Generalised Minimum Induced Drag

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Abstract

Aerodynamic minimum induced drag is achieved for non-negatively-loaded wings, the loading specifically varying elliptically with distance x along the span. However, in some generalisations, the solution can take physically-unacceptable negative values for some x, and it is then necessary to consider a constrained optimisation where non-negativity is demanded. This phenomenon is investigated in detail for some mathematically-idealised drag and lift generalisations, which are such that closed-form analytic solutions are possible. Applications in ship hydrodynamics are also discussed.

1 Introduction

The Munk lifting-surface optimisation problem ([13],[14] p. 303) is to minimise the induced drag

$$D = -\int_{-1}^{1}\int_{-1}^{1} p'(x)p'(\xi)\log|x-\xi|dxd\xi \tag{1}$$

of a thin wing at fixed positive lift

$$L = \int_{-1}^{1} p(x)dx = -\int_{-1}^{1} xp'(x)dx \tag{2}$$

for a "loading" function $p(x)$ defined in $|x| \leq 1$, subject to $p(\pm 1) = 0$.

The Munk problem can be solved by the calculus of variations. The loading slope $p'(x)$ must satisfy the integral equation

$$\int_{-1}^{1} p'(\xi) \log |x - \xi| d\xi = \lambda x \tag{3}$$

for some Lagrange multiplier constant λ. This integral equation reduces to the airfoil equation (see Appendix) upon x-differentiation, and hence solves immediately by inversion of a Hilbert transform.

Thus the Munk problem has an "elliptical-loading" solution

$$p(x) = \frac{2L}{\pi} \sqrt{1 - x^2} \,. \tag{4}$$

with a minimum drag value of $D = 2L^2$, proportional to the square of the lift.

Note that the elliptic solution (4) for the loading $p(x)$ happens to be positive for all $|x| < 1$. However, we shall now discuss some generalisations where it is possible for the solution $p(x)$ to take negative values. This may be unacceptable in applications, in which case an additional non-negativity constraint $p(x) \geq 0$ may have to be imposed. In the following we call problems with such a non-negativity constraint "constrained", and those without it "unconstrained", although strictly speaking all such problems have an equality constraint such as (2).

The simple generalisations first discussed here do not necessarily have any direct physical application, but allow explicit solutions illustrating the effects of the non-negativity constraint. Applications (especially in ship hydrodynamics) are then discussed where although the details are more complicated, insights provided by the simplified problems may be useful.

2 Generalised Lift

The first generalisation is to modify the lift integral, requiring a fixed positive value of a "pseudo-lift"

$$L(\kappa) = \int_{-1}^{1} p(x) \left[1 - 2\kappa x^2\right] dx \tag{5}$$

for some fixed constant κ. The new problem reduces to the Munk problem if $\kappa = 0$.

The problem of minimising D as given by (1), at fixed $L(\kappa)$ with $\kappa \neq 0$, can also be solved easily in explicit analytic form. The integral equation (3) becomes

$$\int_{-1}^{1} p'(\xi) \log|x-\xi| d\xi = \lambda \left[x - \frac{2}{3}\kappa x^3 \right] \tag{6}$$

and again has an explicit solution, namely a positive constant times

$$p(x) = \sqrt{1-x^2} \left[\left(1 - \frac{\kappa}{3} \right) - \frac{2\kappa}{3} x^2 \right] . \tag{7}$$

So long as $\kappa < 1$, the solution (7) remains positive for all $|x| < 1$. However, if $\kappa > 1$, the solution (7) takes negative values near the ends $x = \pm 1$, and this may not be acceptable in some applications.

Suppose now that with $\kappa > 1$ we ask for the minimum drag D within a class of functions $p(x)$ which are not allowed to take negative values anywhere in $|x| \leq 1$. It is almost obvious then that the solution $p(x)$ will be identically zero near the ends, i.e. $p(x) \equiv 0$ in $c \leq |x| \leq 1$ for some constant c satisfying $0 < c < 1$. But now we have the same problem as before, except that the ranges of integration are $(-c, c)$ rather than $(-1, 1)$. Furthermore, the new problem can be scaled so that it reduces exactly to the old one, with a different value of κ.

That is, set $\bar{x} = x/c$ and $\bar{p}(\bar{x}) = cp(x)$. Then $D = \bar{D}/c^2$ and $L(\kappa) = \bar{L}(\bar{\kappa})$ where $\bar{D}$ and $\bar{L}$ are as before but written in terms of the scaled variables, and $\bar{\kappa} = c^2\kappa$. Now it is clear that the new solution remains non-negative in $|\bar{x}| \leq 1$ (i.e. in $|x| \leq c$) if and only if $\bar{\kappa} \leq 1$. Although every such solution is feasible, the one with the minimum actual drag D will be the one with the largest value of c. If at any given value of $\kappa > 1$, we start with $c = 1$, so $\bar{\kappa} = \kappa$, and reduce c (and hence reduce $\bar{\kappa}$) until the solution first becomes feasible, i.e. until $\bar{\kappa} = 1$, we find $c^2 = 1/\kappa$.

In summary, when $\kappa > 1$ the minimum drag within a family of non-negative loadings is achieved by a loading $p(x)$ proportional to the scaled solution at $\bar{\kappa} = 1$, namely

$$p(x) = \begin{cases} (1 - x^2/c^2)^{3/2} & \text{if } |x| \leq c \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where $c = \kappa^{-1/2}$. This solution holds (and is non-negative) for all $\kappa > 1$. An important characteristic of this solution is that it and its first derivative both vanish at the boundary $|x| = c$ between positive and identically-zero solutions.

It is also worth emphasising the elementary point that the constrained non-negative solution is not just obtained by ignoring (i.e. setting to zero) the predicted negative values of the unconstrained solution. For example, the unconstrained solution is negative for $\sqrt{(3-\kappa)/2\kappa} < |x| < 1$ whereas the constrained solution is zero for $1/\sqrt{\kappa} < |x| < 1$.

3 Generalised Drag – Simplest Case

As a second example, suppose now we retain and hold fixed the true aerodynamic lift L given by the original integral (2), but now attempt to minimise a "pseudo-drag" integral

$$D(\kappa) = -\int_{-1}^{1}\int_{-1}^{1} p'(x)p'(\xi)\left[\log|x-\xi| + \frac{2}{3}\kappa(x-\xi)^4\right] dx d\xi \tag{9}$$

for a given constant κ. This expression differs from the aerodynamic induced drag D given by the integral (1) only by addition of a non-singular quartic polynomial term to the logarithmic kernel.

This is perhaps the simplest non-trivial change to the drag. Note for example that if the power "4" in (9) were replaced by "2", the optimisation would be unchanged from the Munk problem, since the change in the drag D would then be proportional to the lift L.

Again the new minimisation problem reduces to the Munk problem if $\kappa = 0$, and can be solved explicitly for any κ. The new integral equation replacing (3) is

$$\int_{-1}^{1} p'(\xi)\left[\log|x-\xi| + \frac{2}{3}\kappa(x-\xi)^4\right] d\xi = \lambda x \tag{10}$$

with a modified kernel. However the extra terms arising from that kernel can be transferred to the right, and ($p'(\xi)$ being an odd function) reduce to a cubic in x as in (6). Hence the solution is again of the form (7), namely an even quadratic in x times the Munk solution, now specifically

$$p(x) = \frac{2L}{\pi}\sqrt{1-x^2}\left[\left(1+\frac{\kappa}{3}\right) - \frac{4\kappa}{3}x^2\right] . \tag{11}$$

with residual minimum drag

$$D = 2L^2(1 + 2\kappa - \kappa^2/3) . \tag{12}$$

The solution (11) again takes negative values near $|x| = 1$ when $\kappa > 1$. On the other hand, when $\kappa < -3$, negative loading occurs near the centre $x = 0$, with the loading remaining positive near the ends. Thus the unconstrained problem yields non-negative loadings over the whole interval $|x| \leq 1$ only in the range $-3 < \kappa < 1$, and we need to consider the effect of a non-negativity constraint both for $\kappa > 1$ and for $\kappa < -3$.

3.1 Constrained optimisation

If $\kappa > 1$ we can again assume that the constrained problem has a solution that is identically zero for $|x| \geq c$, and scale back to the original problem, this time with $\bar{\kappa} = c^4\kappa$. The constrained optimum then demands $\bar{\kappa} = 1$, so that the loading has the same form (8), with $c = \kappa^{-1/4}$. The resulting minimum drag value is then

$$D = \frac{16}{3} L^2 \kappa^{1/2} \tag{13}$$

which is as expected greater in $\kappa > 1$ than the unconstrained minimum given by (12); e.g. the constrained minimum at $\kappa = 4$ is $D = 32L^2/3$ compared to the unconstrained $D = 22L^2/3$.

If $\kappa < -3$ it seems reasonable to assume that the constrained optimum loading will be identically zero in a central region $|x| \leq c$ for some c, so that the new optimisation problem occurs on a domain consisting of the two separated segments $-1 < x < -c$ and $c < x < 1$. Furthermore, we must expect as in (8) that both p and its derivative p' vanish at the interior boundary point $x = c$.

This is no longer a problem that can simply be solved by scaling the unconstrained problem, although with difficulty it still can be solved by quadratures. Using results for "tandem airfoils" given in the Appendix, we can show that the loading slope is

$$p'(x) = \frac{8\kappa L}{\pi} \operatorname{sgn} x \sqrt{\frac{x^2 - c^2}{1 - x^2}} (x^2 - B) \tag{14}$$

for some constant B. In order to have $p(c) = p(1) = 0$, we must choose

$$B = \frac{\int_c^1 x^2 \sqrt{\frac{x^2-c^2}{1-x^2}} dx}{\int_c^1 \sqrt{\frac{x^2-c^2}{1-x^2}} dx} \tag{15}$$

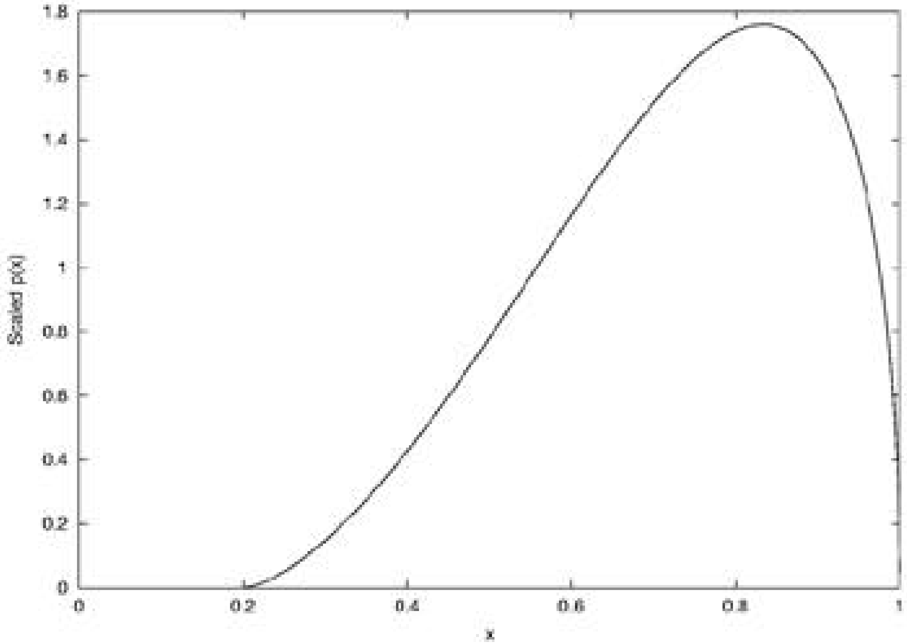

Figure 1: Optimal constrained loading $\pi p(x)/2L$, at $\kappa = -3.9972$ ($c = 0.2$)

Then the lift constraint demands

$$\kappa^{-1} = (1 - c^2)\left[1 - c^2 - 4(1 - B)\right] \tag{16}$$

the expression on the right being a known function of c.

It is easy to see that $B \to 2/3$ so $\kappa \to -3$ as $c \to 0$, joining with the unconstrained solution. Also $\kappa \to -\infty$ as $c \to 1$, so for large negative κ, the loading is nonzero only very close to the ends. Thus (16) provides c as a function of κ, with c varying from 0 to 1 as κ varies from -3 to $-\infty$.

If required the loading $p(x)$ itself can be written in terms of elliptic functions, but it is easy to compute it by direct numerical integration of (15) and (14). Figure 1 shows $\pi p(x)/2L$ in $x > 0$ at $c = 0.2$ or $\kappa = -3.9972$. The constrained minimum drag D in $\kappa < -3$ can also be evaluated if required using the solution (14), but we can be assured that it takes a value greater than the unconstrained minimum (12).

4 Generalised Drag – Second Example

As a second generalisation of the induced drag formula (1), consider

$$D(\kappa) = -\int_{-1}^{1}\int_{-1}^{1} p'(x)p'(\xi)\left[1 + \frac{1}{2}\kappa^2(x-\xi)^2\right]\log|x-\xi|dxd\xi \tag{17}$$

in which the change in the kernel includes the logarithmic factor. Now optimisation of $D(\kappa)$ at fixed L requires us to solve the integral equation

$$\int_{-1}^{1} p'(\xi)\left[1 + \frac{1}{2}\kappa^2(x-\xi)^2\right]\log|x-\xi|d\xi = \lambda x \tag{18}$$

No longer can this integral equation be converted immediately to the airfoil equation. However, it is straightforward to show that the Hilbert transform of the loading $p(x)$ is proportional to $\sin\kappa x$. First define

$$q(x) = \int_{-1}^{1} p'(\xi)(x-\xi)^2\log|x-\xi|d\xi \tag{19}$$

noting that $q(0) = 0$. Then

$$q''(x) = 2\int_{-1}^{1} p'(\xi)\log|x-\xi|d\xi \tag{20}$$

so the integral equation (18) becomes the ordinary differential equation

$$q''(x) + \kappa^2 q(x) = 2\lambda x \tag{21}$$

with solution

$$q(x) = \frac{2\lambda}{\kappa^2}x - \frac{2\pi^2 B}{\kappa^3}\sin\kappa x \tag{22}$$

for some constant B. Then $q''(x) = 2\pi^2(B/\kappa)\sin\kappa x$, and

$$p(x) = -\frac{B}{\kappa}\sqrt{1-x^2}\int_{-1}^{1}\frac{\sin\kappa\xi\, d\xi}{\sqrt{1-\xi^2}\,(x-\xi)} \tag{23}$$

and

$$p'(x) = -\frac{B}{\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}\,\cos\kappa\xi\, d\xi}{x-\xi} \tag{24}$$

where

$$\frac{L}{\pi} = B\int_{-1}^{1}\cos\kappa\xi\sqrt{1-\xi^2}\, d\xi \tag{25}$$

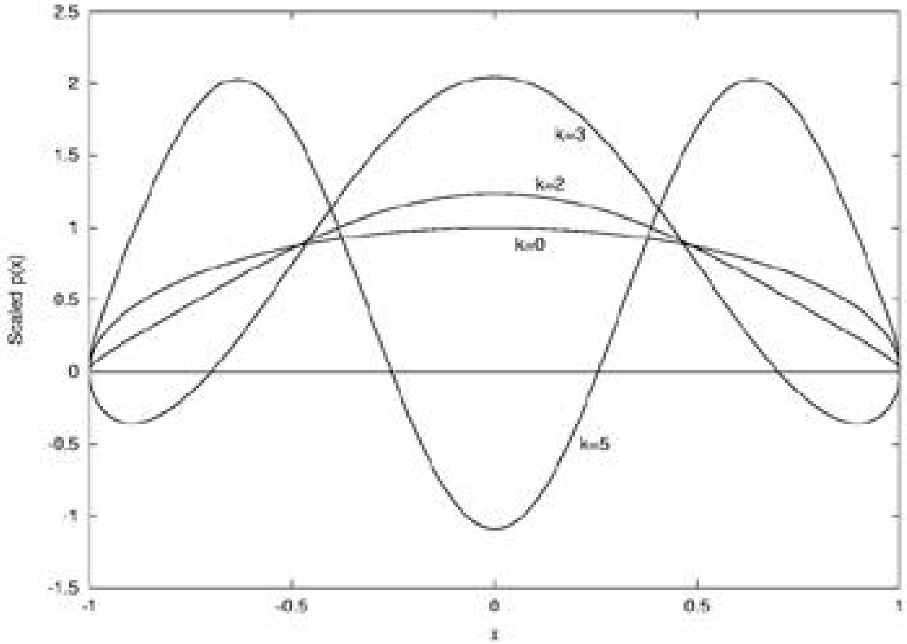

Figure 2: Optimal unconstrained loading $\pi p(x)/2L$, for various κ

Note that as $\kappa \to 0$ we recover the Munk solution (4), with $B = 2L/\pi^2$.

It is straightforward to compute the solution (23) by numerical integration for any κ, and graphs of $p(x)$ are given for various κ in Figure 2. The solution $p(x)$ is positive for all $|x| < 1$ only when $\kappa < 2.4048$. The borderline is where $p'(1) = 0$, and from (24) it can be observed that this happens when $J_0(\kappa) = 0$, where J_0 is a Bessel function ([1] p. 360). Thus $\kappa = 2.4048$ is the first zero of $J_0(\kappa)$.

For $2.4048 < \kappa < 3.8317$ (the first zero of the Bessel function $J_1(\kappa)$), there is a region of negative loading near the ends. For $3.8317 < \kappa < 5.5201$ (the second zero of $J_0(\kappa)$) the pressure is negative near the centre and positive near the ends, etc.

As κ increases further, more and more regions with $p(x) < 0$ are introduced. Thus in the present example, if the non-negativity constraint $p(x) \geq 0$ is required, we expect for large κ to see many zones with $p \equiv 0$ interspersed with many zones with $p > 0$.

However, for $2.4048 < \kappa < 3.8317$, the only zone of identically zero loading will be at the ends, so again we expect $p(x) > 0$ only in a central region $|x| < c$ for some $c = c(\kappa)$. Then the problem can be re-scaled back to the range $|\bar{x}| < 1$ with $\bar{x} = x/c$, on which the same solution (23) applies, and the critical case (highest value of κc allowing fully non-negative solutions) occurs when $p' = 0$ at $\bar{x} = 1$, so $c = 2.4048/\kappa$.

It is also probable that the range $3.8317 < \kappa < 5.5201$ allows constrained non-negative solutions with the non-zero range as $c < |x| < 1$ for some c, but again this solution can no longer be obtained by a simple scaling of the unconstrained solution, and further work is needed on this case.

5 Applications

Although the neither the "pseudo-lift" integral (5) of Section 2, nor the simple quartic-kernel "pseudo-drag" integral (9) of Section 3 have any obvious direct applications in aerodynamics or hydrodynamics, the second "pseudo-drag" integral (17) of Section 4 is relevant to some practical drag minimisation tasks. In particular, ship wave resistance (reviewed by Wehausen [18]; see also [17]) provides a number of relevant generalisations, especially at high speeds.

The most direct equivalent of the present model problems occurs for a large aspect ratio planing surface [8] [2] [3], modelled by a pressure distribution exerting a force per unit length $p(x)$ along the line $|x| < 1$ of the free surface $z = 0$ in a y-directed stream. That is, this planing surface is moving perpendicular to its long x-axis. Then the wave resistance or drag of such a pressure distribution can be written

$$D = \int_{-1}^{1}\int_{-1}^{1} p'(x)p'(\xi)K(\kappa|x-\xi|)dxd\xi \tag{26}$$

where κ is a parameter inversely proportional to the square of the speed, and

$$K(z) = K_0(z) + 2z\left[K_0'(z) - \int K_0(z)dz\right] \tag{27}$$

where K_0 is a modified Bessel function ([1], p. 375). The total lift L is constrained by (2), to balance the weight of the vessel.

For small z we have

$$K(z) = -\log z + A + \frac{3}{4}z^2\log z + Bz^2 + O(z^4\log z) \tag{28}$$

for some (irrelevant) constants A, B. Hence for small κ or high speed, the drag is proportional to

$$D = -\int_{-1}^{1}\int_{-1}^{1} p'(x)p'(\xi)\left[1 - \frac{3}{4}\kappa^2(x-\xi)^2\right]\log|x-\xi|dxd\xi + O(\kappa^4) \qquad (29)$$

which can be compared to (17), where there is exact equivalence, but with the old κ^2 replaced by $-2\kappa^2/3$. Similar solutions therefore apply here, but with imaginary κ, e.g. cosines replaced by hyperbolic cosines. The means that the unconstrained solutions are already non-negative, and no modifications of the domain are necessary for small κ.

As a second example, consider a thin vertical strut $y = \pm f(x)$ in a flow in the x-direction. Michell's integral [11] for this case is proportional to a quadratic form in $f(x)$, namely

$$D = -\frac{\pi}{2}\int_{-1}^{1}\int_{-1}^{1} f(x)f(\xi)Y_0(\kappa|x-\xi|)dxd\xi \qquad (30)$$

where Y_0 is the second kind Bessel function ([1], p. 360). Again, κ is a parameter inversely proportional to the square of the flow speed, and for small κ we have

$$-\frac{\pi}{2}Y_0(z) = -\log z + A + \frac{1}{4}z^2\log z + Bz^2 + O(z^4\log z) \qquad (31)$$

so we again have a pseudo-drag formula similar to (17) with $p'(x)$ replaced by $f(x)$, but with a negative equivalent of the constant κ^2. However, the natural equality constraint for this problem is constancy of strut cross-section area

$$A = 2\int_{-1}^{1} f(x)dx$$

rather than of $\int xf(x)dx$, so there is no immediate equivalent of the full Munk optimisation problem.

Similarly, for a slender ship with section area $S(x)$, the drag is [9, 15] proportional to a quadratic form in the second derivative of S, namely

$$D = -\frac{\pi}{2}\int_{-1}^{1}\int_{-1}^{1} S''(x)S''(\xi)Y_0(\kappa|x-\xi|)dxd\xi \qquad (32)$$

with constancy of volume

$$V = \int_{-1}^{1} S(x)dx = \int_{-1}^{1}\frac{1}{2}x^2S''(x)dx \qquad (33)$$

(assuming a pointed ship with $S(\pm 1) = S'(\pm 1) = 0$). Since now the analogue of the loading slope $p'(x)$ is $S''(x)$, again the equality constraint differs from that in the Munk problem. In both cases (30),(32) with Bessel Y_0 kernel, and also with the planing case (26), the exact solution at general κ can be written in terms of a series of Mathieu functions ([1], p. 721), and is everywhere non-negative.

Interestingly, in the limit $\kappa \to 0$, the formula (32) also gives the drag of a slender body with section area $S(x)$ in supersonic flow of a compressible fluid, and Lighthill [6] discussed the corresponding optimisation problem subject to (33), with solution

$$S(x) = \frac{8V}{3\pi}(1 - x^2)^{3/2} \ . \tag{34}$$

There are ship-hydrodynamic minimum drag problems where the unconstrained optimisation is known to lead to physically-unacceptable negative solutions in some domains. These tend to involve two-variable optimisations, as for example ([18], p. 206) with minimisation of the full Michell integral for a thin ship with equation $y = \pm f(x, z)$, namely

$$D = \kappa^2 \iint dxdz \iint d\xi d\zeta \ f(x, z) \ f(\xi, \zeta) \ K_{ZZ}(\kappa(x - \xi), 0, \kappa(z + \zeta)) \tag{35}$$

where the integral is over the centreplane $y = 0$ of the ship and

$$K(X, Y, Z) = \int_0^{\pi/2} \sec\theta \cos(X \sec\theta) \cos(Y \sec^2\theta \sin\theta) \exp(Z \sec^2\theta) d\theta \tag{36}$$

which is such that $K(X, 0, 0) = -(\pi/2)Y_0(|X|)$, so (35) reduces to (30) when $f(x, z)$ is independent of z.

Physically-impossible negative offsets $f(x, z) < 0$ have been found by Lin et al [7] at some speeds in some (quite small) (x, y) regions near the bow or stern. These regions were at the waterline, so if constrained optimisation yielded zero offsets, this would correspond to submerged bulbous hull ends. Negative section areas near the bow ands stern were also found by Maruo [10] in computations with a prescribed depth dependence, but in that case, replacing the negative section area by a zero value would simply "shorten" the ship.

Similarly for an idealised hovercraft [4, 16] with general non-uniform cushion pressure $p(x, y)$, the wave resistance can be written

$$D = \kappa^2 \iint dxdz \iint d\xi d\zeta \ p(x, y) \ p(\xi, \eta) \ K_{ZZ}(\kappa(x - \xi), \kappa(y - \eta), 0) \tag{37}$$

where the integral is over some region of the free surface $z = 0$. The unconstrained optimum pressures tend to be quite large and highly oscillatory, taking some negative values. While negative pressures are not impossible, they are undesirable in this application, and large oscillatory pressures are certainly not reasonable.

At some speeds, an attempt to minimise D with a non-negative pressure distribution $p(x, y)$ defined on a fixed (x, y) domain leads to an effective reduction in the extent of that domain, because the pressure is then required to vanish near the edges of the domain, as in the present model problem of Section 3 with $\kappa > 1$, or the problem of Section 4 with $2.4048 < \kappa < 3.8317$.

At some other speeds, the optimum hovercraft appears [16] to be a pair of distinct pressure patches at bow and stern, as would arise from requiring a central zero-pressure zone, and this is potentially analogous to $\kappa < -3$ in the first model, or to $3.8317 < \kappa < 5.5201$ in the second model.

The present simple model problems may provide some guidance toward this type of optimisation subject to non-negativity constraints.

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Appendix – tandem airfoil equation

The ordinary airfoil equation is

$$\frac{1}{\pi}\int_{-1}^{1} u(\xi)\frac{d\xi}{x-\xi} = v(x) \tag{38}$$

where $v(x)$ is given for $-1 < x < 1$, and $u(x)$ is to be determined on the same interval. In operator notation, we may write

$$\mathcal{H}u(x) \ = v(x) \tag{39}$$

where $\mathcal{H}$ is a Hilbert transform on the domain $(-1, 1)$.

The airfoil equation has a homogeneous solution ($\mathcal{H}\ u(x) \equiv 0$), namely $u = u_0(x)$ where

$$u_0(x) = (1 - x^2)^{-1/2} \tag{40}$$

For any non-zero $v(x)$, a particular solution is

$$u_1(x) = -u_0(x)\ \mathcal{H}\ v(x)/u_0(x) \tag{41}$$

Then the general solution is

$$u(x) = u_1(x) + Cu_0(x) \tag{42}$$

where C is an arbitrary constant.

The above can be generalised to domains other than $(-1, 1)$. In particular here we are interested in the symmetrically placed "tandem airfoil" domain $(-1, -c) \cup (c, 1)$ where $0 \le c \le 1$. The airfoil equation on that domain is

$$\frac{1}{\pi}\left\{\int_{-1}^{-c} + \int_{c}^{1}\right\} u(\xi)\frac{d\xi}{x-\xi} = v(x) \tag{43}$$

where $v(x)$ is given for $-1 < x < -c$ and $c < x < 1$. This can also be written in the operator form (39) where the appropriate integration range as in (43) is used for the definition of the Hilbert transform.

Here are the solutions on this domain. First note that there are now two homogeneous solutions, namely $u = u_0(x)$ and $u = xu_0(x)$ where

$$u_0(x) = \text{sgn}\, x\ (1 - x^2)^{-1/2}(x^2 - c^2)^{-1/2} \tag{44}$$

The "sgn" factor is important, and makes this $u_0(x)$ an odd function, whereas $u = xu_0$ is even and reduces to the homogeneous solution (40) for the ordinary airfoil equation as $c \to 0$. With this new $u_0(x)$ and the new integration range for $\mathcal{H}$, the particular solution $u_1(x)$ takes the same form (41), and the general solution is

$$u(x) = u_1(x) + Cu_0(x) + Dxu_0(x) \tag{45}$$

where C, D are arbitrary constants.

No proofs of the above results are given here (see [12]). They follow fairly easily by complex analysis, exploiting analyticity of $w(z) = u - iv$ and

$$W(z) = (z^2 - 1)^{1/2}(z^2 - c^2)^{1/2}w(z)$$

in the upper half plane, and allowing $w = O(z^{-1})$ at infinity, so that W grows linearly, with $W \to -iC - iDz$. Physically w is the complex velocity and the two degrees of freedom are needed to account for the separate circulations about the tandem airfoils.

There are a few important special Hilbert transforms. Thus

$$\mathcal{H}\ x^2u_0(x) = -1 \tag{46}$$

so also

$$\mathcal{H}\ \text{sgn}\, x\ \sqrt{\frac{x^2 - c^2}{1 - x^2}} = -1 \tag{47}$$

Similarly

$$\mathcal{H}\ x^4u_0(x) = -x^2 - \frac{1}{2}(1 + c^2) \tag{48}$$

so also

$$\mathcal{H}\ \text{sgn}\, x\ \sqrt{\frac{x^2 - c^2}{1 - x^2}}x^2 = -x^2 - \frac{1}{2}(1 - c^2) \tag{49}$$