On positivity of Fourier transforms

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Abstract

This note concerns Fourier transforms on the real positive line. In particular, we seek conditions on a real function u(x) in x > 0, that ensure that its Fourier-cosine transform $v(t) = \int_0^\infty u(x) \cos xt \, dx$ is positive. We prove first that this is so for all t > 0, if u''(x) > 0 for all x > 0, i.e. that everywhere-convex functions have everywhere-positive Fourier-cosine transforms. We then obtain a complex-plane criterion for some types of non-convex u(x). Finally we consider criteria on u(x) that imply positivity of v(t) for $t > t_0$, for some $t_0 > 0$.

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Introduction

Define for t > 0 the ordinary Fourier-cosine transform

$$v(t) = \int_0^\infty u(x) \cos xt \ dx \tag{1}$$

with inverse

$$u(x) = \frac{2}{\pi} \int_0^\infty v(t) \cos xt \, dt \,,$$
 (2)

with a similar definition for the Fourier-sine transform.

Generally we shall assume here that u(x) is real and smooth in x > 0 and that the Fourier integral (1) converges. In particular, u(x) and all of its derivatives are bounded everywhere in x > 0 and tend to zero as $x \to +\infty$. Meanwhile, u(x) could be bounded at the origin, but more generally could have a weak singularity, with $xu(x) \to 0$ as $x \to 0_+$, i.e. u(x) grows at a rate less than x^{-1} . For Fourier-sine transforms, we can allow a stronger singularity at $x = 0_+$, with any growth rate less than x^{-2} . We shall also generalise the results later, to allow even stronger singularities at the origin. One important class of functions u(x) is "convex", i.e. such that u''(x) > 0for all x > 0, which implies (since $u'(+\infty) = 0$) that u'(x) < 0 and (since $u(+\infty) = 0$) that u(x) > 0. That is, convex functions possessing Fourier transforms are also decreasing and positive. Such convex functions need not be smooth at x = 0, indeed not even bounded so long as they are integrable. In particular, they need not (indeed cannot) have all of their odd-order derivatives zero at $x = 0_+$, and hence do not extend smoothly as even functions into x < 0. We shall show that convex functions have everywhere-positive Fourier-cosine transforms. An elementary convex example is $u(x) = e^{-x}$ with $v(t) = 1/(1 + t^2) > 0$.

However, we are more interested here in non-convex functions u(x) which are bounded, positive and decreasing in x > 0, which extend smoothly as an even function to the whole real line, i.e. all odd-order derivatives vanish at $x = 0_+$, and which usually have a single inflexion point in x > 0. Let us call such functions "bell-shaped" functions.

Some bell-shaped functions have positive Fourier transforms, and some don't. Thus compare $u(x) = 1/(1 + x^2)$, which has transform $v(t) = (\pi/2)e^{-t}$, with $u(x) = 1/(1 + x^4/4)$, which has transform $v(t) = (\pi/2)e^{-t}(\cos t + \sin t)$. One v(t) is positive, the other oscillates between positive and negative values, but both u(x) are bell-shaped and have quite similar graphs. A criterion for discriminating between such bell-shaped functions would be of some value.

Proof of positivity for convex functions

Positivity of Fourier-sine transforms is somewhat easier to prove than that of Fourier-cosine transforms. But by integration by parts we have

$$v(t) = -\frac{1}{t} \int_0^\infty u'(x) \sin xt \, dx \,, \tag{3}$$

given that the assumed convergence requirements $(u \to 0 \text{ as } x \to +\infty \text{ and } xu(x) \to 0 \text{ as } x \to 0_+)$ eliminate the integrated part. That is, the Fourier-cosine transform of u(x) is -1/t times the Fourier-sine transform of its derivative u'(x).

Now let us prove that the Fourier-sine transform of a decreasing function w(x) is positive. That is,

$$\int_{0}^{\infty} w(x) \sin xt \, dx = \sum_{j=0}^{\infty} \int_{2\pi j/y}^{2\pi (j+1)/y} w(x) \sin xt \, dx$$
$$= \frac{1}{t} \sum_{j=0}^{\infty} \int_{0}^{2\pi} w\left(\frac{2\pi j + \theta}{t}\right) \sin \theta \, d\theta$$
$$= \frac{1}{t} \sum_{j=0}^{\infty} \int_{0}^{\pi} \left[w\left(\frac{2\pi j + \theta}{t}\right) - w\left(\frac{2\pi j + \theta}{t} + \frac{\pi}{t}\right) \right] \sin \theta \, d\theta$$
(4)

If w(x) is a decreasing function for all x, the quantity in square brackets is positive for all t and all j, and so is $\sin \theta$ in $(0, \pi)$; hence the Fourier-sine transform is positive. This is essentially a simple geometrical result, each negative halfperiod loop of the sine function contributing less to the sum than the positive half-period loop preceeding it.

Now define w(x) = -u'(x). Then u''(x) > 0 implies w'(x) < 0 so this w(x) is a decreasing function. Therefore its Fourier-sine transform is positive, and hence so is the Fourier-cosine transform of u(x). Thus we have proved that u''(x) > 0 for all x > 0 guarantees v(t) > 0 for all t > 0. That is, convex functions have positive Fourier-cosine transforms.

However, bell-shaped functions are not convex, and it is doubtful if there is any criterion based solely on the real numerical values of u(x) for positive real x, for positivity of the Fourier-cosine transform of bell-shaped functions. Somewhat reluctantly, we must move into the complex plane.

Complex detours

Suppose we can continue the function u(z) into the upper half complex z plane, and that it is an even analytic function of z, real on the real axis, satisfying $\bar{u}(z) = u(\bar{z})$. Then we can write

$$v(t) = \frac{1}{2} \int_{-\infty}^{\infty} u(z) e^{izt} dz .$$
 (5)

Now suppose that $|u(z)| \to 0$ as $\Re z \to \pm \infty$ for some range of positive values of the imaginary part of z, say for $\Im z < p$. Then we can shift the path of integration upward, writing z = x + ip and giving

$$v(t) = \frac{1}{2}e^{-pt} \int_{-\infty}^{\infty} u(x+ip)e^{ixt} dx$$
(6)

$$= e^{-pt} \int_0^\infty \left[\Re u(x+ip)\cos xt - \Im u(x+ip)\sin xt \right] dx .$$
 (7)

Equation (7) expresses v(t) as the sum of a Fourier cosine and a Fourier sine transform, each times an exponential decay factor. Hence if $\Re u(x + ip)$ is convex (and decreasing and positive) and also $-\Im u(x + ip)$ is decreasing (and positive), then v(t) is positive for all t > 0.

An example is $u(z) = 1/\sqrt{1+z^2}$ where we can take p = 1. Then $\Re u(x + i) = R \cos \theta$ and $-\Im u(x + i) = R \sin \theta$, where $R = x^{-1/2}(x^2 + 4)^{-1/4}$ and $\tan 2\theta = 2/x$. These functions have the required properties, which proves that v(t) is positive for all t > 0. In fact, $v(t) = K_0(t)$ is a modified Bessel function ([1], p. 376) which is indeed positive and decays exponentially for large t.

Non-integrable singularities

The above analysis is valid as it stands if u(z) is integrable along the whole line z = x + ip, including the case of bounded u(z). However, it is of no use for the present purpose if u(z) is bounded as $z \to ip$, because then evenness of u(z) necessarily implies that $\Im u(ip) = 0$, so $-\Im u(x + ip)$ cannot be decreasing and positive for x > 0. Thus we are only interested in choices of p such that u(z) has a singularity at z = ip on the imaginary axis, and no other singularity closer to the origin. The above example $u(z) = 1/\sqrt{1+z^2}$ has an (integrable) inverse square root branch point at z = i.

But what if the nearest singularity is stronger than that? For example, $u(z) = 1/(1 + z^2)$ is not integrable through the simple pole at z = i, nor is $u(z) = (1 + z^2)^{-\alpha}$ for any $\alpha \ge 1$. Nevertheless these happen to be functions with positive Fourier-cosine transforms. We would like to be able to prove that statement using methods like those in the previous section. For the present, we shall only discuss the simple-pole case $\alpha = 1$; although a similar analysis can be performed for stronger singularities, it requires generalisation of the concept of a Fourier transform to non-integrable functions.

Thus we now assume that as $z \to ip$ we have

$$u(z) \to U_0 \left[i(z - ip) \right]^{-1} \tag{8}$$

for some real constant U_0 . The example $u(z) = 1/(1+z^2)$ has $U_0 = 1/2$. Note that when (8) holds, only the imaginary part of u is singular as $x \to 0_+$ on the line z = x + ip, with $-x\Im u(x + ip) \to U_0$, but $x\Re u(x + ip) \to 0$. Hence both Fourier integrals in (7) converge in spite of the non-integrable character of the singularity in u(z). Nevertheless we must modify (7) to take account of the pole.

The necessary modification is simply to allow the path of integration to pass below the pole, on a semicircle of vanishingly small radius. The net effect is to add a term proportional to the residue at the pole, so (7) becomes

$$v(t) = e^{-pt} \left[\int_0^\infty \left[\Re \ u(x+ip) \cos xt - \Im \ u(x+ip) \sin xt \ dx + U_0 \frac{\pi}{2} \right].$$
(9)

For example, suppose $u(z) = 1/(1+z^2)$ and p = 1. Then

$$v(t) = e^{-t} \left[\int_0^\infty \frac{1}{x^2 + 4} \cos xt \, dx + \int_0^\infty \frac{2}{x(x^2 + 4)} \sin xt \, dx + \frac{\pi}{4} \right] \,. \tag{10}$$

Since the coefficient of $\sin xt$ is positive and decreasing, the Fourier-sine integral in (10) is positive. Although the coefficient of $\cos xt$ is not convex, we no longer need the Fourier-cosine integral to be positive (though it is!), so long as it is overwhelmed by the positive correction term $\pi/4$. This is clearly so, since (replacing $\cos xt$ by -1), the Fourier-cosine integral can be seen to be greater than $-\pi/4$. Hence v(t) > 0. Of course, given that we can actually evaluate this $v(t) = (\pi/2)e^{-t}$ and the other Fourier integrals in (10), this appears a clumsy way to prove something obvious, but is important in principle, in that it does not depend on a knowledge of the exact integrals, so generalises to more complicated functions.

Positivity only for $t > t_0$

In fact, in some applications it is neither necessary nor desirable to insist that v(t) > 0 for all t > 0, and it may be enough to show that there is a finite $t_0 > 0$ such that v(t) > 0 for all $t > t_0$. Can we find criteria on u(x) for this to be true, and if so, can we estimate t_0 ? Only preliminary discussions of this generalised task are given here.

Assuming validity of (7), i.e. ruling out for the time being non-integrable singularities, on integration of the first term of (7) by parts, v(t) can be expressed as a single Fourier-sine integral

$$v(t) = e^{-pt} \int_0^\infty F(x;t) \sin xt \, dx \tag{11}$$

where

$$F(x;t) = -\Im u(x+ip) - \frac{1}{t}\frac{d}{dx}\Re u(x+ip)$$
(12)

$$= \Re \left[iu(x+ip) - \frac{1}{t}u'(x+ip) \right] . \tag{13}$$

Now if (in any range of t values) the function F(x;t) is a decreasing (and positive) function of x for all x > 0, then v(t) is positive for that range of t. This is true for all t when the two terms of (12) are both decreasing and positive for all x > 0, as in the examples already given.

However, suppose it is not true for all t, but only for $t > t_0$, for some $t_0 > 0$. Then in particular it must be true for large t, when the second term of (12) tends to zero, so the first term $F(x; \infty) = -\Im u(x+ip)$ of (12) must be decreasing and positive for all x > 0. If the second term was also decreasing and positive for all x > 0, we would have $t_0 = 0$ as above, so let us assume that this is not so for some x values. Then there is still a chance of finding a finite t_0 such that the sum of the two terms of (12) is decreasing and positive for all x > 0. This will be possible if the second term of (12) is bounded (together with its derivative) in x > 0, and does not become asymptotically large relative to the first term, either as $x \to 0_+$ or as $x \to \infty$.

For example, consider

$$\int_0^\infty \frac{\sin xt - x\cos xt}{1 + x^2} dx = e^{-t} \operatorname{Ei}(t)$$
(14)

where Ei is the exponential integral ([1], p. 230). Now

$$\int_{0}^{\infty} \frac{\sin xt - x \cos xt}{1 + x^{2}} dx = \int_{0}^{\infty} F(x; t) \sin xt \, dx$$
(15)

where

$$F(x;t) = \frac{1}{1+x^2} + \frac{1}{t} \frac{1-x^2}{(1+x^2)^2}$$
(16)

is positive and decreasing for all x if $t > t_0 = 1$. This is a conservative estimate of t_0 , since in fact Ei(t) > 0 for all t > 0.37253.

There is a potential application to the celebrated Riemann hypothesis [2]. This hypothesis might well be true if $v(t) = V'(t)^2 - V(t)V''(t)$ could be proved positive for all $t > t_0$, where $V(t) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ is a real-valued scaling of the Riemann zeta function ([1], p. 807) on its critical line s = 1/2 + it. Numerical evidence [4] is that this is so with $t_0 \approx 5.9009$, but a proof is elusive. The inverse Fourier transform of this v(t) is the bell-shaped function

$$u(x) = \frac{1}{4} \int_0^\infty y^2 U\left(\frac{x+y}{2}\right) U\left(\frac{x-y}{2}\right) dy , \qquad (17)$$

where

$$U(x) = -2e^{-x/2} + 4e^{x/2} \sum_{n=1}^{\infty} e^{-n^2 \pi e^{2x}}$$
(18)

is the (also bell-shaped) inverse Fourier transform of the (sign-oscillatory) Riemann function V(t) [3]. The nearest singularity of u(z) is at $z = i\pi/2$, so we could try $p = \pi/2$ in the above. However, there also appear to be many other singularities along the line $z = x + i\pi/2$, which may or may not be integrable. Further study of u(z) near that line would seem to be of value.

References

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