Lifting Surfaces with Circular Planforms

E.O. Tuck and L. Lazauskas

The University of Adelaide
SA 5005
AUSTRALIA

February 3, 2005

Abstract

A general lifting surface computer program is used for the special case where the planform is a circle, in order to compare lift and pitching moment with semi-analytic solutions.

Introduction

Wings with truly circular planforms are not common in aeronautics or hydrodynamics. However, there are disc-like projectiles (Potts and Crowther 2002) or planing surfaces with sporting and recreational applications, at least one full-size circular-winged aircraft (Benningfield 2004), and recent "micro air vehicle" designs (Grasmeyer and Keennon 2001) with circular or near-circular wings. Modern marine propellers (Kerwin and Lee 1978) also tend to fill much of their circular profile.

In the present paper we are mainly concerned with analytical and numerical results for thin impermeable lifting surfaces with circular planforms. There are some semi-analytic approaches in the literature (e.g. Jordan 1973, Hauptman and Miloh 1986) which yield exact or nearly exact results for flat discs at small angle of attack. These solutions are reviewed and compared with direct numerical solutions (Tuck 1993) of the lifting-surface integral equation.

Solutions for non-flat discs with small twist or camber are also discussed, including choice of twist or camber to minimise induced drag, or to achieve a favourable placement of the centre of pressure. Axisymmetrically cambered discs at zero angle of attack are paid special attention, including an example with a (small) positive lift acting at the disc centre.

Lifting surface summary

The jump in pressure $\Delta p(x,y)$ across a thin lifting surface $z = f(x,y)$ with a general planform B in a stream U satisfies the lifting-surface integral equation (LSIE)

$$\int\!\!\int_B \Delta p(\xi,\eta) W(x-\xi, y-\eta)\, d\xi d\eta = 4\pi\rho U^2 f_x(x,y)$$

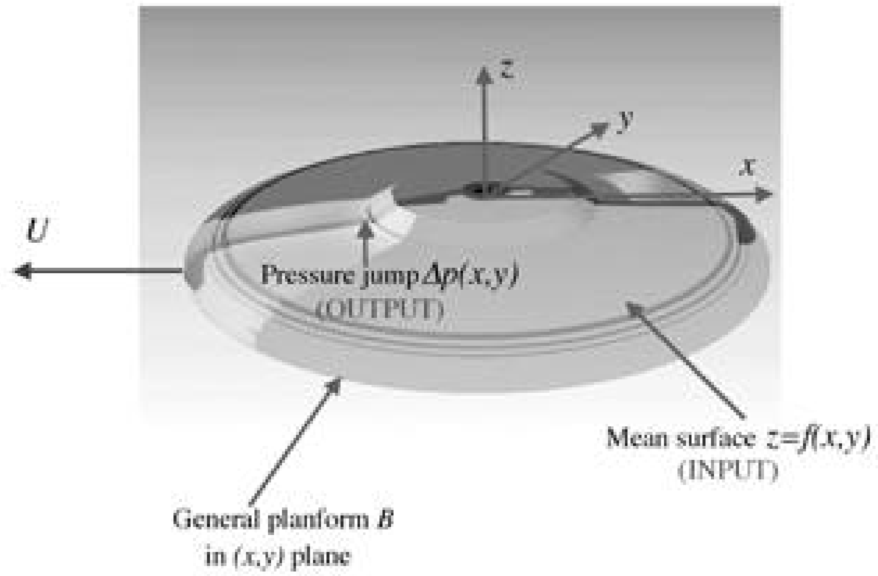

Figure 1: Lifting-surface geometry and loading.

where

$$W(x,y) = \frac{1}{y^2}\left[1 + \frac{x}{\sqrt{x^2+y^2}}\right]$$

is the downwash induced by a unit horse-shoe vortex at the origin. The LSIE needs to be solved subject to a Kutta condition that Δp vanish at the trailing edge of B. Reasonably accurate (about 3 to 4 figure) numerical codes are available to solve the LSIE, yielding the output loading $\Delta p(x,y)$ for any input mean-surface shape $f(x,y)$, on any given planform B. This input-output task is illustrated in Figure 1.

Once such solutions for $\Delta p(x,y)$ are available, the net lift L is just the (double) integral of $\Delta p(x,y)$ over B, namely the integral

$$L = \int_{-b}^{b} F(y)dy$$

over the span $2b$, of the chordwise-integrated loading or "lift per unit span"

$$F(y) = \int \Delta p(x,y)dx \ .$$

Results for lift are presented in terms of the lift coefficient

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 B}$$

using B here for the actual area of the planform.

The strength of the trailing vortex sheet at each y in the wake $|y| < b$ is proportional to the spanwise rate of change $F'(y)$ of the loading. The resulting induced drag is

$$D_I = -\frac{1}{4\pi\rho U^2}\int_{-b}^{b}\int_{-b}^{b} F'(y)F'(\eta)\ \log|y-\eta|\ dyd\eta$$

and is minimised at fixed lift L (for any planform B) by the elliptic loading

$$F(x) = \frac{2L}{\pi b^2}\sqrt{b^2 - y^2} \; .$$

The pitching moment M about $x = 0$ is the double integral of $x\Delta p(x, z)$ over B, and the centre of pressure is then at $x = x_P = M/L$. If a particular cambered surface $z = f_0(x, z)$ is rotated through a small angle of attack α, so $f(x, z) = f_0(x, z) - \alpha x$, then both the lift L and moment M are linear in α, e.g.

$$L = L_0 + L_1\alpha$$

where L_0 is the lift due to camber f_0, and L_1 is the lift per unit angle of attack for a flat plate $f_0 = 0$.

Another important output varying across the span is the strength $Q(y)$ of the leading-edge singularity. If the leading edge is at $x = x_L(y)$, then the loading becomes infinite as $x \to x_L(y)$, specifically

$$\Delta p(x, y) \to \frac{Q(y)}{\sqrt{x - x_L(y)}}$$

and then there is a leading-edge suction force, contributing a negative drag

$$D_S = -\frac{\pi}{4\rho U^2}\int_{-b}^{b} Q(y)^2 dy \; .$$

Numerical solution of the LSIE

The LSIE is amenable to direct numerical solution on approximation of the integral over B by a numerical quadrature, and inversion of the resulting system of linear algebraic equations. The most popular such method is the "vortex lattice method" (e.g. Kerwin and Lee 1978), in which the smooth loading $\Delta p(x, y)$ is replaced by a finite number of point loads. More generally, if the loading is locally approximated by a polynomial of order m, the method could be called an order-m method. The case $m = 0$ corresponds to approximating the loading as piecewise constant on some grid, and is particularly convenient for implementation (Tuck 1993). Since point loads are the derivative of step loads, the vortex lattice method can be said to have $m = -1$. In spite of its lower formal order, the vortex lattice method gives remarkably good results for the lift, comparable to that of the piecewise-constant method. All low-order methods have difficulty capturing the leading-edge singularity accurately, and various special correction procedures have been used to improve this situation (e.g. Tuck and Standingford 1997).

There are also geometrical difficulties for planforms B with curved edges, irrespective of the formal order of the numerical method. Essentially all numerical methods must subdivide B into a large but finite number of small sub-domains. The most convenient subdomain shape is rectangular, and very accurate representation of rectangular planforms is therefore possible. Both the vortex lattice method and piecewise-constant methods can easily achieve 6 or 7 figure accuracy for the lift of rectangular wings.

However, if the boundary of B is curved, rectangular sub-domains can fit to it only approximately, and this is an additional source of numerical error. If the rectangular sub-domain assumption is abandoned (e.g. in favour of trapezoidal panels), there are numerical consequences involving loss of accuracy, and it is probably more effective simply to greatly

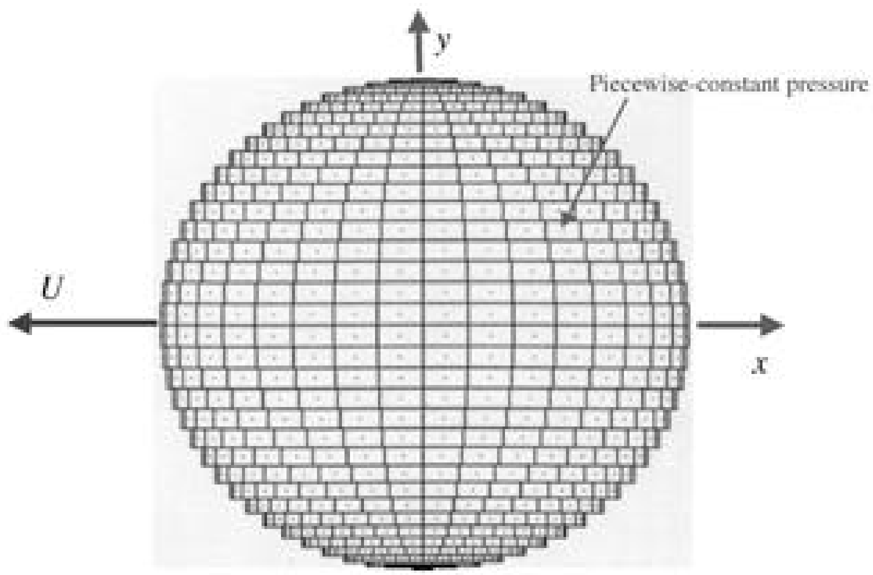

Figure 2: Rectangular panels for a circular wing. The grid shown is 18 by 36; computations were actually done on a 50 by 100 grid.

increase the number of rectangular panels. In practice, because of this problem, only about 3-figure accuracy can be expected for curved planform boundaries for most output quantities, which is satisfactory for most purposes, but needs checking relative to more accurate methods. The program used in the present study is essentially that described in Tuck and Standingford (1997) and will be referred to as the "TS" program. It was run with 5000 panels, namely on a grid of 50 x-points by 100 y-points, but making use of y-symmetry to reduce to 2500 unknowns. Figure 2 gives an indication of such a panelisation for a circular planform, although for the sake of clarity, only an 18 by 36 grid is shown.

Series solutions for circular planforms

For the special case where B is the unit circle

$$x^2 + y^2 \leq 1$$

it is possible to construct very accurate solutions, in the form of infinite series. This is best seen indirectly, by noting that our real task is to solve the 3D Laplace equation exterior to a circular disc which is the limiting form of the oblate spheroid

$$z = \pm\epsilon\sqrt{1 - x^2 - y^2}$$

as $\epsilon \to 0$. Hence we can write $\Delta p(x, y)$ as an infinite series whose terms are fundamental solutions of Laplace's equation in oblate spheroidal polar coordinates, which involve associated Legendre functions.

The remaining difficulty involves the boundary conditions on the disc, and in particular the Kutta trailing edge condition. Efforts must be made to preserve the correct leading and trailing edge behaviour within the terms of the series, with a change-over at the wingtips. This is quite difficult, since such a (Cartesian) distinction between the leading and trailing halves of the wing does not fit easily into the oblate spheroidal co-ordinate system.

Solutions of this series type have been obtained by Robinson and Laurmann (1956), Jordan (1973), Hauptmann and Miloh (1986), Boersma (1989), and others, with various assumptions made about the mean surface function $f(x, y)$.

Flat discs at angle of attack

For the flat disc $f_0(x, y) = 0$, or $f(x, y) = -\alpha x$, very accurate results have been obtained by Jordan (1973) and confirmed by Boersma (1989). The lift coefficient (per unit angle of attack) is $C_L/\alpha = 1.79002$ and the centre of pressure is at $x_P = -0.52086$, all 5 decimal places of accuracy being reliable, and more available. By contrast, the lift coefficient predicted by lifting-line theory is $C_L/\alpha = 2.444$, and the 2D "quarter-chord" centre of pressure is $x_P = -0.5$. These results are incorrect because they are valid only at high aspect ratio, and a circle is a low-aspect-ratio wing.

The general purpose LSIE solver TS with 5000 panels produces $C_L/\alpha = 1.79078$ (error 0.04%) and $x_P = -0.52194$ (error 0.2%).

This special flat-plate case is of more general importance, as by linearity it also gives the slope $dC_L/d\alpha = 1.79002$ of the curve of lift versus (small) angle of attack α, for an arbitrary (non-flat) thin surface.

Hauptman-Miloh twisted discs

Hauptman and Miloh (1986) constructed a simplified series solution which nevertheless gives a very accurate approximation to the above solution for a flat disc. In fact, the Hauptman-Miloh result is the exact solution for a "twisted" disc

$$f(x, y) = -\alpha x\,[1 - g(y)]$$

where $g(y)$ is a relatively small quantity. An explicit formula and table of values for this function $g(y)$ is given by Boersma (1989), there called $W_1(y)$. Since $g(y)$ tends (logarithmically weakly) to $+\infty$ as $y \to 1$, the wingtips have a locally large and negative apparent angle of attack. However, since the local chord is tending to zero, even this weak wingtip infinity does not cause the over-all shape to differ much from a flat disc. In fact, this distortion of the disc is so small that the Hauptmann-Miloh lift coefficient $C_L/\alpha = 1.79075$ is within 0.04% of the true flat-plate value. However, the centre of pressure at $x_P = -.52360$ is not quite so close to the flat-plate value, differing by 0.5%. These values are almost as close to the exact flat-plate values as are the numerical results given by the TS program for the flat plate.

It must be pointed out that closeness of lift to the exact flat plate value is a function of the definition of the angle of attack α, which is non-unique for non-flat wings. For example, one could demand that $g(y)$ had zero mean, so that $\alpha = \bar{\alpha}$ is the mean angle of attack averaged over the span, or more simply could demand $g(0) = 0$, so that $\alpha = \alpha_0$ is the angle of attack at mid-span $y = 0$. In fact, Hauptman and Miloh (1986) did neither of these things, instead in effect averaging the boundary condition over the span. This appears to have had the effect of ensuring that the lift is very close to that for the flat plate, and it is not then surprising that the moment is not quite as close. In contrast, if we use instead the angle of attack α_0 at mid-span, then $C_L/\alpha_0 = 1.75471$, which is 2% less than that of a flat-plate with constant angle of attack $\alpha = \alpha_0$, because of the reduced wingtip incidence.

It is important to note that the twist, while small, must be allowed to vary in proportion to the angle of attack in order to maintain this exact solution. For example, if we fix the shape of this particular twisted disc at some angle of attack α_1, and then change the angle of attack from α_1 to α_2 by simply rotating it without further distortion, the change in lift is given by the exact (Jordan) coefficient 1.79002 times the change $\alpha_2 - \alpha_1$ in angle of attack, so the new lift is not $C_L = 1.79075\alpha_2$. Clearly this is only a minuscule distinction in the present case, but is important in principle.

Another interesting point about this type of twist distortion is that the local axis of rotation for the twist at each fixed y can be shifted x-wise arbitrarily. This is because the aerodynamics of thin lifting surfaces depends only on the longitudinal slope f_x. Hence the same flow arises for the shape $f(x, y) = -\alpha\,[x - xg(y) + h(y)]$, for any $h(y)$. For example, if we choose $h(y) = x_L(y)g(y)$ where $x = x_L(y)$ is the leading edge, then

$$f(x, y) = -\alpha\,[x - (x - x_L(y))g(y)]\,,$$

in which case there is no local distortion of the flat disc at the leading edge, and the twisting distortion is concentrated near the trailing edge.

Twisted discs with elliptic loading

Robinson and Laurmann (1956) also produced accurate series-type solutions for a nearly-flat disc at a small positive angle of attack. However, they used a constraint that the chordwise-integrated loading $F(y)$ vary exactly elliptically across the span. This means that the resulting wing is optimal from the point of view of minimisation of induced drag. Again this constraint results in a small twist $g(y)$, but this twist is different from that of the Hauptman-Miloh disc. A quadrature formula for, and graph of, the effective local angle of attack (there called "$-S(y)/C$ ") is given on p. 266 of Robinson and Laurmann (1956). The magnitude of the Robinson-Laurmann twist is about the same as that for the Hauptman-Miloh disc, but the sign is opposite, i.e. this disc has a *positive* logarithmically-infinite local angle of attack at its wingtips.

The actual lift of the Robinson-Laurmann disc is given by $C_L/\alpha = 1.75960$, differing by 1.7% from the Jordan flat plate value. However, a different convention for definition of angle of attack was adopted for this disc, in that (unlike the Hauptman-Miloh disc) the mean value of the Robinson-Laurmann twist function $g(y)$ is specifically forced to be zero, so that $\alpha = \bar{\alpha}$ is the mean angle of attack. Again, if we use instead the angle of attack α_0 at mid-span, then $C_L/\alpha_0 = 1.82747$, which is 2% greater than the flat-plate value due to the increased wingtip incidence. It would be possible to define yet another "pseudo-mean" angle of attack as with the Hauptman-Miloh disc, such that the error from the flat plate value was less, indeed as small as desired. However, no matter what averaging convention is adopted for definition of angle of attack, again the magnitude of the twist must vary in proportion to that angle of attack, if it is desired to maintain exact elliptic loading.

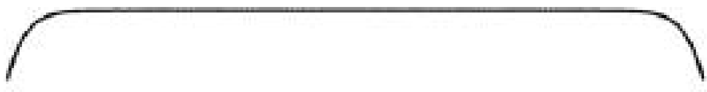

Figure 3: Shape of the $n = 20$ monomial

Axisymmetrically cambered discs

Twist as introduced in the previous sections disturbs the axisymmetry of the circular planform. An alternative departure from a flat disc at zero angle of attack has

$$f(x, y) = g(r)$$

for some shape function $g(r)$, where $r = \sqrt{x^2 + y^2}$, maintaining axisymmetry. This is particularly relevant if the disc must spin about its axis while in flight, as in some sporting and recreational applications. However, it is not our present aim to model closely actual current recreational disc designs. These tend to have large local slopes at their rims, from which the flow is likely to separate, leading to nonlinear aerodynamics (Potts and Crowther 2002).

For definiteness let us normalise in such a way that $g(0) = 0$ and $g(1) = -\beta$, where β is a small thickness parameter. The paraboloid of revolution $g(r) = -\beta r^2 = -\beta x^2 - \beta y^2$ is of particular interest, noting that since only the longitudinal slope contributes to the linearised aerodynamics, the term "$-\beta y^2$ " can be ignored. Hence this gives the same lift and moment as for a circular disc with parabolic x-wise camber $f(x, y) = -\beta x^2$, which has been studied by previous authors, including Hauptman and Miloh (1986). Boersma (1989) gives $C_L/\beta = 1.86469$ and $x_P = 0.47064$, and these results, obtained using a complete series, are expected to be accurate. Again, the simplified series used by Hauptman and Miloh (1986) is exact for a particular slightly-twisted paraboloid, and is a good approximation for the true paraboloid, but now with errors of the order of 0.5%. The values given by TS are $C_L/\beta = 1.86842$ (error 0.2%) and $x_P = 0.47102$ (error 0.08%).

Power n	Lift C_L/β	Moment C_M/β	$x_P = C_M/C_L$
2	1.86842	0.88006	0.47102
4	2.99223	1.40910	0.47092
10	5.19459	2.44468	0.47062
20	7.62819	3.58616	0.47012

Table 1: Lift and moment on monomial axisymmetric discs.

More generally, consider the "monomial" family of shapes $g(r) = -\beta r^n$ for some power n. The above paraboloid is the case $n = 2$. As n increases, the slope of the body is more and more concentrated near its rim $r = 1$; Figure 3 shows the shape for $n = 20$. Table 1 gives TS results for some members of this family. Note how little the centre of pressure changes within this family, remaining very near to $x_P \approx 0.47$ for all n. This seems characteristic of most convex axisymmetric bodies.

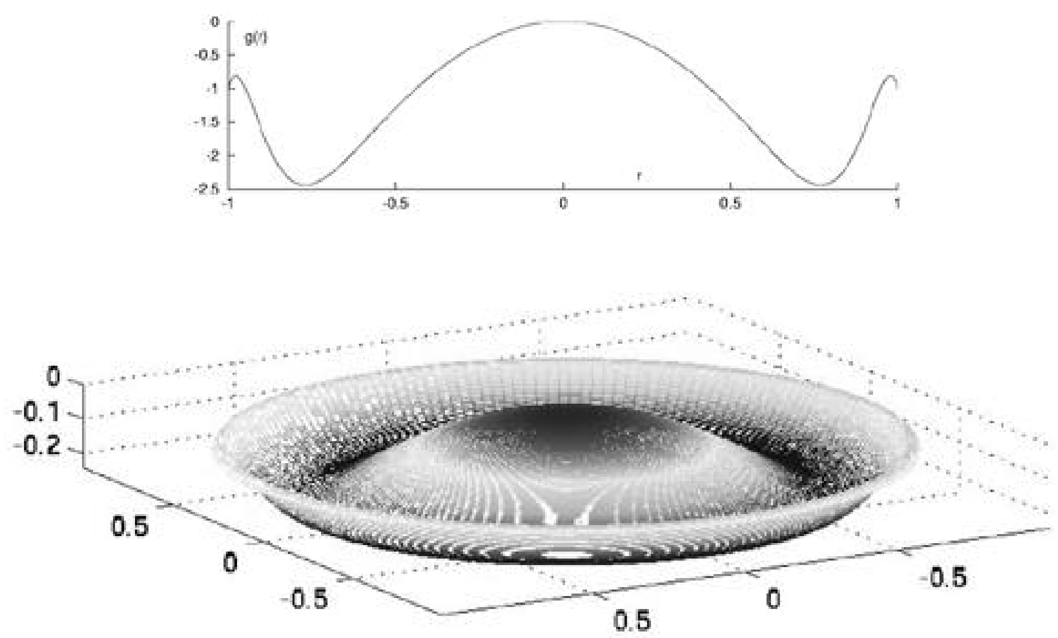

Figure 4: An axisymmetric disc with almost zero pitching moment and positive lift.

Axisymmetric discs with zero pitching moment.

Although concave-down axisymmetric discs at zero angle of attack tend to have negative (leading-edge down) pitching moments, corresponding to centres of pressure in the trailing half of the disc, it is easy by a linear combination of two or more such shapes (with at least one "upside-down") to eliminate the pitching moment entirely, so moving the centre of pressure to the axis. This is a highly desirable property for any spinning device, as it eliminates gyroscopic forces inducing roll.

Unfortunately, since fully concave-down shapes seem to have centres of pressure that are very close to each other (all with $x_P \approx 0.47$), elimination of pitching moment in this way also simultaneously almost eliminates lift! However, some apparent success retaining a small positive lift can be achieved by a linear combination of three of the special monomial shapes discussed above. For example (with $\beta = 0.1$)

$$g(r) = -0.521r^2 + 0.921r^{10} - 0.500r^{20}$$

has $C_L = 0.00332$ and $C_M = 0.00005$. Figure 4 shows this (non-convex) shape, both as a plot of $g(r)$ and in perspective view.

Conclusion

In the present paper we have shown agreement to about three figures between a general-purpose computer program run with up to 5000 panels and previous semi-analytic solutions for circular lifting surfaces. The latter are essentially exact, though in some cases only exact for discs that are slightly twisted, the small differences in lift due to this twist being dependent on the definition of the angle of attack. We have also studied various axisymmetrically distorted discs, noting that for convex discs the centre of pressure varies little from a point at about 47% of the radius aft of the centre. Nevertheless, it is possible to design a non-convex axisymmetric disc with a small positive lift acting at its centre.

References

BENNINGFIELD., D. "It's a bird! It's a plane! It's a ... flying pancake?", *Smithsonian Magazine*, August 2004.

BOERSMA, J. "Note on the lifting-surface problem for a circular wing in incompressible flow", *Quarterly Journal of Mechanics and Applied Mathematics* **42** (1989) 55–64.

HAUPTMAN, A. and MILOH, T. "On the exact solution of the linearized lifting-surface problem of an elliptic wing", *Quarterly Journal of Mechanics and Applied Mathematics* **39** (1986) 41–66.

GRASMEYER, J.M. and KEENNON, M.T. "Development of the Black Widow micro air vehicle", *AIAA paper 2001-0127*, January 2001.

JORDAN, P.F. "Exact solutions for lifting surfaces", *AIAA journal* **11** (1973) 1123–1129.

KERWIN, J.E. and LEE, C.-S., "Prediction of steady and unsteady marine propellor performance by numerical lifting surface theory", *Transactions of the American Society of Naval Architects and Marine Engineers* **86** (1978) 218–253.

POTTS, J.R. and CROWTHER, W.J., "Frisbee aerodynamics", *AIAA paper 2002-3150*, June 2002.

ROBINSON, A. and LAURMANN, J.A., *Wing theory*, Cambridge University Press, 1956.

TUCK, E.O. "Some accurate solutions of the lifting surface integral equation", *Journal of the Australian Mathematical Society, Series B* **35** (1993) 127–144.

TUCK, E.O. and STANDINGFORD, D.W.F., "Numerical solution of the lifting surface equation", *Boundary Element Technology XII*, Knoxville, Tenn., 9–11 April 1997. Proceedings ed. J.I. Frankel, C.A. Brebbia and M.A.H. Aliabadi, Computational Mechanics Publications, Southampton, pp. 195–206.