Computation and Minimisation of Ship Waves

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1 Introduction

Ships make waves. Sometimes these waves are like Figure 1. Sometimes they are like Figure 2. In fact, these pictures represent the two extremes: the ambient sea is seldom quite as calm as in Figure 1, and most ships avoid seas as rough as in Figure 2. Most of the life of most ships is spent in seas nearer to the calm state than the rough state. This paper will therefore concentrate on ships moving steadily forward at constant speed into a totally calm sea. The effect of (small to moderate) ambient waves can be added later if required.

Ships and other objects (including animals) moving on or near a water surface are influenced by three dominant forces, namely inertia, gravity and viscosity. Waves are essentially a balance between the first two of these. Viscosity (i.e. friction) in fact plays only a very small role in most of the fluid domain surrounding the body, as is the case for most flows at human scales (metres or more), whether or not a free surface is present. However, viscosity dominates very close to the surface of the body, in a thin (centimetres or less) boundary layer, where it provides a significant fraction of the total drag. In the present paper we shall for the most part neglect viscosity. Again, the effect of viscosity can be added later if required.

Indeed this additive characteristic of viscous and wave effects on ships was the main thrust of the mid-19th century work of the pioneering naval architect William Froude, and is the basis for use of model towing tanks for ship design. Essentially a small scaled model is used to determine wave drag, which is then scaled up to full scale by what is now called "Froude scaling" (speeds proportional to the square root of length), and then viscous effects are added via an empirical skin friction formula dependent on wetted surface area. Froude's experimental work on this scaling principle was done long before the concept of the boundary layer, e.g. as

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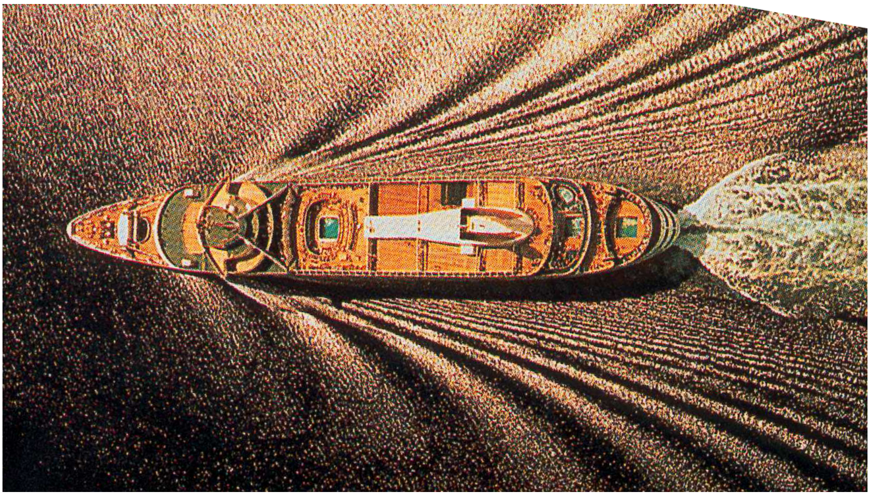

Figure 1. *A cruise ship in a calm sea*

formalised by Ludwig Prandtl in 1912 gave a rational justification for it.

Once viscosity is neglected, the mathematical and computational task is to solve a boundary-value problem for Laplace's equation, subject to a Neumann boundary condition on both the body surface and the free water surface. However, the latter surface is unknown in advance, so we must add a further boundary condition that the pressure (obtained from the flow via Bernoulli's equation) be constant on the free surface, in order to simultaneously determine that free surface. The combination of two (nonlinear) boundary conditions on the (unknown) free surface may be called the Stokes conditions, and the resulting boundary-value problem a "Neumann-Stokes" problem. Although at first sight one might consider this to be a straightforward task, the parenthetic qualifiers "nonlinear" and "unknown" in the previous sentence give indications of trouble ahead, and it was late in the 20th century before anything close to a complete or useful solution of the Neumann-Stokes problem was possible.

2 Thin Ships

Meanwhile, however, even in the 19th century, a remarkable approximate solution to the Neumann-Stokes problem was presented by the Australian mathematician John Henry Michell [11]. Michell assumed that the ship was thin, lying close to a vertical plane, and therefore was able to approximate the Neumann boundary condition on the hull by a simpler "Michell" boundary condition on that plane. Not only that, but thinness of the ship guaranteed that it made small waves, and hence the nonlinear Stokes conditions on the unknown free surface could be replaced by a linear "Kelvin" condition on the known undisturbed free surface, which is a horizontal plane. The resulting "Michell-Kelvin" problem, having only plane co-

Figure 2. *A tanker in heavy seas*

ordinate boundaries, was immediately solvable analytically by Fourier methods.

Michell's main concern was for the wave resistance, or inviscid drag D due to the energy left behind in the wave system. His final result can be written

$$D = \frac{2}{\pi}\,\rho U^2 k_0^4 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \sec^5\theta \left| \iint_W \mathrm{d}x\,\mathrm{d}z \;\; Y(x,z) e^{ik_0 x \sec\theta + k_0 z \sec^2\theta} \right|^2 \tag{1}$$

Here $y = \pm Y(x,y)$ is the hull surface, in a frame of reference with x from bow to stern, y to starboard, and z upward from the free-surface (so $z < 0$ here). The (x,z) integral in (1) is over the centreplane W of the ship, and ρ is the water density, U the ship speed and $k_0 = g/U^2$ where g is gravity. The form of Michell's integral quoted here is for pointed hulls with $Y = 0$ at the ends; if this condition is not met, e.g. for transom-hulled vessels, some extra terms must be added. The water is assumed to be of infinite depth.

Numerical evaluation of Michell's integral is a triple quadrature, a daunting task in 1898, and not entirely a simple one in 2003. Nevertheless Michell himself did present one numerical result, for an idealised choice of $Y(x,z)$ such that the (x,z) integral could be evaluated analytically, so he only had the θ integration to compute numerically by Simpson's rule. More than 100 years later, we can evaluate the full Michell triple integral in less than a twentieth of a second with high precision, for real ships whose offsets $Y(x,z)$ are prescribed as data.

The 20th century history of Michell's integral is a somewhat sad one, reviewed by Wehausen [20] and Tuck [15]. Though published in a very well-known journal, it lay almost unknown for a quarter of a century, until Wigley [23] and Havelock [7] eventually gave it credit in the 1920s, and then in long series of papers used Michell's integral to explore the way in which various changes in simplified ship forms affected wave resistance. Later, Weinblum [21] and others [2] developed pre-computer numerical tools for its evaluation for actual ships, and used it in drag minimisation studies [22]. With the dawn of the computer age, this sort of work was carried a little further [9], but by then the approximate nature of the original formula was a cause for (misplaced) concern, and the power of the computer was increasingly turned instead toward the much more difficult task of solving more exact models, such as the Neumann-Stokes problem, or even models including viscosity. However, Michell's integral remains a useful and often remarkably accurate estimator of wave resistance, see [20], [15],[19].

Michell also provided formulae enabling determination of flow quantities other than wave resistance. In particular, the actual wave elevation $z = Z(x, y)$ made by the thin ship $y = \pm Y(x, z)$ (with pointed ends) can be written as the quadruple integral

$$Z(x,y) = \frac{1}{\pi^2}\iint_W \mathrm{d}\xi\,\mathrm{d}\zeta\;\; Y(\xi,\zeta)\,\Re\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\int_0^\infty \mathrm{d}k \\ \frac{k^2}{k-k_0\sec^2\theta}\exp\bigl[-ik(x-\xi)\cos\theta - iky\sin\theta + k\zeta\bigr] \tag{2}$$

The path of k-integration in (2) passes above the pole at $k = k_0 \sec^2\theta$, so guaranteeing that waves occur only for $x > 0$.

Clearly (2) is much harder to compute from than (1), not least because of the extra k integral, and Michell himself made no attempt to compute ship waves in 1898. Indeed until computers arrived, this was an almost impossible task, and even then little progress was made until late in the 20th century.

An important advance was made by Newman [12] who in effect reduced a significant part of the k and θ integration task to evaluation of a Chebyshev polynomial approximation. The speed at which this can be done is such that computation of $Z(x, y)$ averages, for each (x, y) point, a time not more than that taken for one evaluation of the wave resistance D. Such speed is necessary, since to capture a realistic representation of a complete ship wave field with its complex mix of superposed waves, short and long, near and far, transverse and diverging, requires about 100,000 points. This takes about 5 minutes to compute on a current 2 GHz PC. There are many other factors influencing total computer times for such detailed wave fields, such as the balance between near-field and far-field contributions, the efficiency of handling rapid oscillations near $|\theta| = \pi/2$, and the extent to which parts of the work done at one point can be saved and used again at other points. For further discussion of these matters, see [18],[13],[19].

Figure 3 is an example of a computation performed in this way for a destroyer hull at 30 knots full-scale speed. This figure is simply produced by colouring about 100,000 (x, y) pixels a shade of blue whose lightness is proportional to the computed value of $Z(x, y)$, with the deepest troughs nearly black and the highest crests nearly

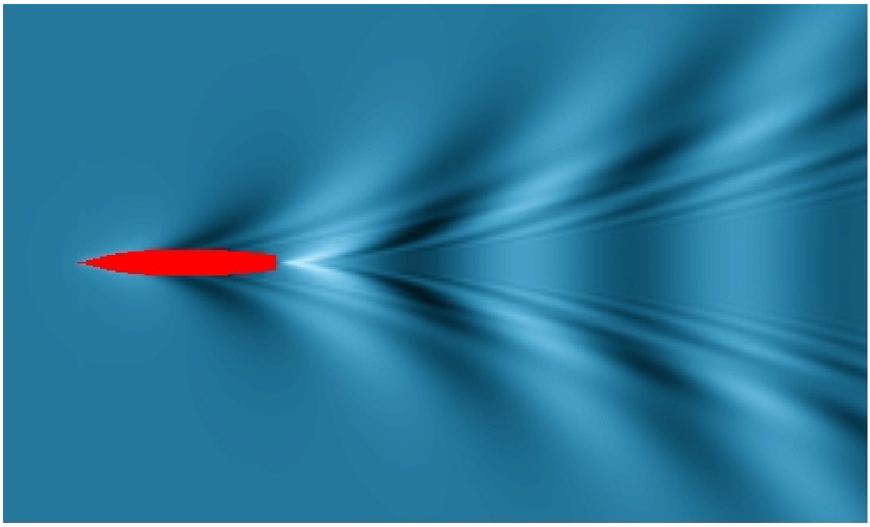

Figure 3. *Free-surface elevation due to destroyer at 30 knots*

white. The red outline of the ship itself is less well represented pictorially, but is modelled quite smoothly in the computation.

3 Pressure Distributions

Another type of travelling disturbance that potentially makes small waves, allowing linearisation of the free-surface boundary conditions, is a surface pressure distribution with a prescribed small excess $p = P(x, y)$ over atmospheric pressure. This could directly be a moving meteorological disturbance or the pressure exerted by a hovercraft, or indirectly be a model for a vessel of small draft, on the hull of which the hydrodynamic pressure is $P(x, y)$.

The wave resistance of such a pressure field is

$$D = \frac{k_0^3}{2\pi\rho g} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \sec^5\theta \left| \iint_W \mathrm{d}x\,\mathrm{d}y \;\; P(x,y) e^{ik_0 \sec^2\theta (x\cos\theta + y\sin\theta)} \right|^2 \tag{3}$$

and the wave field $z = Z(x, y)$ is

$$Z(x,y) = \frac{1}{2\pi^2\rho g} \iint_W \mathrm{d}\xi\,\mathrm{d}\eta \;\; P(\xi,\eta)\, \Re \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^\infty \mathrm{d}k \; \frac{k^2}{k - k_0\sec^2\theta} \exp\big[-ik(x-\xi)\cos\theta - ik(y-\eta)\sin\theta\big] \tag{4}$$

The formulae (3) and (4) are similar to the corresponding Michell formulae (1) and (2), a double integral over a portion W of the plane $z = 0$ replacing a double integral over a portion of the plane $y = 0$, and similar numerical procedures enable efficient computation of 100,000-point wave fields, for pressures $P(x, y)$ inputted as data. For examples, see [19].

4 Flat Ships

Michell used "thinness" (i.e. small beam) of the hull to linearise the Neumann-Stokes problem for a ship. This is not the only way that we can restrict the hull shape to ensure small waves. In particular, another option is "flatness" (i.e. small draft). If the equation of the ship is now written $z = Z_0(x, y)$, we therefore assume Z_0 is small.

In one mathematical sense, flatness is more attractive as an approximation than thinness, since in the formal limit as $Z_0 \to 0$, the only boundary is where $z = 0$. Physically, there is no longer any disturbance "in" the water. Again, Fourier methods allow an immediate (partial) solution, and the disturbed flow field can be seen to be caused by an apparent travelling pressure distribution $p = P(x, y)$ on the free surface. Since we now have efficient tools as above to compute the wave resistance D and wave field $z = Z(x, y)$ for such pressure distributions, this solves the flat-ship problem once the pressure $P(x, y)$ is known.

But unfortunately, given the hull shape $Z_0(x, y)$, the corresponding pressure distribution $P(x, y)$ is not immediately known. An early attempt at a flat-ship theory was that of Hogner [8] (see also [20], pp.170–171) who assumed in effect that $P(x, y) = -\rho g Z_0(x, y)$. This is just a "hydrostatic" approximation, since it is the fluid pressure acting on the hull of the vessel when it is at rest, noting that in general $Z_0 < 0$ so $P > 0$. One might expect that the Hogner theory would be valid (if at all) for relatively low speeds, where the hydrostatic approximation retained some validity.

In fact, the correct condition ultimately determining the pressure $P(x, y)$ is just $Z(x, y) = Z_0(x, y)$. That is, over the limiting portion W of the plane $z = 0$ that is occupied by the ship, we require that the apparent free-surface elevation produced by the pressure $P(x, y)$ is identical to the elevation (usually negative) of the corresponding point of the ship's surface. Equating $Z(x, y)$ as computed from (4) to the input $Z_0(x, y)$ gives an integral equation for the unknown $P(x, y)$ on the portion W of the (x, y) plane, and hence the basic task in solving the flat-ship problem is to solve such an integral equation. This makes the flat-ship problem very much more difficult computationally than the thin-ship problem.

However, in principle we have all the tools required. It is convenient to envisage the problem in discrete form on some (x, y) grid, with a vector $\mathbf{p} = \{P(x, y)\}$ of pressures corresponding to a vector $\mathbf{z} = \{Z(x, y)\}$ of elevations. Then (4) tells us that $\mathbf{z} = \mathbf{A}\mathbf{p}$, for some matrix $\mathbf{A}$ obtained by discretising the kernel function. The most straightforward way to find the elements of the matrix $\mathbf{A}$ is just to run the code that determines $Z(x, y)$ many times, using some basis set of $\mathbf{p}$-vectors. Now given $\mathbf{z} = \{Z_0(x, y)\}$, we simply have to invert that system of linear equations to

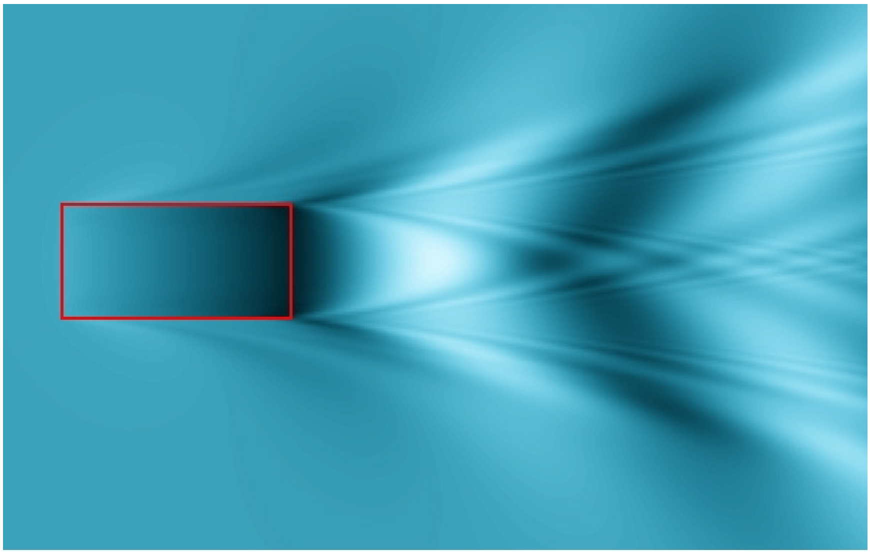

Figure 4. *Free-surface elevation due to "flat" plate at* $F = 0.5$

find $\mathbf{p} = \mathbf{A}^{-1}\mathbf{z}$.

There are a number of difficulties with this approach, encountered and appreciated by many who have attempted to solve this problem in the past, e.g. [14],[5],[4]. From the computational point of view, the matrix $\mathbf{A}$ appears to be ill-conditioned, and most authors have seen grid-scale oscillations in the output pressures. Physically, this is associated with short diverging waves with crests nearly parallel to the ship's track that spring almost uncontrollably from each discontinuity induced in the pressure distribution by numerical approximation of the integrals. Partial cures are possible by various averaging techniques, but it remains difficult to produce smooth pressure outputs.

Interestingly, though, perhaps it is not always necessary to do so. If the pressure $P(x, y)$ depends over-sensitively on the hull shape $Z_0(x, y)$, it follows that the hull shape $Z_0(x, y)$ is not very sensitive to the pressure $P(x, y)$. The present author's experience is that even when the inversion code produced somewhat "wobbly" pressures $P(x, y)$, a back-substitution showed that this pressure nevertheless yielded quite smooth and believable wave fields $Z(x, y)$ agreeing with $Z_0(x, y)$, and these results were essentially unchanged when the wobbles in $P(x, y)$ were smoothed away, either by eye or systematically by averaging or least square techniques.

5 Planing Surfaces

A "planing surface" is in principle just another name for a flat ship. However, it is appropriate in introducing this alternate terminology to highlight special considerations which are of particular importance in the application to actual planing boats,

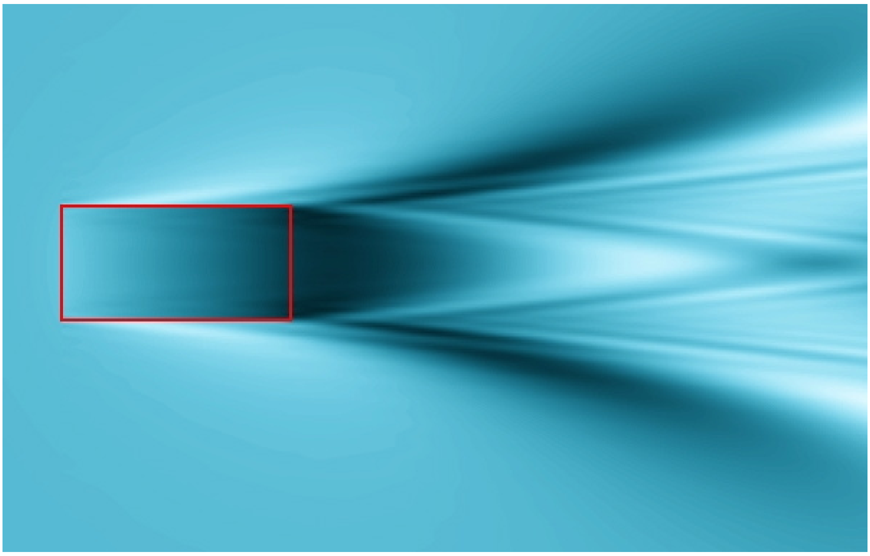

Figure 5. *Free-surface elevation due to "flat" plate at* $F = 1.0$

an important feature of which is their high speed. Indeed, it is generally felt that planing is a state that is achieved for any vessel only, if at all, when the speed is sufficiently high.

At low speed the vessel moves forward with its hull wetted to an extent that is little changed from its static state. As the speed increases, for some hulls there may come a point where (often quite suddenly and simultaneously): significant hydrodynamic lifting forces are induced, the trim and wetted domain change dramatically, the boat lifts out of the water, with a decrease in drag or increase in speed at fixed propulsive force, and the flow near the stern changes its character. If there is a transom (a suddenly cut off stern end) the flow will detach smoothly from the edges of the transom rather than wetting it, but even if there is a pointed stern, smooth detachment and de-wetting may occur prior to the actual stern ending. In any case, once this happens, there is a "hole in the water" at atmospheric pressure immediately behind the stern, and the smoothness of the detachment of the water from the stern to form this hole indicates that the pressure must take the atmospheric value not just on the surface of the water, but also at that detachment location on the hull.

Mathematically, this means that any pressure $P(x, y)$ that is an acceptable output from the code described above for a flat ship must, at least at speeds above that where planing commences, vanish at the trailing end of the wetted domain. This is an extra condition, equivalent to the Kutta condition of aerodynamics, which is not in general satisfied by the unique inversions of (4) for a given $Z = Z_0(x, y)$ defined on a given wetted portion W of the (x, y) plane.

Nor indeed should we expect it to be satisfied! A given hull shape $Z_0(x, y)$

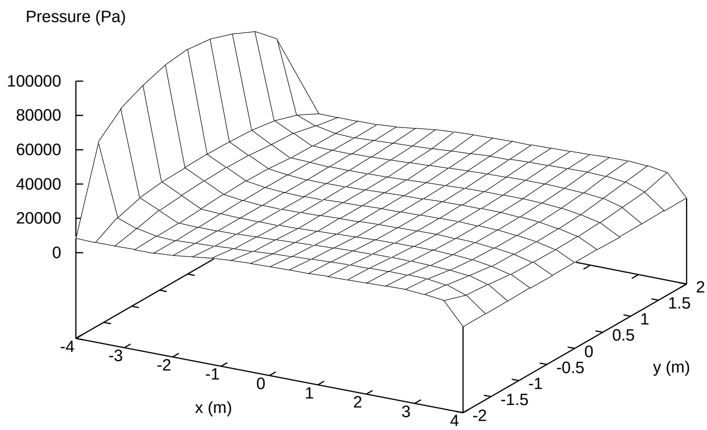

Figure 6. *Pressure on "flat" plate at* $F = 1$

will be wetted when planing to an extent that must be determined by the flow, and hence the domain W over which the integral equation is solved must be determined as part of the solution, not fixed. Although this dependency of wetted domain on hydrodynamics holds in principle for any surface-piercing body at any speed, the change in wettedness accompanied by changes in trim and draft (or "squat") is relatively small for conventional ships at low speeds. However, it is an essential feature of planing that *large* changes in wettedness and trim occur as the speed increases, and for example the wetted length of a planing boat at high speed can be only a small fraction of that at rest. This requirement to determine the domain W of integration in the (x, y) plane, simultaneously with solution of the integral equation for $P(x, y)$ on that domain, presents a real challenge to any computational procedure for solving the planing problem.

Alternatively, if we choose to fix the wetted domain W, we must allow an equivalent degree of freedom in the hull shape $Z_0(x, y)$. This can be computationally more convenient, though less relevant to real planing boat design. For example, if we set $Z_1(x, y) = Z_0(x, y) + C(y)$, then the program determines $C(y)$, thus distorting the input hull $Z_0(x, y)$ (by bending it about a longitudinal axis) to a new shape $Z_1(x, y)$ on the same wetted domain W, such that the Kutta condition $P(x, y) = 0$ is satisfied at the trailing end. Whether we like it or not, the program then solves for the flow about the boat $z = Z_1(x, y)$, not the input boat $z = Z_0(x, y)$!

Although much more work is needed to bring this sort of planing surface program to a useful state, Figures 4 and 5 show examples of free-surface elevations $Z(x, y)$ computed in this way for a flat plate $Z_0(x, y) = -\alpha x$ at constant angle of attack α on a fixed 2:1 rectangular domain W. These are for two different speeds, namely for Froude numbers $F = 0.5$ and $F = 1.0$ respectively, where $F = U/\sqrt{gL}$, with L the length. In fact, at both speeds the function $C(y) \approx -0.7\alpha L/2$ is almost constant, so to a large extent the true body $z = Z_1(x, y)$ is still a flat plate, but simply shifted down by the flow without distortion, with the forward 30% of the plate above the undisturbed water level (but still wetted). The scale of the highest crests and deepest troughs shown is about $\pm 1.5\alpha L/2$, i.e. about twice the maximum plate submergence at the stern. Note the transverse waves (e.g. as marked by the dark trough for $F = 0.5$) at low speeds, disappearing at the higher speed $F = 1$ to leave a mainly diverging pattern. Note also the familiar "rooster tail" of white (highest crest) water behind the stern, which moves further downstream as the speed increases.

Figure 6 shows an example at $F = 1$ of the corresponding pressure distribution $P(x, y)$, for a plate with $L = 8$ metres and $\alpha L/2 = 1$ metre. The basic "bedstead" shape of the pressure function $P(x, y)$ seems to be smooth in both directions, with a leading-edge singularity (inverse square root and only crudely approximated here by a large finite pressure) modelling a splash, finite non-zero values at the sides, and a square-root approach to zero at the trailing edge. However, the rapidly-varying character of the diverging waves tends to compromise this smoothness computationally, and requires averaging techniques in the solution of the integral equation to avoid spurious rapid oscillations in the output pressure.

In fact, the speeds in these examples are still relatively low for application to actual planing boats (though high relative to conventional ship speeds), and the large waves made at these speeds are undesirable. There is some interest in design of large vessels which must operate at such speeds, and hence interest in minimising these waves. On the other hand, at even higher true planing speeds, wave effects are essentially negligible irrespective of hull shape, and the planing surface behaves like a lifting surface in aerodynamics. There are then computational methods [3] using aerodynamic lifting-surface theory for analysing the flow, but even these have seen little application so far to real planing-boat design, which remains largely empirical.

6 Design, Detection and Optimisation

The above describes flow *analysis* tools for a given body, and in particular for given planing surfaces reveals computational challenges that are far from resolved. Somewhat different challenges are presented by *design* tasks, where the body shape is (fully or partially) output rather than input. In some cases, the design task can be simpler than the analysis task.

If the body shape is to be determined, then something else must be given. One possibility is to design a body to produce a given wave field. Although this is a somewhat unlikely *design* task, it is essentially the same as a *detection* task. That is, we may wish to determine the shape or character of a vessel that is not readily observable directly, by examining the wave field that it produces. Although this is a very interesting and potentially important application, it will not be further discussed here.

Another design objective is *optimisation*. In particular, it has always been one of the main tasks of the naval architect to minimise the drag on a vessel, which is the sum of a viscous component and a wave component. Viscous drag is largely insensitive to hull shape, being mostly just proportional to the total wetted surface area. Hence so long as this wetted area is kept as small as possible, the residual design effort goes into minimising the wave resistance D.

For conventional thin ships, Michell's integral (1) is very well adapted to this task, being a positive definite quadratic functional in the hull shape function $Y(x,z)$, and there have been a number of optimisation studies [9], [20] pp. 205–214, of its minimisation subject to various constaints, the most important of which is fixed displacement volume $2\iint Y\,\mathrm{d}x\,\mathrm{d}z$. The results have not been too satisfactory, with unacceptable features like negative offsets $Y(x,z)$, corrugations in the designed hull at low speeds, and end singularities at higher speeds. Some of these features are perhaps inevitable: corrugations are a fruitless attempt to cancel waves that are much shorter than the ship, while the end singularities are a fruitless attempt to place as much of the displacement as far apart as possible in order to cancel waves that are much longer than the ship. Features such as negative offsets can be eliminated by including non-negativity constraints in the optimising code. This area is one that perhaps could benefit from further research, in an era of cheap high-speed computing capability, and sophisticated optimisation algorithms.

Unconventional vessels allow more scope for optimisation, and we discuss here some drag minimisation issues for multihulls (catamarans, trimarans, etc) and for pressure distributions modelling hovercraft or planing surfaces.

7 Optimal Spacing for Multihulls

Michell's thin-ship theory provides a linear relationship (2) between hull shape Y and waves Z, which can (with some reservations) be extended by simple linear superposition to collections of separate thin hulls. [The reservations are associated with vertical-axis vortices and side forces induced by each hull on every other hull, but these are generally felt to be negligible for vessels of sufficiently small draft.] The wave resistance D depends quadratically rather than linearly on the hull shape function Y, but the Michell integral (1) can be written

$$D = \frac{2}{\pi}\,\rho U^2 k_0^4 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \sec^5\theta\, \left|\Omega(\theta)\right|^2 \tag{5}$$

involving the modulus squared of a complex spectrum function $\Omega(\theta)$ which for a single hull is given by

$$\Omega(\theta) = \iint_W \mathrm{d}x\,\mathrm{d}z\;\; Y(x,z)e^{ik_0 x\sec\theta + k_0 z\sec^2\theta} \tag{6}$$

and thus depends linearly on the hull shape Y.

Suppose now that we have N hulls, with the jth hull having equation

$$y = b_j + Y_j(x - a_j, z)\;, \tag{7}$$

on its centreplane W_j, so that its centre point is located at $(x,y) = (a_j, b_j)$. Then the linearly superposed spectrum is

$$\Omega(\theta) = \sum_{j=0}^{N} e^{-ik_0\sec^2\theta(a_j\cos\theta + b_j\sin\theta)} \iint_{W_j} \mathrm{d}x\,\mathrm{d}z\;\; Y_j(x,z)e^{ik_0 x\sec\theta + k_0 z\sec^2\theta}\;. \tag{8}$$

A simple but instructive special case is where all Y_j functions are proportional to each other, i.e.

$$Y_j(x,z) = \beta_j Y_0(x,z) \tag{9}$$

for constants β_j. Physically, each hull has the same shape and centreplane W, but a varying beam proportional to β_j. Then

$$\Omega(\theta) = \Omega_0(\theta)\, F(\theta) \tag{10}$$

where

$$\Omega_0(\theta) = \iint_W \mathrm{d}x\,\mathrm{d}z \;\; Y_0(x,z) e^{ik_0 x \sec\theta + k_0 z \sec^2\theta} \tag{11}$$

and

$$F(\theta) = \sum_{j=0}^{N} \beta_j e^{-ik_0 \sec^2\theta (a_j \cos\theta + b_j \sin\theta)} \tag{12}$$

Now since D depends only on the modulus squared of Ω, for a fixed basic hull shape Y_0 and hence a fixed basic spectrum Ω_0, it depends only on the modulus squared of $F(\theta)$.

It is therefore possible to perform quite simple optimisation studies by minimising $|F(\theta)|^2$ as a function of the arrangement pattern (a_j, b_j) and beams β_j, divorcing this from the separate question of the optimum choice of the shape $Y_0(x,z)$ of the hulls. Such a minimisation is potentially confused by the dependence of $F(\theta)$ on the wave angle θ, but any choice which reduces $|F(\theta)|^2$ over a significant range of angles θ is likely to be desirable when this quantity is integrated to give the net wave resistance.

A systematic set of such studies was reported in [16]. The simplest example is a conventional (side-by-side) catamaran $N = 2$, where we can set $\beta_1 = \beta_2 = 1/2$, $a_1 = a_2 = 0$ and $b_1 = -b_2 = b$, where $2b$ is the hull centreplane separation. Then

$$F(\theta) = \cos(k_0 b \sec^2\theta \sin\theta) = C(\theta). \tag{13}$$

Good hull-separation choices then are such that $C(\theta) = 0$ for some $\theta = \theta_0$ of importance, e.g. where the basic spectrum $\Omega_0(\theta)$ has a maximum. The smallest such separation is therefore $2b = (\pi/k_0)\cos^2\theta_0/\sin\theta_0$. Importantly, however, nothing can be done by adjusting the hull separation of conventional catamarans to eliminate or even reduce *transverse* waves having $\theta = 0$, since $F(0) = 1$ irrespective of the separation $2b$.

A somewhat more interesting example is a "staggered" trimaran $N = 3$, which is essentially the above catamaran, but with a third centrally placed hull having $a_3 = -s$, $b_3 = 0$ and $\beta_3 = 1$, located a distance s ahead of the side hulls. We have chosen the central hull to have exactly half of the total displacement of the vessel, so its beam is twice that of each of the side hulls, which is optimal for cancellation of transverse waves. Then

$$F(\theta) = C(\theta) + e^{ik_0 s \sec\theta} \tag{14}$$

where $C(\theta)$ is again given by (13). Indeed, transverse waves are easily seen to be totally cancelled, with $F(0) = 0$, if the (smallest) longitudinal stagger is chosen to be $s = \pi/k_0$. Physically, this simply means that the stagger s is half of the transverse wavelength $2\pi/k_0$, so the transverse waves made by the central hull are exactly out of phase (and equal in amplitude because of the choice of displacement) to the transverse waves made by the side hulls. With that choice of s, we have finally

$$\left|F(\theta)\right|^2 = 1 + C(\theta)^2 + 2C(\theta)\cos(\pi \sec\theta) \tag{15}$$

and our remaining task is to choose the best separation $2b$, in order to make $|F(\theta)|^2$ as small as possible over a large range of θ values.

It is not hard to see (e.g. graphically) that it is optimal to make both $C(\theta)$ and $\cos(\pi\sec\theta)$ vanish simultaneously at some angle θ_0. A good choice appears to be $\sec\theta_0 = 5/2$ ($\theta_0 \approx 66°$) and $k_0 b \sec^2\theta_0 \sin\theta_0 = \pi/2$, so $k_0 b/\pi = b/s = \cos^2\theta_0/2\sin\theta_0 \approx 0.262$. With that choice, the spectral energy $|F(\theta)|^2$ remains below about a quarter of that for a monohull, for a large range of angles, about $|\theta| < 71°$. Notably the condition $b/s \approx 0.262$ indicates that the trimaran forms an "arrow" configuration with angle about 29°, so lying inside the Kelvin angle of 39°. Actual computations from Michell's integral [16] confirm this to be a configuration with very low wave resistance near the design speed $U = \sqrt{gs/\pi}$. Other examples with especially low wave resistance are studied in [16], including a laterally asymmmetric (staggered) catamaran ($N = 2$), and a tetra-hull ($N = 4$) in a "diamond" arrangement.

8 Optimal Pressure Distributions

Design of pressure distributions $P(x, y)$ to minimise an objective such as wave resistance may be a worthwhile task for vessels such as hovercraft or surface-effect ships, where there is some element of direct control of the pressure distribution. However, it is potentially of even greater significance for planing surfaces, where the pressure $P(x, y)$ is that felt under the hull due to the hydrodynamics of the flow. If we can make such an optimal choice of $P(x, y)$, then this leads immediately to a hull shape $Z_0(x, y)$, without the need to solve the integral equation (4).

The formula (3) allows wave resistance minimisation with respect to choice of pressure $P(x, y)$, subject to a fixed weight constraint of given $\iint P \, \mathrm{d}x \, \mathrm{d}y$, and there were early studies e.g. [10], [1], and more recently [6] on such optimisations. Typically the results are oscillatory, and in particular possess unacceptable large swings between positive and negative pressures.

An additional constraint of importance in practice, both directly for hovercraft and indirectly for planing boats, is non-negativity of the pressure, i.e. the inequality constraint $P(x, y) \geq 0$. When this constraint is implemented [17], perhaps not surprisingly, we find zones of zero pressure, replacing but not necessarily coinciding with the negative-pressure zones of the unconstrained solutions, interspersed with zones of positive pressures. In effect, this implies that the optimal pressure distribution is not a single patch, but rather a "multi-patch", analogous to a multi-hull ship as above.

At relatively high speeds the optimal positive pressure is confined to the extreme bow and stern ends. The end pressure varies in magnitude smoothly across the width, decreasing toward zero at the sides. In fact, an aerodynamic analogy (with exact equivalence between wave resistance and induced drag in the limit as $F \to \infty$) shows that at sufficiently high speed this lateral variation must be elliptic.

In the case of a 2:1 rectangle, so long as $F > 0.96$ the optimum consists *only* of these two bow and stern pressure patches, and the best result is achieved in the apparently unrealistic limit where their longitudinal extent is zero and their magnitude is large. Such Froude numbers are high enough that transverse wave cancellation by bow-stern interaction is ineffective, and the best that can be achieved is to place the disturbing pressures as far apart longitudinally as possible. At lower speeds, it is desirable to include a third patch of positive pressure, located amidships, and this patch grows in importance and length as the speed is reduced, with the end lines withering away, until the optimal configuration is a single patch.

However, the simple bow-stern pressure-line optimum holds in a speed range near $F = 1$ of considerable interest, and has a wave resistance typically less than a quarter of that for a constant-pressure patch bearing the same total weight. For a hovercraft modelled by a constant-pressure patch, this occurs at "hump" speed, which is a barrier that has to be overcome before the vessel moves to a higher operating speed where wave-making is not a factor, so there is only a relatively minor operational importance attached to reducing the size of this hump. In the indirect application to a planing surface, on the other hand, the reduced wave resistance may be a real benefit for large boats incapable of reaching these higher speeds, whose operating speed may of necesssity be near $F = 1$.

At first sight, the conclusion that the optimal pressure distribution is highly concentrated at the bow and stern of the vessel is disappointing, and perhaps not achievable. However, at least for the planing application, it suggests that a "tandem" configuration consisting of two distinct planing surfaces, each of high aspect ratio (small length relative to width), may be optimal.

Throughout the present paper we have until now discussed only fully three-dimensional bodies and flows. However, it is interesting that this conjectured optimum points toward locally two-dimensional flows near each of these two high-aspect-ratio planing surfaces, and (especially since on the small local lengthscale the Froude number is high and wave effects locally negligible) there is scope for simplified two-dimensional flow analysis. It should however be noted that although the *far-field* waves made by the tandem configuration are small, so yielding low wave resistance, this does not necessarily imply a small free-surface elevation everywhere, and indeed, there are indications (see e.g. [19]) of a large trough between the bow and stern patches.

9 Acknowledgement

This work was supported by the Australian Research Council, and also by the Surveillance Systems Division of DSTO Australia. Contributions from and discussions with David Scullen and Leo Lazauskas are gratefully acknowledged.

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