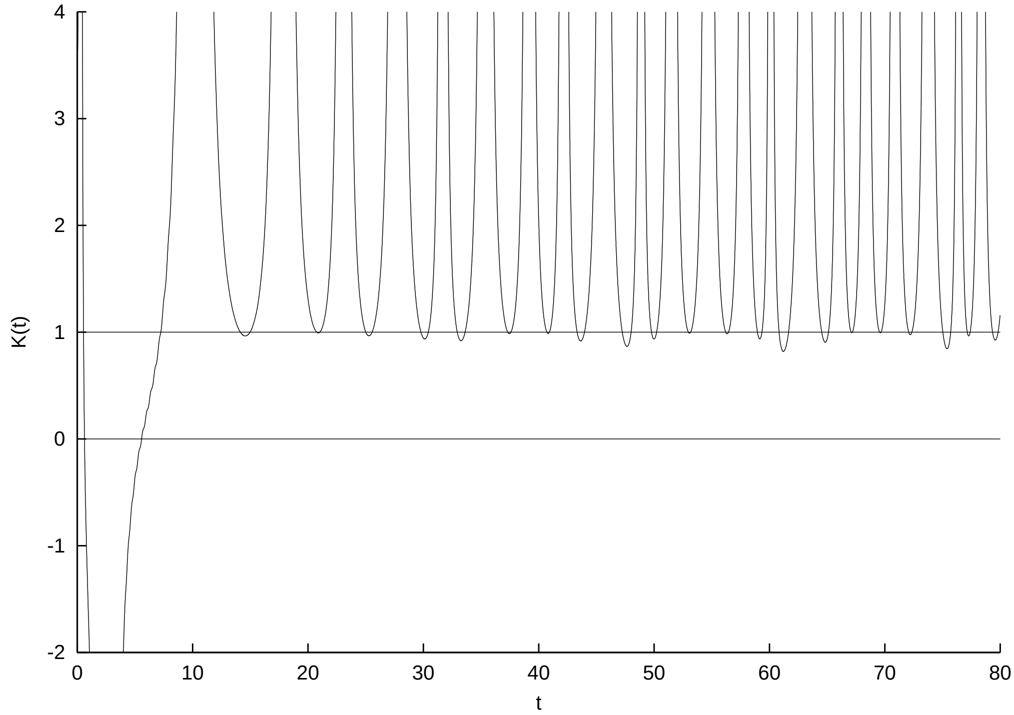
Abstract

The real Hardy function $Z(t)$ is proportional to the Riemann zeta function on its critical line, and the unproved Riemann hypothesis is that $Z(t)$ has only real zeros. For any function $Z(t)$, reality of its zeros may be equivalent to $W(t) > 0$, where $W(t) = Z'(t)^2 - Z(t)Z''(t)$. If $Z(t)$ is the Hardy function, this seems to be true computationally, at least for $5.901 < t < 30,000,000$. The ratio $K(t) = W(t)/Z'(t)^2$ is particularly interesting, having no maxima, and local minima which all seem to lie in the range $0.349 < K(t) < 1$. Even $K(t) > 0$ would prove the Riemann hypothesis, but in fact computations up to $t = 30,000,000$ suggest the stronger result $K(t) > 0.349$. Sadly, such computations don't constitute a proof!

Before even defining $Z(t)$ or explaining why we might be interested in

$$K(t) = \frac{Z'(t)^2 - Z(t)Z''(t)}{Z'(t)^2}$$

let me show you the graph of $K(t)$ and samples of the lowest of its minima. This $K(t)$ is an interesting function, irrespective of its definition and its applications. It is a real function, defined and easily computable for all real positive t.

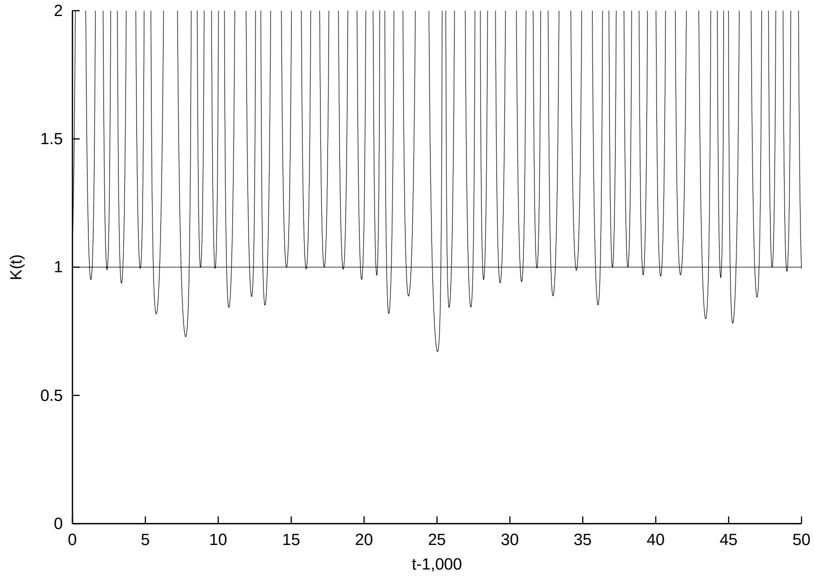


Plot of $K(t)$, for $0 < t < 80$

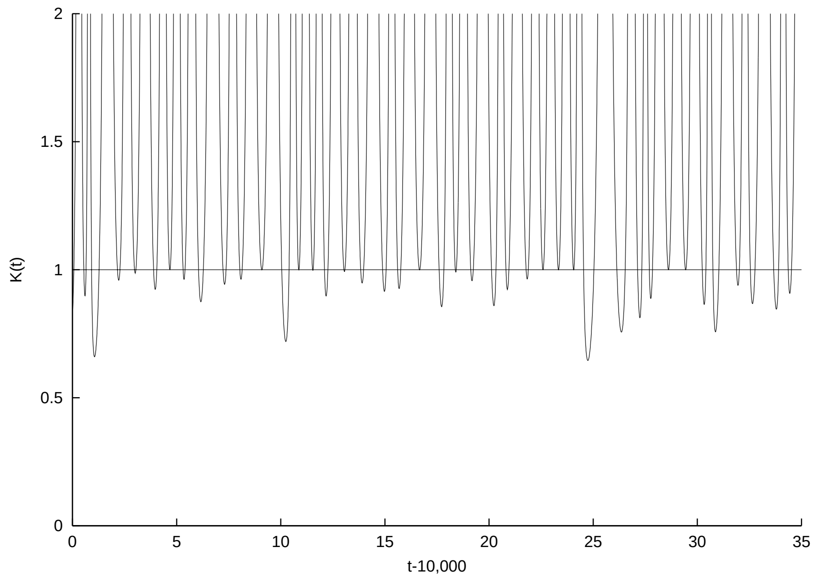
Here is its graph for $0 < t < 80$. It starts out with some negative values, but for all $t > 6$ seems to stay positive. There are no maxima, but many minima. In between these minima there are points where $K(t) \to +\infty$; these are just the zeros of the quantity $Z'(t)$ that appears in the denominator, i.e. the turning points of $Z(t)$. Remarkably, all minima seem to have $0 < K(t) < 1$; most of them in fact are only just below 1.

Now this pattern continues for ever as t increases. Here are graphs near $t = 1,000$, near $t = 10,000$, near $t = 100,000$, near $t = 1,000,000$, and near

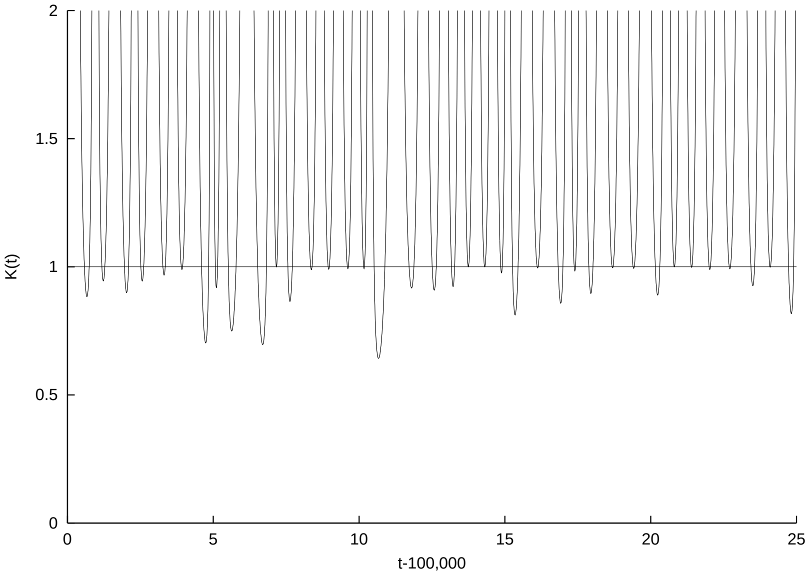
$t = 10,000,000$. Pretty similar and pretty boring! These are quite typical; does anything more interesting happen anywhere?



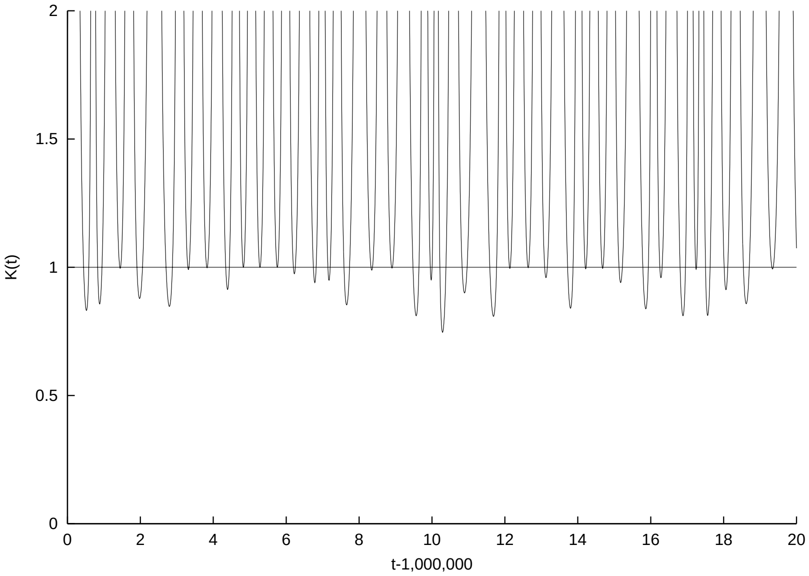
Plot of $K(t)$, for $t \approx 1,000$



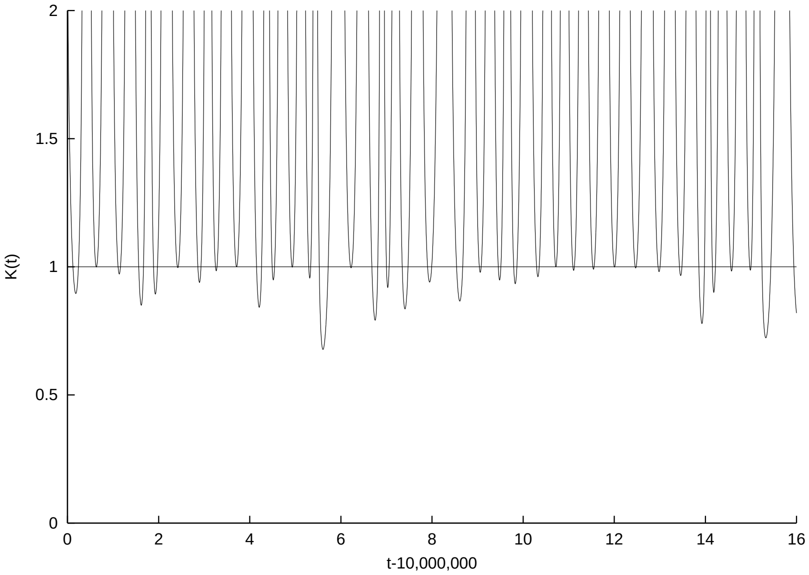
Plot of $K(t)$, for $t \approx 10,000$



Plot of $K(t)$, for $t \approx 100,000$

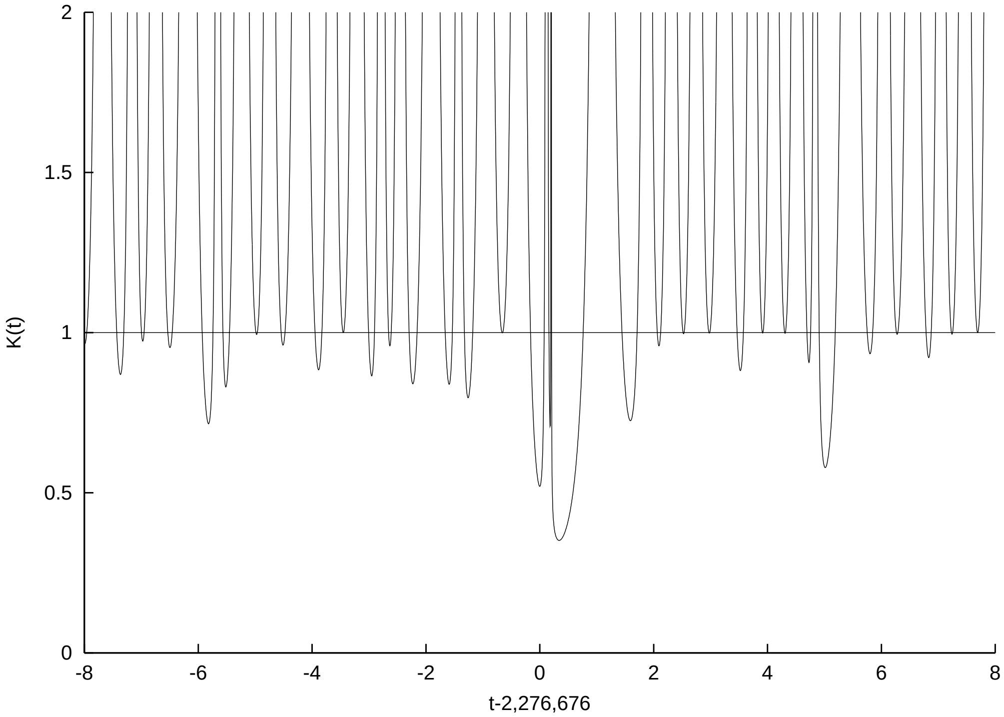


Plot of $K(t)$, for $t \approx 1,000,000$

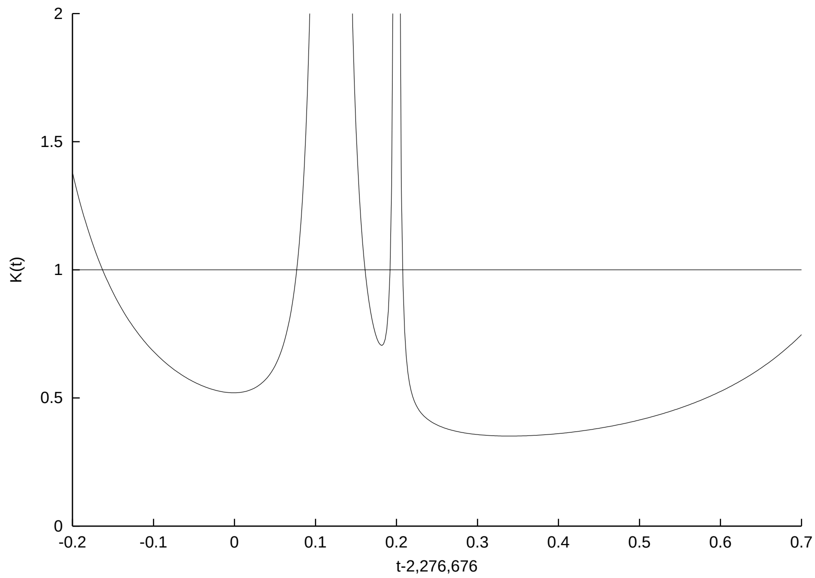


Plot of $K(t)$, for $t \approx 10,000,000$

Well, something a bit more interesting happens near some particular values of t. Here is a graph near $t = 2,276,676$. It is still much the same, except that one of the minima is significantly below 1, of value about 0.35. This minimum is relatively broad compared to most others, although there is a sharp but higher minimum nearby. Since the above graph is a bit cluttered near this sharp minimum, we also show a magnified view in the following graph.

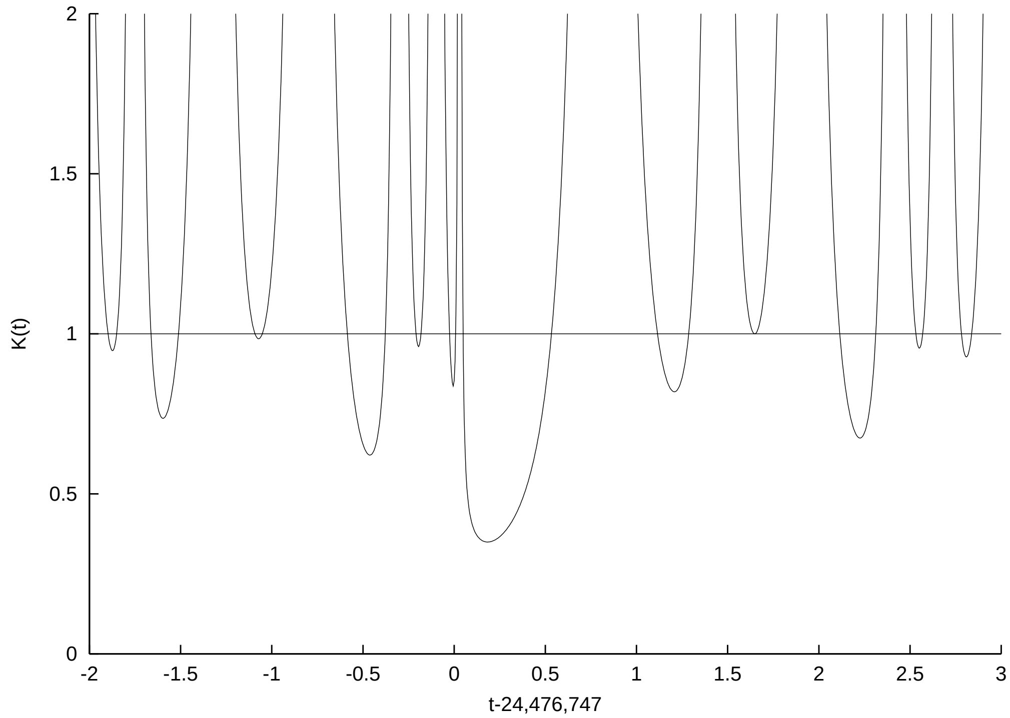


Plot of $K(t)$, for $t \approx 2,276,676$



Plot of $K(t)$, for $t \approx 2,276,676$ (magnified view)

Similarly, here is the neighbourhood of $t = 24,476,747$. In fact, this particular graph shows the lowest minimum that I have found so far, of value 0.34961.



Plot of $K(t)$, for $t \approx 24,476,747$

Here tabulated in increasing size are the lowest 30 minima that I have found so far, after computing up to $t = 30$ million. I find this table rather fascinating. It does seem very likely that there is a positive lower bound not much less than 0.35. There is no pattern in the t values where these special minima occur, and in particular they do not become more or less frequent as t increases. There are really remarkably few of these low minima, e.g. less than 40 out of a total of more than 30 million local minima have $K < 0.37$!

$$
\begin{aligned}
K(24,476,747.18) &= 0.34961 \\
K(22,641,125.64) &= 0.34983 \\
K(2,276,676.34) &= 0.35122 \\
K(9,338,720.85) &= 0.35218 \\
K(13,449,079.34) &= 0.35549 \\
K(22,723,124.58) &= 0.35565 \\
K(24,782,255.51) &= 0.35661 \\
K(14,253,736.74) &= 0.35748 \\
K(11,151,603.67) &= 0.35752 \\
K(6,820,051.73) &= 0.36064 \\
K(17,095,484.13) &= 0.36106 \\
K(18,481,859.38) &= 0.36135 \\
K(28,655,617.95) &= 0.36277 \\
K(4,998,855.58) &= 0.36297 \\
K(22,965,339.64) &= 0.36321 \\
K(10,385,792.57) &= 0.36349 \\
K(10,775,253.85) &= 0.36392 \\
K(22,383,080.74) &= 0.36402 \\
K(20,672,095.87) &= 0.36426 \\
K(3,745,331.66) &= 0.36501 \\
K(5,393,528.30) &= 0.36514 \\
K(17,185,753.77) &= 0.36617 \\
K(29,477,788.25) &= 0.36668 \\
K(18,046,379.01) &= 0.36789 \\
K(19,642,897.25) &= 0.36867 \\
K(7,723,713.96) &= 0.36875 \\
K(11,155,215.54) &= 0.36905 \\
K(8,694,161.99) &= 0.36915 \\
K(14,147,334.67) &= 0.36938 \\
K(26,998,373.84) &= 0.36982
\end{aligned}
$$

What's going on? Can we prove that all minima are in the range 0.349 to 1? Indeed, can we even prove the weaker result that $K > 0$ for $t > 6$? If we could, we would have proved the Riemann hypothesis!

It's time to define and discuss $Z(t)$ and hence $K(t)$.

For the present purpose, I choose to define

$$Z(t) = 2^{-3/4}\pi^{-1/4}t^{1/4}e^{\pi t/4}u(t)$$

where

$$u(t) = \pi^{-1/4-it/2}\Gamma(1/4 + it/2)\zeta(1/2 + it)$$

and $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

Riemann himself proved that $u(t)$ is real when t is real, and hence so is $Z(t)$.

The above $Z(t)$ is "almost" the Hardy function, the usual definition of the Hardy function having a different scale factor multiplying $u(t)$, but which asymptotes to the above rapidly for large t. Indeed, as the examples above suggest, our main interest is in large values of t, for which the Riemann-Siegel sum

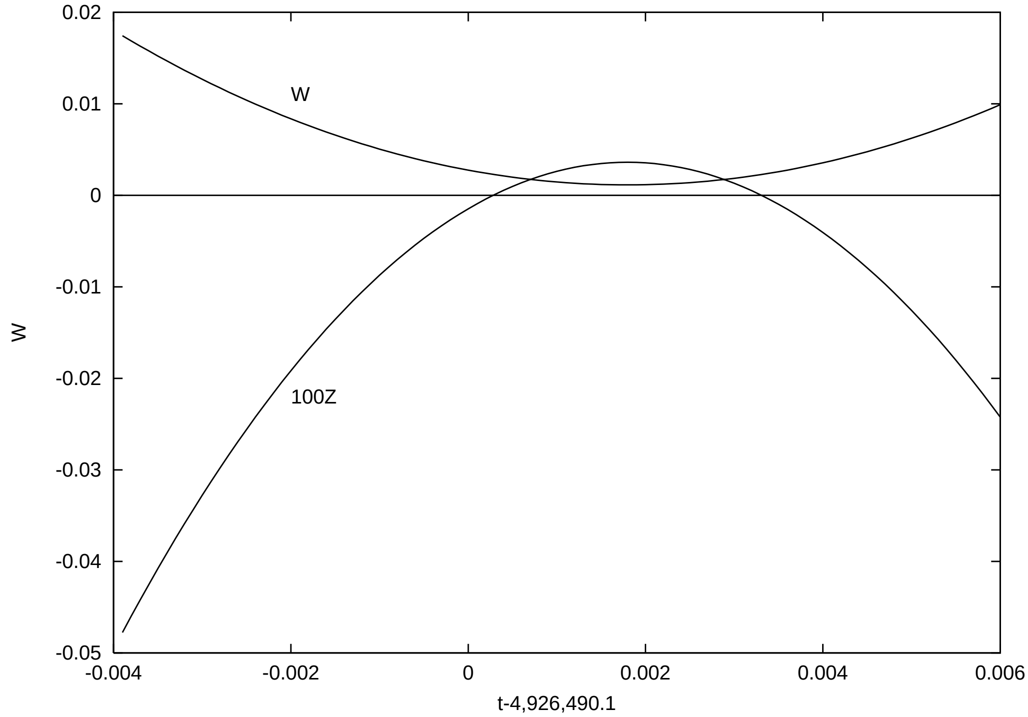
$$Z(t) = 2\sum_{k=1}^{[\sqrt{t/2\pi}\,]} k^{-1/2}\cos\left(t\log\frac{\sqrt{t/2\pi}}{k} - \frac{1}{2}t - \frac{\pi}{8}\right) + O(t^{-1/4})$$

enables very rapid computation, especially when using good estimates for the error term.

Riemann's (unproved) hypothesis is that all non-trivial zeros of $\zeta(s)$ have real part 1/2, or equivalently that all zeros of $Z(t)$ are real. I believe that reality of zeros is equivalent (for any function $Z(t)$, not just the Hardy function) to the statement that the quadratic combination of derivatives

$$W(t) = Z'(t)^2 - Z(t)Z''(t)$$

is positive. This equivalence is also unproved, though something close to it has been proved by Lagarias, and I can construct plausible arguments; some day perhaps I will make these into a proper proof of equivalence. It is not necessary for $W(t)$ to be positive for all $t > 0$, and it may be sufficient that there be a $t_0 > 0$ such that $W(t) > 0$ for all $t > t_0$. For the Hardy function defined as above, $t_0 \approx 5.901$.



Plot of $W(t)$ and $Z(t)$, for $t \approx 4,926,490.1$

Finally, why have I looked at the ratio $K(t) = W(t)/Z'(t)^2$ rather than $W(t)$ itself? Because most of the time, when $W(t)$ is small, so is $Z'(t)$. The quantity $W(t)$ takes many quite small values in my computations up to $t = 30,000,000$, e.g. $W(4,926,490.101792) \approx 0.0011416$. The following graph shows $W(t)$ and $Z(t)$ near this point. Clearly the minimum of $W(t)$ is very close indeed to the local maximum of $Z(t)$, with $Z'(4,926,490.101794) = 0$. This maximum lies almost half-way between two very close zeros of $Z(t)$, namely $Z(4,926,490.10030) = 0$ and $Z(4,926,490.10330) = 0$. My computations of the locations of these two zeros have been confirmed to at least 11 figures, by comparison with a program available on the internet from G.R. Pugh. Since the minimum of $W(t)$ is so close to a zero of $Z'(t)$, the ratio $K(t)$ is not particularly small in this neighbourhood, and hence this ratio provides a more robust measure of positivity of $W(t)$.

Indeed, as I have shown you, the ratio $K(t)$ seems to have a positive lower bound of not much less than 0.35. I have of course spent considerable effort attempting to prove analytically these computational observations, without success. I hope this talk might stimulate some of you to try.