Criteria for real zeros

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Abstract

Given an analytic function $f(z)$ that is real when z is real, define $w = f'(x)^2 - f(x)f''(x)$. Then $w > 0$ guarantees that $f(x+iy)$ cannot vanish for $0 < |y| < b$, for some $b \propto \sqrt{w}$.

Suppose $F(y)$ is a real even function of y such that $F(0) \geq 0$ and $F''(0) > 0$. Suppose also that in an interval $0 < y < a$, either the fourth derivative $F''''(y)$ is non-negative, or, if it does takes some negative values, it has a lower bound $-M$ for some $M > 0$. Then $F(y) > 0$ for $0 < |y| < \min(a, b)$, where $b = \sqrt{12F''(0)/M}$.

Proof of the above result is trivial, based on the $\mathrm{O}(y^4)$ Lagrange remainder for the Taylor series. Note that the result quoted is independent of the actual value of $F(0)$. If $F(0) > 0$, we can obtain a slightly stronger result (i.e. a bigger value of the interval size b), but $F(0) = 0$ is the worst case.

Now apply that result to $F(y) = |f(x+iy)|^2$, which is such that $F''(0) = 2w$, and the required criterion follows immediately.