

LONGITUDINAL WAVES IN SLENDER MOONPOOLS

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SUMMARY

An asymptotic slender body theory is presented for water waves in elongated moonpools. There are nontrivial three-dimensional effects for those modes involving mainly longitudinal motion, in which an apparent flux to “infinity” in each cross-section is modelled by a line of sources in an outer region, and then is matched to an inner two-dimensional free-surface flow.

1. INTRODUCTION

In a closed elongated pool of water, with length much greater than its width, the longest-period waves will involve mainly longitudinal motion, with a nearly-uniform free-surface elevation across the cross-section. In such a fully-closed system, as the local free-surface elevation and hence the area occupied by fluid in a particular cross-section changes, there must be instant compensation in the form of longitudinal fluid motion, in order to conserve mass.

On the other hand, if the pool has access via an open bottom to unbounded water, as with a “moon-pool” or ice-hole, there can be an apparent mass source or sink at infinity in each cross-section. In some “pumping” modes, this apparent 2D flux becomes a real three-dimensional transfer of mass to infinity. In other “sloshing” modes there is still zero net 3D flux, but the conversion of changes in section area to longitudinal fluid motion occurs slowly, effectively allowing longitudinal waves to behave in some ways as if they were lateral waves. Then not only is there a non-trivial cross-sectional variation of the amplitude of the longitudinal wave, but its frequency is higher, formally now of the same order of magnitude as that of the shorter lateral modes.

An asymptotic slender body theory for such waves is presented here and compared with fully 3D computations. We use a co-ordinate system (s, x, y) where s is along the pool, x across it, and y is vertically upward from the equilibrium free surface. In the present paper we restrict attention to rectangular holes $|x| < b$, $|s| < \ell$, in a rigid sheet of zero thickness occupying the rest of the plane $y = 0$. This models a moonpool of length 2ℓ and width $2b$ in a fixed vessel of large length $\gg 2\ell$ and beam $\gg 2b$, and small draft $\ll 2b$, or equivalently an elongated hole in an ice sheet, the water depth being infinite in both cases.

Assuming a velocity potential $\phi(s, x, y)e^{i\omega t}$ corresponding to angular frequency ω , we have to solve Laplace’s equation

$$\phi_{ss} + \phi_{xx} + \phi_{yy} = 0 \quad (1)$$

in $y < 0$, subject to

$$\phi_y = \frac{\omega^2}{g}\phi \quad (2)$$

on the free part of $y = 0$, and $\phi_y = 0$ on the rest.

We shall use matched expansions to provide an asymptotic solution for slender pools with $2b \ll 2\ell$, or large aspect ratio ℓ/b . The outer expansion has the scale ℓ and the flow is three-dimensional but possesses no free surface. On the other hand, the inner expansion has the scale b and the flow is two-dimensional in the cross-section $s = \text{constant}$, with a free surface.

Molin [1] has provided a numerical method for solution of some problems of the present type for arbitrary aspect ratio ℓ/b . The present asymptotic approach, though limited by the requirement that the aspect ratio be large, is somewhat simpler and potentially of wider applicability than the numerical method of Molin [1]. For example, in extended work [2] we have retained the same outer solution, but generalised to arbitrary (longitudinally uniform) inner geometry, in particular allowing non-zero drafts and curved boundaries for the containing vessel. Generalisations of the outer geometry such as allowing finite beam of the containing vessel with a free surface at infinity, or allowing longitudinal variations in the width of the moonpool, are also straightforward.

2. OUTER EXPANSION

In the outer region as $b/\ell \rightarrow 0$ with $x, y, s = O(\ell)$, the pool shrinks to a cut of zero width in an otherwise-rigid plane boundary $y = 0$. We assume that the flow

is then generated by a line of 3D Rankine sources of (to-be-determined) strength or volume flux $q(s)$ per unit length along the line $x = y = 0$, from $s = -\ell$ to $s = +\ell$. The outer velocity potential is thus given by

$$\phi = -\frac{1}{4\pi} \int_{-\ell}^{\ell} \frac{q(t) dt}{\sqrt{(s-t)^2 + r^2}} \quad (3)$$

where $r^2 = x^2 + y^2$.

We need the inner expansion of the outer expansion, i.e. the small- r expansion of (3), which then is approximated by an apparent 2D source of local strength $q(s)$, i.e.

$$\phi = \frac{q(s)}{2\pi} \log \frac{r}{2\ell} + f(s) + O(r^2). \quad (4)$$

The additive term $f(s)$ determines the longitudinal variation of the axial velocity, and is a functional of the source strength function $q(s)$. This functional dependence is in general non-local, with $f(s)$ at any particular value of s requiring a knowledge of $q(s)$ for all s , and can be expressed in various ways, e.g. as convolution integrals [3]. We use here a representation in which $q(s)$ is written in a Fourier-Legendre series

$$q(s) = \sum_{j=0}^{\infty} q_j P_j(s/\ell) \quad (5)$$

where $P_j(t)$ is the j -th degree Legendre polynomial. Then

$$f(s) = \frac{1}{2\pi} \sum_{j=0}^{\infty} q_j P_j(s/\ell) \left[\sigma_j - \log \sqrt{1 - s^2/\ell^2} \right] \quad (6)$$

where

$$\sigma_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}, \quad \sigma_0 = 0. \quad (7)$$

We have particular cause in the present paper to be interested in choices of $q(s)$ such that $f(s)$ is determined locally from $q(s)$, with the functions $f(s)$ and $q(s)$ proportional to each other, such that the ratio

$$\frac{f(s)}{q(s)} = \frac{\lambda}{2\pi} \quad (8)$$

is a constant, independent of the co-ordinate s . This choice means that the whole velocity potential given by (4) has an s -variation proportional to $q(s)$, irrespective of the value of r . This property is needed for matching to the inner expansion. Thus substituting (8) in (6) and setting $t = s/\ell$, we need to solve

$$\sum_{j=0}^{\infty} q_j P_j(t) \left[\sigma_j - \log \sqrt{1 - t^2} \right] = \lambda \sum_{j=0}^{\infty} q_j P_j(t) \quad (9)$$

in $|t| < 1$, for the coefficients q_j and the constant λ , which plays the role of an eigenvalue.

Numerical solution of (9) is immediate by truncating to N terms and collocating at an appropriate set

of N values of t , giving a generalised eigenvalue problem which can be solved for large $N \approx 100$ by standard numerical methods. The first few eigenvalues $\lambda = \lambda_n$, for $n = 0, 1, 2, 3, 4$, are

$$\lambda_n = 0.2332, 1.4437, 1.9409, 2.2833, 2.5317. \quad (10)$$

The lowest eigensolution $n = 0$, corresponding to $\lambda = \lambda_0 = 0.2332$, has an amplitude that is one-signed along the pool, and this and all even modes $n = 0, 2, 4, \dots$ potentially allow non-zero net vertical flux of volume across the whole free surface. Actual non-zero flux occurs only when such longitudinal modes are combined with lateral modes which have a similar ‘‘pumping’’ property.

On the other hand, all odd longitudinal modes $n = 1, 3, 5, \dots$ are antisymmetric with respect to s , and hence give zero net volume flux across the whole free surface. However, when these odd longitudinal modes are combined with (even) lateral pumping modes, there is an apparent 2D flux at each section $s = \text{constant}$, which is converted three-dimensionally into longitudinal motion. In particular, the mode $n = 1$ with $\lambda = \lambda_1 = 1.4437$ has a single node at $s = 0$, and is the fundamental ‘‘sloshing’’ mode. The actual mode shapes are quite non-sinusoidal, and in particular have large end slopes.

In summary, when we have chosen $q(s)$ to be proportional to one of the above eigensolutions, the inner expansion of the outer solution (4) becomes the statement that

$$\phi \rightarrow \frac{q(s)}{2\pi} \left[\log \frac{r}{2\ell} + \lambda \right] \quad (11)$$

when r/ℓ is small, and $\lambda = \lambda_n$ takes one of the above eigenvalues, $n = 0, 1, 2, \dots$

3. INNER EXPANSION

We now assume that $x, y = O(b)$, while s remains $O(\ell)$, and also that the frequency ω is high, such that $k = \omega^2 b/g = O(1)$. Then to leading order ϕ satisfies the 2D Laplace equation with respect to (x, y) , and the full free-surface condition (2) is retained. The co-ordinate s plays only a parametric role and can be suppressed in the inner problem. Meanwhile, matching requires that the outer expansion of this inner solution agrees with the inner expansion of the outer solution, i.e. (11) becomes the ‘‘2D far-field’’ boundary condition for the inner solution as $r/b \rightarrow \infty$.

The solution of the 2D Laplace equation in $y \leq 0$, with a to-be-determined vertical velocity distribution $V(x) = \phi_y(x, 0_-)$ across $y = 0$, $|x| < b$, is

$$\phi = -\frac{1}{\pi} \int_{-b}^b V(\xi) \left[\log \frac{\sqrt{(x-\xi)^2 + y^2}}{2\ell} + \lambda \right] d\xi, \quad (12)$$

which satisfies the far-field condition (11) with flux

$$q = -2 \int_{-b}^b V(\xi) d\xi. \quad (13)$$

Setting $y = 0$ in (12) and implementing the free-surface boundary condition (2) gives an integral equation for $V(x)$, namely

$$V(x) = -\frac{\omega^2}{\pi g} \int_{-b}^b V(\xi) \left[\log \frac{|x - \xi|}{2\ell} + \lambda \right] d\xi. \quad (14)$$

To solve this integral equation numerically, we expand in a Fourier-Chebyshev series

$$V(x) = \sum_{j=0}^{\infty} v_j \frac{\cos j\theta}{\sin \theta} \quad (15)$$

where $x = b \cos \theta$. Then (14) becomes

$$\sum_{j=0}^{\infty} v_j \frac{\cos j\theta}{\sin \theta} = k \left[v_0 \beta + \sum_{j=1}^{\infty} v_j \frac{\cos j\theta}{j} \right], \quad (16)$$

where

$$\beta = \log \frac{4\ell}{b} - \lambda. \quad (17)$$

Equation (16) is now also immediately amenable to numerical solution as a generalised eigenvalue problem with eigenvalue k , by truncating and collocating. The lateral eigenvalues $k = k_m$, $m = 0, 1, 2, \dots$, can then be found for each specification of the constant β , which is determined as in (17) by the values of the longitudinal eigenvalue λ and the aspect ratio ℓ/b .

However, the odd-numbered modes $m = 1, 3, 5, \dots$ only involve odd-numbered terms in the series (15), with $v_0 = 0$. Hence they are independent of β and so also of these parameters λ and ℓ/b , and yield purely two-dimensional lateral sloshing modes of motion. For example, the lowest such frequency takes the well-known (see e.g. [4]) value $k_1 = 2.006119$.

On the other hand, all even-numbered modes $m = 0, 2, 4, \dots$ have $v_0 \neq 0$ in (16) and hence do depend on β and hence on λ and ℓ/b . This is especially significant for the fundamental lateral mode $m = 0$, which specifies a wave elevation which is one-signed across the section. For example, with $\lambda = 1.4437$ (one-noded longitudinal sloshing mode) and aspect ratio $\ell/b = 16$, we have $k_0 = 0.5523$, which gives a value $\omega^2(2\ell)/(\pi g) = 5.624$ in close agreement with Figure 9 of [1], at the limiting draft ratio $h/b = 0$. Good agreement with that Figure is also seen for the less-slender case $\ell/b = 4$, where $k_0 = 1.0646$ and $\omega^2(2\ell)/(\pi g) = 2.711$.

The case $\ell/b = 4$ was also studied by Newman and Lee [5] by a full solution (using the code WAMIT) for the response of a barge of finite length, beam and draft containing a moonpool of that aspect ratio. For drafts h such that $h/b = 0.25$ and $h/b = 0.125$ respectively, Newman and Lee [5] found $k_0 = 0.92$ and $k_0 = 1.00$, giving reasonable indications of an approach as $h/b \rightarrow 0$ to the above value $k_0 = 1.0646$. This was for the first longitudinal sloshing mode $n = 1$; similar trends hold for the lowest longitudinal

pumping mode $n = 0$ and also for higher longitudinal modes $n = 2, 3, 4$.

Other longitudinal modes can be specified simply by changing the input value of λ within the set of eigenvalues (10) already found from the outer expansion, modifying the constant β , and then re-solving the inner eigenvalue problem (16) for $k = k_m(\beta)$. In fact, the present theory allows output data to be presented in a single graph of $k = k_m(\beta)$, as in the attached Figure 1. The horizontal axis of this Figure contains scales for recovering the dependence of k on the actual aspect ratio ℓ/b , separately for each longitudinal mode n .

4. CONCLUSIONS

In the present paper we have used ideas from aerodynamic slender body theory to construct an asymptotic solution valid for large aspect ratio ℓ/b for waves in elongated moonpools. The application of these slender body ideas in this context is unusual. In particular, the type of outer expansion used here, where eigensolutions for the source distribution are chosen so that the whole velocity potential due to a line of sources is proportional to a single function $q(s)$ of the coordinate s measured along the “body”, is new, and may have applications elsewhere.

From the point of view of application to the actual moonpool problem, the present solution suffers from the defect relative to previous solutions that it is approximate, applying only to elongated pools. However, in compensation, the results can be expressed very compactly, and allow systematic study of effects of various geometric parameters. The present paper only discusses a special inner geometry with zero draft and an infinite plane rigid sheet surrounding the moonpool, but is easily generalised [2] to allow arbitrary inner geometry.

5. REFERENCES

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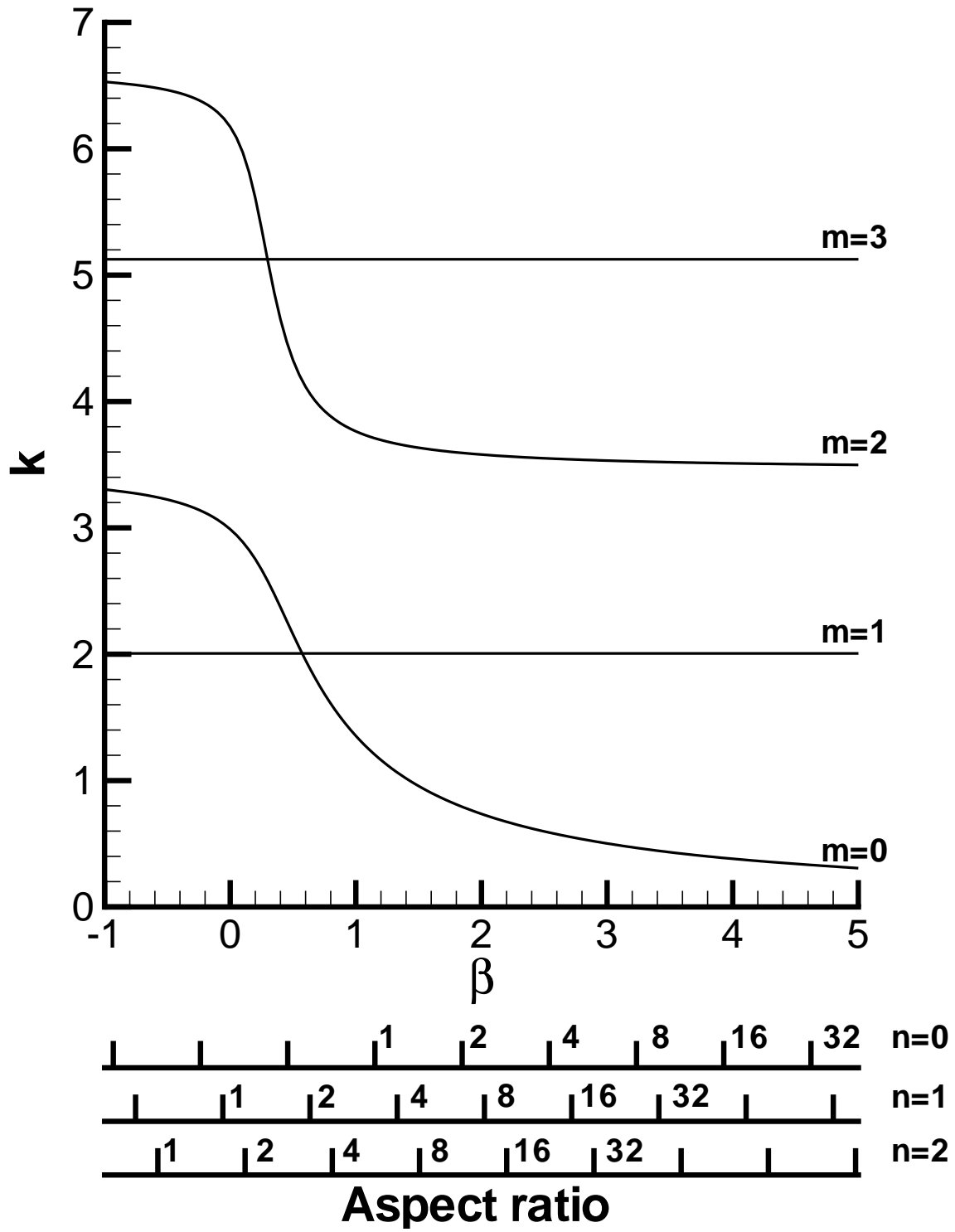


Figure 1: Non-dimensional wavenumber $k = \omega^2 b/g$ as a function of parameter $\beta = \log(4\ell/b) - \lambda$. Separate curves $k = k_m$ are for different lateral modes $m = 0, 1, 2, \dots$. Odd lateral modes $m = 1, 3, \dots$ are two-dimensional, and hence independent of β . Also shown are the corresponding scales for the moonpool's aspect ratio ℓ/b , a different scale applying to each longitudinal mode $\lambda = \lambda_n$, $n = 0, 1, 2, \dots$.