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TWO-DIMENSIONAL FINITE-DEPTH PLANING HYDROFOIL UNDER GRAVITY

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ABSTRACT

The problem addressed in the paper is that of two-dimensional irrotational flow past a finite-depth planing hydrofoil under gravity. Linearized planing surface theory is applied. A simple numerical approach is used to obtain data on hydrodynamic coefficients and flow pattern for various ranges of input parameters. These data are partly verified through the analysis of two limiting cases of the considered problem: first, the infinite depth, Froude number being finite and second, finite depth with very high Froude numbers.

1 INTRODUCTION

The linearized problem of a 2D hydrofoil planing under gravity is well-known and its theory was addressed by many authors. The first references are the papers by Sretensky [14], Wagner [19], Sedov [11], Kochin [7], and Y.S. Chaplygin [1], where some numerical results were produced for the planing flat plate. Haskind [6] extended Sedov's approach to finite depth, but gave no numerical results. In [9] some results for the free surface elevations are presented. Linearized planing surface theory was also treated by Maruo [8]. Squire [13] analyzed a planing flat plate using a method similar to that by Sretensky and Maruo, and Cumberbatch [2] used a method similar to Sedov's.

Tuck [15, 16, 17, 18] used a procedure involving numerical solution to the planing integral equation. In a paper by the present author [4], Sedov's approach was used to analyze the influence of a spoiler upon the hydrodynamic coefficients of a planing hydrofoil. Lately [3] the method of singular integral equations along with discrete vortices method was applied to the problem under consideration.

The linear theory adequately describes the flow pattern for small incidence angles and curvatures of the hydrofoils everywhere except in the vicinity of the leading and trailing edges. The stagnation point, the spray jet, and the spoiler occur in these regions and therefore the perturbations can not be considered to be small there. Asymptotic analysis has shown that the linear solution loses its correctness in the proximity of the leading edge at distances of $\mathrm{O}(\alpha^2)$, where α denotes the incidence angle. The linear solution is also not correct in the vicinity of the trailing edge with the spoiler. The scale of the region of invalidity is of $\mathrm{O}(\bar{\varepsilon})$, where $\bar{\varepsilon}$ denotes the relative spoiler length [10].

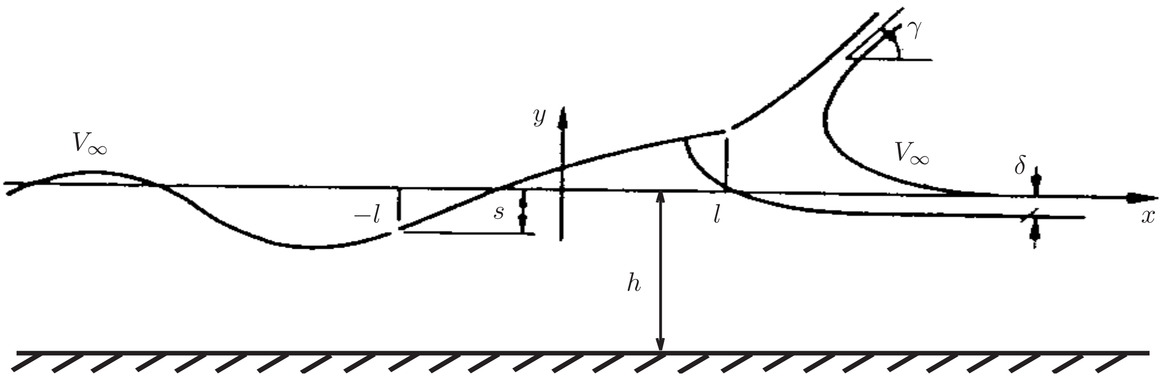

Figure 1: Sketch of the flow pattern around a finite-depth planing hydrofoil under gravity.

2 STATEMENT OF THE PROBLEM

The planing hydrofoil $y = \eta(x)$, $x \in [-l; l]$ is assumed to be in the uniform finite-depth flow of an ideal incompressible fluid under gravity. V_∞ and p_0 denote the stream velocity and pressure at infinity. A cartesian coordinate system (x, y) has its origin at the middle point of the foil's wetted length projection onto the stream direction, which coincides with the x–axis direction, the y–axis being directed vertically upwards, see Fig. 1. The depth of the incoming flow at upstream infinity is h. The problem is rendered non-dimensional by dividing all the length variables by half of the wetted length projection l, and all velocity variables by V_∞. The Froude number based on the length l is introduced: $Fr = V_\infty/\sqrt{gl}$, where g denotes the gravity acceleration.

The problem is considered to be solved when the velocity potential function $\varphi(x, y)$ is found. The harmonic function $\varphi(x, y)$ is the real part of an analytical function of complex potential $w(z) = \varphi + i\psi$, where $z = x + \mathrm{i}y$, and has to satisfy the boundary kinematic condition on the wetted length

$$\frac{\partial\varphi}{\partial n} = -\sin\theta(x)\,,$$

the dynamic condition on the free surface

$$\frac{\partial\varphi}{\partial x} - \frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right) - \nu y = 0\,,$$

and the impermeability condition on the bottom $y = -h$

$$\frac{\partial\varphi}{\partial y} = 0\,,$$

where $\partial/\partial n$ denotes the derivative in the normal direction, $\theta(x) = \eta'(x)$ denotes the tangential angle to the foil at point x, and $\nu = gl/V_\infty = 1/(2Fr^2)$. Finally, at infinity we require, following [11], that $w(z) \sim (A_1 + \mathrm{i}\,A_2)\,\mathrm{e}^{-\mathrm{i}\,\nu z}$ as $x \to +\infty$)

Assuming $\theta(x)$ is small, we can neglect the second-order terms in the above boundary conditions and summarise the boundary-value problem for linearized finite-depth planing as:

$$\begin{aligned}
&\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad \text{for} \quad -h < y < 0\,,\\
&\frac{\partial\varphi}{\partial y} = -\theta(x) \quad \text{at } y = 0\,,\ |x| < 1\\
&\frac{\partial\varphi}{\partial x} - \nu y = 0 \quad \text{at } y = 0\,,\ |x| > 1\\
&\frac{\partial\varphi}{\partial y} = 0 \quad \text{at } y = -h
\end{aligned} \tag{1}$$

Additionally, $\mathrm{d}w/\mathrm{d}z$ should be finite everywhere in the flow domain except in the vicinity of the leading edge $z = 1$. In other words, if we introduce the pressure coefficient $p(x)$ which is connected to the velocity potential through the relation (in the framework of the linear theory)

$$p(x) = \frac{\partial\varphi}{\partial x} - \nu y\,,$$

then $p(x) \sim \mathcal{B}/\sqrt{1-x}$ as $x \to 1$.

3 THE FINITE-DEPTH PLANING EQUATION

Kashkind [6] was the first to obtain an expression for the relationship between a smooth distribution of pressure $p(x)$ acting in the interval $|x| < 1$, $y = 0$ and the free-surface elevation $\eta(x)$ on the same interval. This expression can be used to formulate the finite-depth planing equation as follows

$$\eta(x) + \mathcal{C} = \int_{-1}^{1} p(\xi)\,\mathcal{K}(x-\xi)\,\mathrm{d}\xi\,, \tag{2}$$

where the kernel function has the form

$$\mathcal{K}(x-\xi) = -\frac{1}{\pi}\int_0^\infty \frac{\cos\lambda(x-\xi)\,\sinh(\lambda h)}{\nu\,\sinh(\lambda h) - \lambda\,\cosh(\lambda h)}\,\mathrm{d}\lambda - \\ - \mathcal{H}(\nu h - 1)\,\frac{\sinh(\lambda_0\,h)\,\cosh(\lambda_0\,h)}{\nu\,h - \cosh^2(\lambda_0\,h)}\,\sin\lambda_0(x-\xi)\,. \tag{3}$$

In the above expression λ_0 is the real positive root of the equation

$$\lambda - \nu\,\tanh(\lambda\,h) = 0 \tag{4}$$

and $\mathcal{H}(x)$ denotes the unit step function. This means that the second term in (3) contributes only if $\nu h > 1$. Note that equation (4) has only pure imaginary roots if $\nu h < 1$. The integral equation (2) should be solved in the $\infty - 0$ solution class, which means that $p(x)$ has to have an inverse square-root singularity at the leading edge of the planing hydrofoil, and has to behave like $\sqrt{1+x}$ in the vicinity of the trailing edge. The absence of the second term in (3) when $\nu h < 1$ corresponds to the absence of an infinite train of trailing waves.

The role of the unknown constant $\mathcal{C}$ in the integral equation (2) is discussed in [15, 16, 17, 18]. In fact, the correct choice of $\mathcal{C}$ ensures that the Kutta–Joukowsky condition at the trailing edge $p(-1) = 0$ is satisfied, and hence a unique solution to (2) exists.

4 NUMERICAL SOLUTION

A simple numerical solution readily results from the assumption that $p(x)$ is a step-wise constant function, a discretization of the wetted length $-1 < x < 1$ being introduced. Such an approach was successfully used in [18] for the infinite-depth planing flow problem.

Assume that $p(\xi) \approx p_j$ when $\xi_{j-1} < \xi < \xi_j$, $j = 1, \ldots, N$ and $\xi_0 = -1$, $\xi_N = 1$. Then, using the collocation method, the integral equation (2) can be re-written in the form

$$\eta(x_i) + \mathcal{C} = \sum_{j=1}^{N} p_j \int_{\xi_{j-1}}^{\xi_j} \mathcal{K}(x_i - \xi)\,\mathrm{d}\xi\,, \tag{5}$$

where x_i, $i = 1, \ldots, N$ denotea collocation points at which the integral equation (2) is considered to be satisfied. In the present paper the Chebyshev grid is applied with

$$\xi_j = -\frac{1}{2}\left(1 - \cos\frac{\pi j}{N}\right), \quad x_i = -\frac{1}{2}\left(1 - \cos\frac{\pi(i-0.5)}{N}\right).$$

In fact, the collocation points x_i, $i = 1, \ldots, N$ are the zeros of the Chebyshev polynomial of the first kind $T_N(x)$ while ξ_j, $j = 1, \ldots, N-1$ are the zeros of the Chebyshev polynomial of the second kind $U_{N-1}(x)$. Such a choice of discretization along with the simple condition $p(-1) = 0$ accounts for the fact that we seek a solution in $\infty - 0$ solution class.

Note that in the case of infinite depth, comparison of this numerical method with that based on Fourier series expansion of the pressure distribution coefficient $p(x)$, see [5], demonstrated a good agreement for a wide range of the flow parameters.

5 NUMERICAL RESULTS AND DISCUSSION

Consider two important limiting cases of the finite-depth planing problem under gravity, first, the infinite depth $h \to \infty$, Froude number being finite $Fr < \infty$ and second, $h < \infty$, $Fr \to \infty$. Both cases are well studied and therefore can be used as a verification of the numerical results obtained in the section.

5.1 Limiting case I: infinite depth, finite Fr

This corresponding problem was addressed in many works, some references to which are mentioned in the Introduction to the present paper. Note that $\nu h \gg 1$ and hence the second term in (3) vanishes. Thus, as $h \to \infty$, the kernel of the governing integral equation reduces into

$$\mathcal{K}_\infty(x-\xi) = -\frac{1}{\pi} \int\limits_0^\infty \frac{\cos\lambda(x-\xi)}{\nu-\lambda}\,\mathrm{d}\lambda + \sin\nu x\,. \qquad (6)$$

After a little algebra this expression can be shown to coincide with well-known kernel formula for the infinite-depth planing.

5.2 Limiting case II: finite depth, infinite Fr

Sedov [12] was the first to consider the corresponding linear problem of the planing hydrofoil on the surface of a fluid of finite depth, gravity influence being neglected. With the assumption that the planing hydrofoil makes only small perturbations to the incoming flow, and in the notation of the present paper, the perturbed conjugate velocity $\omega(z) = u - \mathrm{i}v$ can be written in the form

$$\omega(z) = \frac{2\,\mathrm{i}}{\pi}\sqrt{\frac{\sinh 2(\zeta+a)}{\sinh 2(\zeta-a)}} \times \\ \times \int\limits_{-a}^{a} \frac{\theta(t)}{\sinh 2(t-\zeta)} \sqrt{\frac{\sinh 2(\zeta-a)}{\sinh 2(\zeta+a)}}\,\mathrm{d}t\,, \qquad (7)$$

where $a = \pi/(4h)$ and $\zeta = \pi z/(4h)$. The free-surface elevation is obtained by integration of (7)

$$y_{fs}(x) = -\mathrm{Im} \int\limits_x^{\pm\infty} \omega(t)\,\mathrm{d}t$$

and the height of the trailing edge is given by

$$\mathcal{C}_\infty = -\mathrm{Im} \int\limits_1^\infty \omega(t)\,\mathrm{d}t - \eta(1)\,.$$

5.3 Pressure distribution and flow pattern

An arc of a parabola with the topography of the wetted portion $y(x) = 2d(1-x^2) + \alpha x$ was chosen for calculations.

Figures 2 and 3 demonstrate the pressure coefficient $p(x)$ for the planing parabolic hydrofoil with $\alpha = 10^{\mathrm{o}}$, $d = 0.05$ and $\nu = 2$ when $h = 1$ and $h = 2$ correspondingly. Here 30 collocation points were used. In the same graphs, $p(x)$ is also shown for the case of infinite depth. It is seen that in the case of $h = 2$ both lines practically coincide.

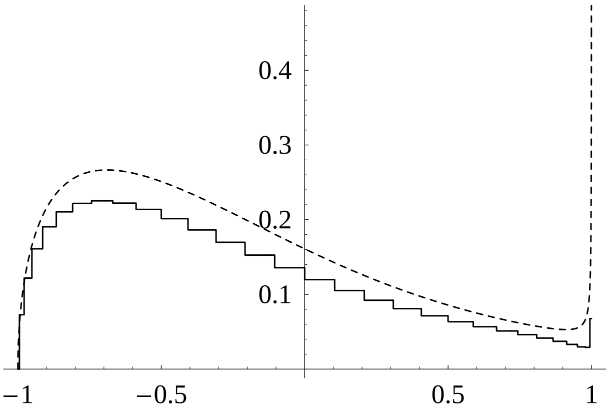

Figure 2: Pressure distribution for $h = 1$ (solid line) and $h \to \infty$ (dashed line); $\alpha = 10^{\mathrm{o}}$, $d = 0.05$, $\nu = 2$ ($Fr = 0.5$) and $M = 30$.

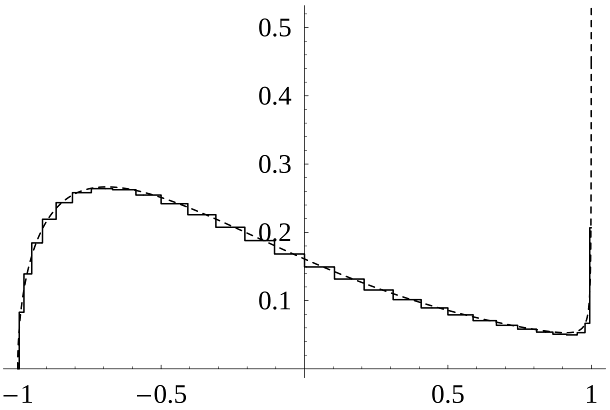

Figure 3: Pressure distribution for $h = 2$ (solid line) and $h \to \infty$ (dashed line); $\alpha = 10^{\mathrm{o}}$, $d = 0.05$, $\nu = 2$ ($Fr = 0.5$) and $M = 30$.

Flow pattern for the same set of the flow parameters are depicted in Figs. 4 and 5, shift of the trailing edge $\eta(-1) + \mathcal{C}$ being -0.3446 and -0.3351 correspondingly. Again, in the case of $h = 2$ flow pattern is close to that in the case of $h \to \infty$ (shift of the trailing edge is -0.3279). In both figures the value of νh is greater than 1 and therefore an infinite train of trailing waves is present. Next two figures 6 and 7 demonstrate flow patterns with no trailing waves for $\nu h < 1$. Though the Froude number is not very high, both flow patterns are similar to those in the case of $Fr \to \infty$.

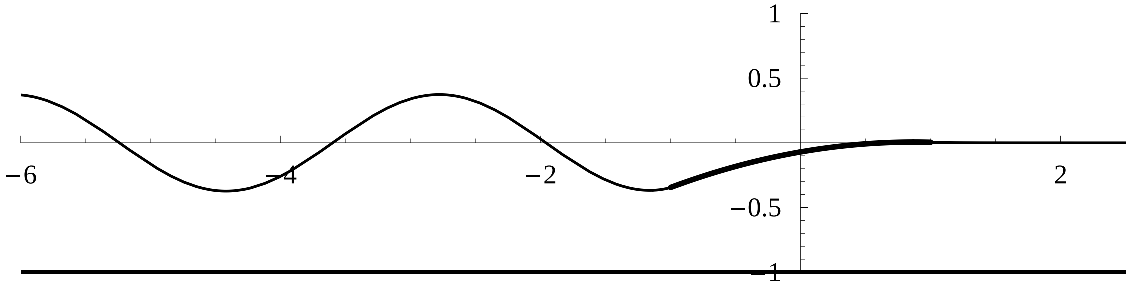

Figure 4: The flow pattern for $h = 1$: $\alpha = 10^{o}$, $d = 0.05$, $\nu = 2$ ($Fr = 0.5$) and $M = 30$. Shift of the trailing edge $\eta(-1) + \mathcal{C} = -0.3446$.

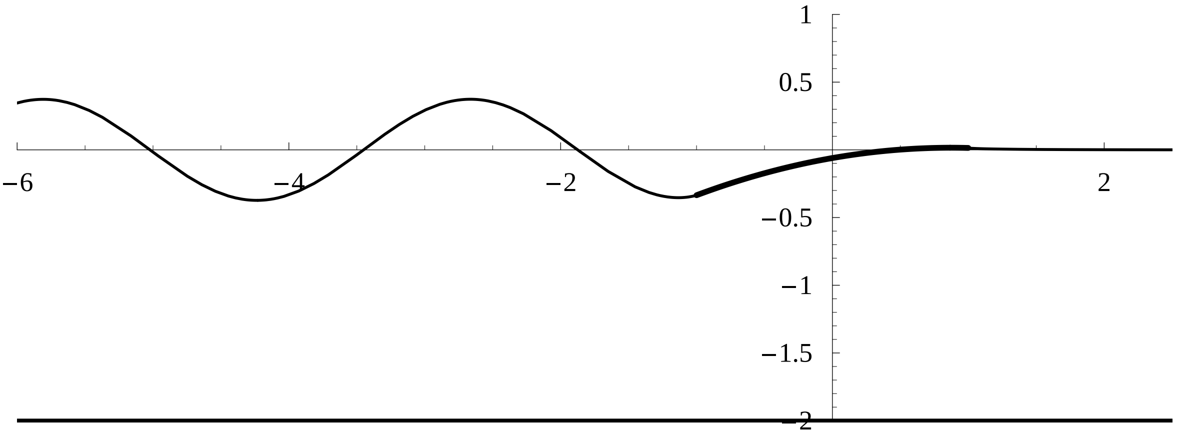

Figure 5: The flow pattern for $h = 2$: $\alpha = 10^{o}$, $d = 0.05$, $\nu = 2$ ($Fr = 0.5$) and $M = 30$. Shift of the trailing edge $\eta(-1) + \mathcal{C} = -0.3351$.

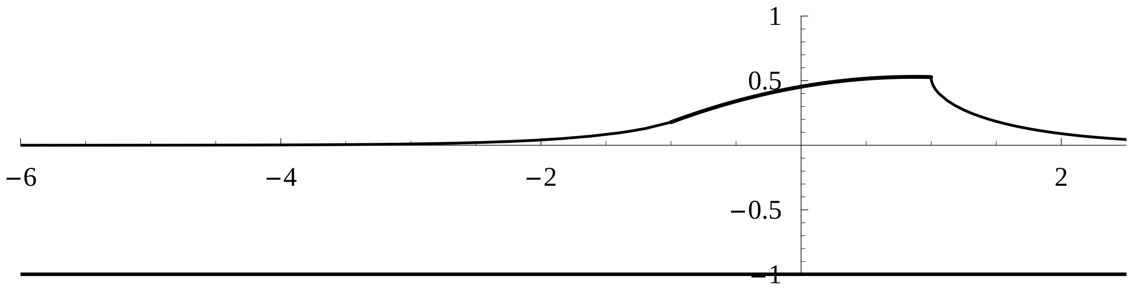

Figure 6: The flow pattern for $h = 1$: $\alpha = 10^{o}$, $d = 0.05$, $\nu = 0.25$ ($Fr = \sqrt{2}$) and $M = 30$. Shift of the trailing edge $\eta(-1) + \mathcal{C} = 0.1787$. $\nu h < 1$ – no trailing waves.

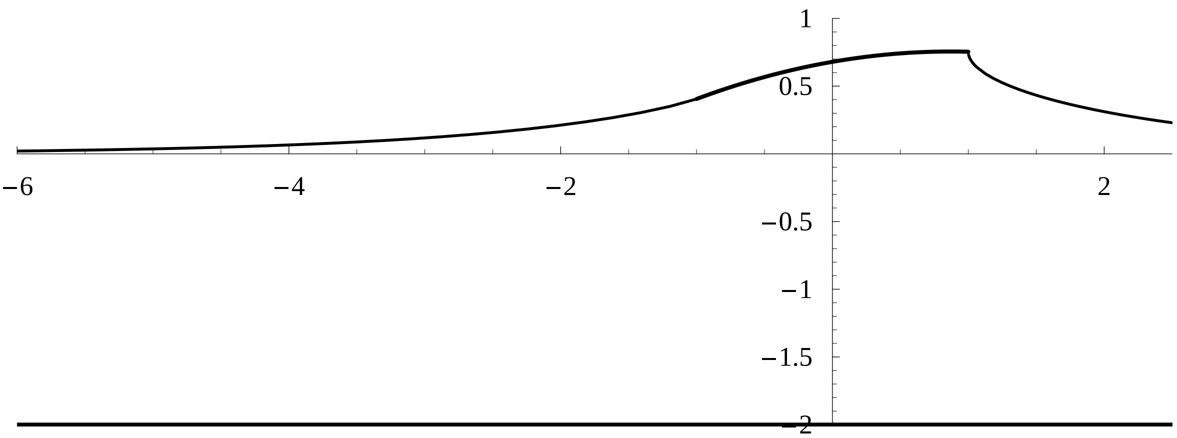

Figure 7: The flow pattern for $h = 2$: $\alpha = 10^{o}$, $d = 0.05$, $\nu = 0.25$ ($Fr = \sqrt{2}$) and $M = 30$. Shift of the trailing edge $\eta(-1) + \mathcal{C} = 0.4049$. $\nu h < 1$ – no trailing waves.

5.4 Hydrodynamic coefficients

Derivatives of the lift coefficient with respect to α and d for the planing hydrofoil $y(x) = 2d(1-x^2)+\alpha x$ versus Froude number Fr are depicted in Figs. 8 and 9 correspondingly for $h = 1$, $h = 2$ and $h \to \infty$. In fact, total lift can be calculated using a simple expression

$$C_L = \frac{\partial C_L}{\partial \alpha}\,\alpha + \frac{\partial C_L}{\partial d}\,d\,.$$

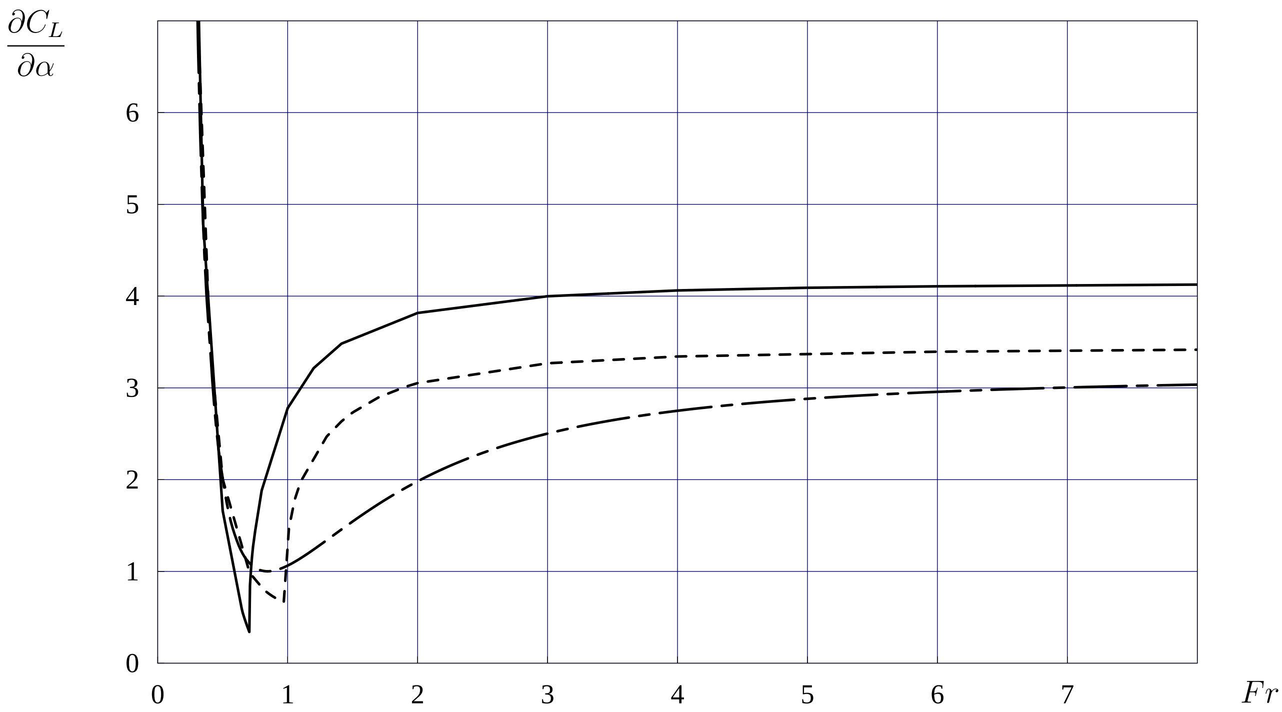

Figure 8: Derivative of the lift coefficient with respect to α for the planing hydrofoil $y(x) = 2d(1-x^2)+\alpha x$ versus Froude number Fr. Depth $h = 1$ (solid line), $h = 2$ (dashed line) and $h \to \infty$ (chain-dotted line).

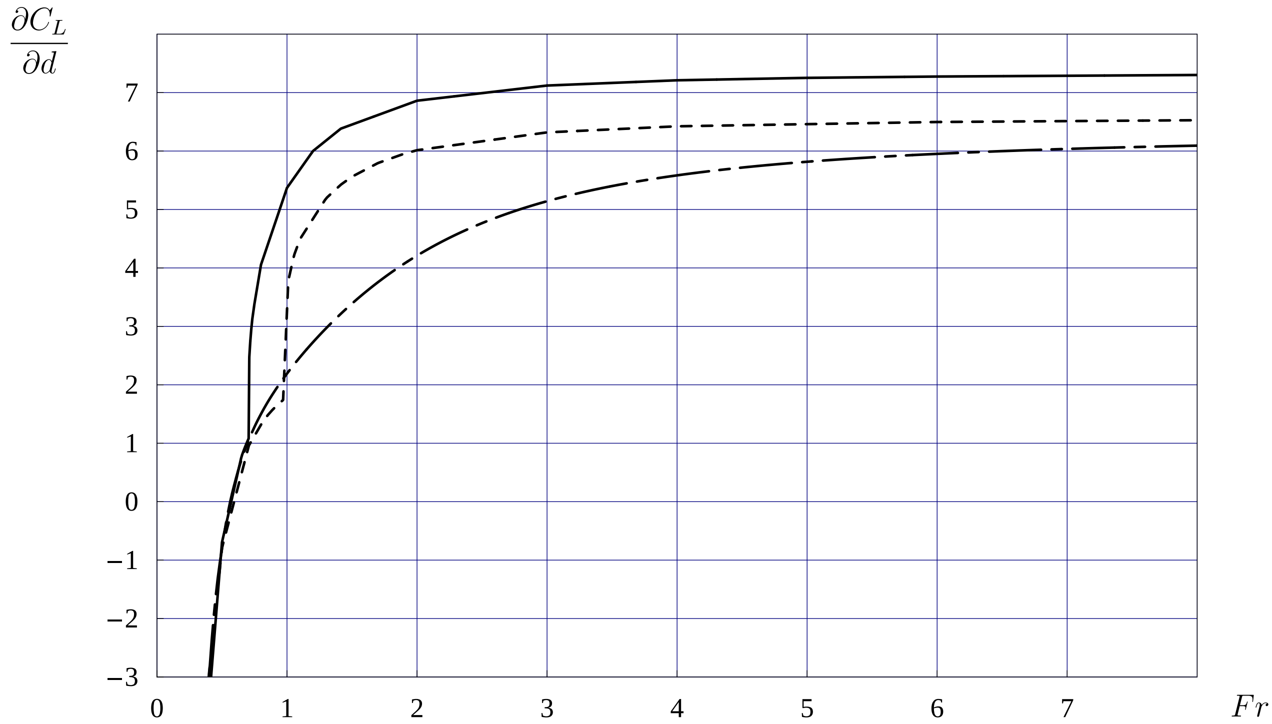

Figure 9: Derivative of the lift coefficient with respect to d for the planing hydrofoil $y(x) = 2d(1-x^2)+\alpha x$ versus Froude number Fr. Depth $h = 1$ (solid line), $h = 2$ (dashed line) and $h = \infty$ (chain-dotted line).

6 CONCLUSIONS

Integral equation and some corresponding numerical results has been derived for the lift coefficient, pressure distribution and free surface shape of the finite-depth planing hydrofoil of an arbitrary topography under gravity for a wide range of the flow parameters.

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