The Comparison of two constructions of the Refined Analytic Torsion on Manifolds with Boundary

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Outline

1. Analytic torsion
2. Refined analytic torsion
3. Comparison theorem for refined analytic torsions

Flat bundle

- (M, g^M) a closed Riemannian manifold, $\dim(M) = m$.
- (E, ∇, h^E) a flat complex vector bundle over M, i.e. $\nabla^2 = 0$.
- In general $dh^E(u,v) = h^E(\nabla u, v) + h^E(u, \nabla' v)$, $\forall u, v \in C^\infty(M, E)$.
- ∇' is called the dual connection on E
- If ∇ Hermitian (i.e. h^E flat), then $\nabla' = \nabla$.

Fact

- (E, ∇) is flat $\iff \exists\, \rho : \pi_1(M) \to GL(n, \mathbb{C})$ s.t. $E = \widetilde{M} \times_\rho \mathbb{C}^n$, where $\widetilde{M}$ is the universal covering of M.
- (E, ∇) is flat and h^E is flat $\iff \exists\, \rho : \pi_1(M) \to U(n)$ s.t. $E = \widetilde{M} \times_\rho \mathbb{C}^n$.

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Hodge theory

- de Rham complex:

$$0 \to \Omega^0(M,E) \xrightarrow{\nabla} \Omega^1(M,E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m(M,E) \to 0$$

- de Rham theorem:

$$H^p(M,E) \cong H^p_{dR}(M,E) = \frac{\mathrm{Ker}(\nabla|_{\Omega^p(M,E)})}{\mathrm{Im}(\nabla|_{\Omega^{p-1}(M,E)})}$$

- Hodge Laplacian:

$$\Delta_p = \nabla\nabla^* + \nabla^*\nabla : \Omega^p(M,E) \to \Omega^p(M,E),$$

where ∇^* is the adjoint of ∇ w.r.t. $< \cdot, \cdot >$ on $\Omega^\bullet(M,E)$ induced from g^M and h^E.

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ζ-regularized determinant

- For $s \in \mathbb{C}$, $\operatorname{Re} s > m/2$, the ζ-function

$$\zeta_{\Delta_p}(s) := \operatorname{Tr}(\Delta_p)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}[\exp(-t\Delta_p) - \dim \operatorname{Ker} \Delta_p] dt$$

converges. Moreover, it has a meromorphic continuation to $\mathbb{C}$. In particular, it is regular at $s = 0$.

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$$\operatorname{Det} \Delta_p := \exp(-\zeta'_{\Delta_p}(0)).$$

- Formally,

$$\operatorname{Det} \Delta_p = \text{''} \prod_{\lambda_k > 0} \lambda_k \text{''}$$

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Determinant line

- V: n-dim. vector space, $\det V := \wedge^n V$ complex line.
- volume element: $[v] = v_1 \wedge \cdots \wedge v_n \in \det V$,
  where $\{v_i\}$ orthornormal basis for V.
- determinant line of cohomology groups:

$$\det H^\bullet(M,E) = \otimes_p \left(\det H^p(M,E)\right)^{(-1)^p},$$

- For $[h_i] \in \det H^i(M,E)$,

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- Scalar Ray-Singer torsion

$$T(M, g^M, h^E) := \exp\left(\frac{1}{2}\sum_{p=0}^{m}(-1)^{p+1}\cdot p\cdot \mathrm{Det}\,\Delta_p\right).$$

- Ray-Singer torsion

$$\rho^{\mathrm{RS}}(\nabla) \; := \; \rho(\nabla, g^M)\cdot T(M, g^M, h^E) \in \det H^\bullet(M, E).$$

- Ray-Singer metric $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(M,E)}$ on $\det H^\bullet(M, E)$

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Cheeger-Müller theorem

- If $\dim M$ odd, $\|\cdot\|^{RS}_{\det H^\bullet(M,E)}$ does not depend on g^M, h^E a topological invariant.
- If $\dim M$ even, M orientable, h^E flat, then $T(M, g^M, h^E) = 1$.
- If $\dim M$ even, h^E unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$) , then $\|\cdot\|^{RS}_{\det H^\bullet(M,E)}$ does not depend on g^M, a topological invariant.
- Ray-Singer conjecture:The Ray-Singer torsion coincides with the Reidemeister torsion.
- h^E flat, Cheeger(1978), Müller(1978), RS conj. holds
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$\partial M \neq \phi$

- Impose relative and absolute boundary conditions for Δ.
- h^E flat, g^M product structure near ∂M: Lott-Rothenberg(1978), Lück(1993), Vishik(1995), Hassell(1998)
- h^E flat, but without assuming product structure near ∂M: Dai-Fang(2000)
- Most general case: Brüning-Ma(2006)
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Odd signature operator

- (M^m, g^M) a closed oriented Riemannian manifold, $m = 2r - 1$.
- (E, ∇, h^E) a complex flat vector bundle over M.
- Define the Chirality operator by

$$\Gamma := i^r(-1)^{\frac{k(k+1)}{2}} * : \Omega^k(M,E) \to \Omega^{m-k}(M,E),$$

  where $*$ is the Hodge star operator. Then $\Gamma^2 = \mathrm{Id}$,
- In general $\nabla^* = \Gamma\nabla'\Gamma$. If ∇ Hermitian, then $\nabla^* = \Gamma\nabla\Gamma$.
- The odd signature operator

$$\mathcal{B} := \Gamma\nabla + \nabla\Gamma : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$$

  not necessarily self-adjoint.
- Note that, if ∇ Hermitian, $\mathcal{B}^2 = (\Gamma\nabla + \nabla\Gamma)^2 = \Delta$.

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- The odd signature operator

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- (M^m, g^M) a closed oriented Riemannian manifold, $m = 2r - 1$.
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Graded determinant of $\mathcal{B}_{\text{even}}$

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The graded determinant of $\mathcal{B}_{\text{even}}$ is defined as

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η-invariant

- The η-function of $\mathcal{B}_{\text{even}}$ is defined as

$$\eta(s, \mathcal{B}_{\text{even}}) = \sum_{\text{Re}\,\lambda > 0} \lambda^{-s} - \sum_{\text{Re}\,\lambda < 0} (-\lambda)^{-s}.$$

- $\eta(s, \mathcal{B}_{\text{even}})$ holomorphic for $\text{Re}\, s$ large and admits a meromorphic extension to $\mathbb{C}$. In particular, $s = 0$ is a regular point.
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- Denote by $\eta_{\text{trivial}}(g^M)$ the η-invariant for $M \times \mathbb{C}$

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Relation with η-invariant

Proposition

- If
$$\xi = \frac{1}{2}\sum_{p=0}^{m}(-1)^{p+1}\cdot p\cdot \log \operatorname{Det} \mathcal{B}^2|_{\Omega^p(M,E)}$$
then
$$\operatorname{Det}_{gr}(\mathcal{B}_{\text{even}}) = e^{\xi - i\pi(\eta(\mathcal{B}_{\text{even}})+\cdots)}.$$
- In particular, if ∇ is acyclic (i.e. $H^\bullet(M,E)=0$) and Hermitian, then
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Refined analytic torsion

- Volume element:

$$\begin{aligned}\rho_\Gamma(\nabla, g^M) &= (-1)^R \cdot [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_{r-1}]^{(-1)^{r-1}} \\ &\otimes [\Gamma h_{r-1}]^{(-1)^r} \otimes [\Gamma h_{r-2}]^{(-1)^{r-1}} \otimes \cdots \otimes [\Gamma h_0]^{(-1)}\end{aligned}$$

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$$\rho_{an}(\nabla) := \rho_\Gamma(\nabla, g^M) \cdot \mathrm{Det}_{gr}(\mathcal{B}_{even}) \cdot e^{i\pi \cdot \mathrm{rk}\, E \cdot \eta_{trivial}(g^M)},$$

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Some properties of refined analytic torsion

- (Braverman-Kappeler 2007): The refined analytic torsion is closely related to the Farber-Turaev torsion, a refinement of the Reidemeister torsion.
- If E acyclic and ∇ Hermitian, then $|\rho_{\text{an}}(\nabla)| = \rho^{\text{RS}}(\nabla)$ and $\text{Ph}(\rho_{\text{an}}(\nabla)) = -\pi\rho(\nabla)$, where $\rho(\nabla) = \eta(\mathcal{B}_{\text{even}}) - \operatorname{rank} E \cdot \eta_{\text{trivial}}(g^M)$.
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- When $\partial M \neq \phi$, Vertman(2009) and Huang-Lee(2010) in two different independent constructions.
- Braverman-Vertman (2015): A gluing formula for refined analytic torsion on connected components of the complex representation space which contain a unitary point.
- The essential ingredient in the definition of the refined analytic torsion is the twisted de Rham complex with a chirality operator and the odd signature operator associated to the complex.
- Roughly speaking, Vertman considers

$$\Omega^{\bullet}_{\mathrm{rel}}(M,E) \oplus \Omega^{\bullet}_{\mathrm{abs}}(M,E), \quad \widetilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}, \quad \widetilde{\mathcal{B}}_{\mathrm{even}} := \begin{pmatrix} 0 & \mathcal{B}_{\mathrm{even}} \\ \mathcal{B}_{\mathrm{even}} & 0 \end{pmatrix}$$

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Vertman's approach

Lemma

$\mathrm{Spec}(\widetilde{\mathcal{B}}_{\mathrm{even,rel/abs}})$ is symmetric w.r.t. 0. Hence $\eta\left(\widetilde{\mathcal{B}}_{\mathrm{even,rel/abs}}\right) = 0$.

Proof.

$$\begin{pmatrix} 0 & \mathcal{B}_{\mathrm{even}} \\ \mathcal{B}_{\mathrm{even}} & 0 \end{pmatrix} \begin{pmatrix} \phi_{\mathrm{rel}} \\ \psi_{\mathrm{abs}} \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{\mathrm{even}}\psi_{\mathrm{abs}} \\ \mathcal{B}_{\mathrm{even}}\phi_{\mathrm{rel}} \end{pmatrix} = \lambda \begin{pmatrix} \phi_{\mathrm{rel}} \\ \psi_{\mathrm{abs}} \end{pmatrix}.$$

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Boundary conditions for $\mathcal{B}$

- Now assume $\partial M = Y \neq \phi$ and g^M is a product metric near Y.
- Trivialize E along the normal direction near Y by using ∇.
- Assume ∇ is Hermitian.
- For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near Y,

$$\phi = \nabla^Y \phi_1 + \phi_2 + du \wedge \big((\nabla^Y)^* \psi_1 + \psi_2\big),$$

  where $\phi_2, \psi_2 \in \mathrm{Ker}\,\Delta_Y$.
- We define

$$\mathcal{K} = \{\phi_2 \,|\, \nabla\phi = \Gamma\nabla\Gamma\phi = 0\}, \quad \Gamma^Y\mathcal{K} = \{\psi_2 \,|\, \nabla\phi = \Gamma\nabla\Gamma\phi = 0\}.$$

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$$\mathcal{K} \cong \mathrm{Im}(\iota^* : H^\bullet(M, E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M$$

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  where $\phi_2, \psi_2 \in \operatorname{Ker} \Delta_Y$.
- We define

$$\mathcal{K} = \{\phi_2 \,|\, \nabla\phi = \Gamma\nabla\Gamma\phi = 0\}, \quad \Gamma^Y\mathcal{K} = \{\psi_2 \,|\, \nabla\phi = \Gamma\nabla\Gamma\phi = 0\}.$$

- Then

$$\mathcal{K} \cong \operatorname{Im}(\iota^* : H^\bullet(M,E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M$$

  and

$$H^\bullet(Y, E|_Y) \cong \operatorname{Ker} \Delta_Y = \mathcal{K} \oplus \Gamma^Y\mathcal{K}.$$

Projections $\mathcal{P}_-$ and $\mathcal{P}_+$

- Hodge decomposition:

$$\Omega^\bullet(Y, E|_Y) = \operatorname{Im}\nabla^Y \oplus \operatorname{Im}(\nabla^Y)^* \oplus \mathcal{K} \oplus \Gamma^Y\mathcal{K}.$$

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$$\operatorname{Im}\mathcal{P}_- = \begin{pmatrix} \operatorname{Im}\nabla^Y \oplus \mathcal{K} \\ \operatorname{Im}\nabla^Y \oplus \mathcal{K} \end{pmatrix}, \qquad \operatorname{Im}\mathcal{P}_+ = \begin{pmatrix} \operatorname{Im}(\nabla^Y)^* \oplus \Gamma^Y\mathcal{K} \\ \operatorname{Im}(\nabla^Y)^* \oplus \Gamma^Y\mathcal{K} \end{pmatrix}$$

- Define the realization $\mathcal{B}_{\mathcal{P}_-}$ by $\mathcal{B}$ with domain

$$\operatorname{Dom}(\mathcal{B}_{\mathcal{P}_-}) = \{\psi \in \Omega^\bullet(M, E) | \mathcal{P}_-(\psi|_Y) = 0\},$$

and similarly, for $\mathcal{B}_{\mathcal{P}_+}$.

- Note that

$$\Gamma^Y\mathcal{P}_-\Gamma^Y = \mathcal{P}_+, \quad \Gamma\mathcal{B}_{\mathcal{P}_-}\Gamma = \mathcal{B}_{\mathcal{P}_+}.$$

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A cochain complex with a chirality operator Γ

- Cochain complexes $\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_{0/1}}(M,E),\nabla,\Gamma\right)$:

$$0 \to \Omega^{0}_{\mathcal{P}_{\mp}}(M,E) \xrightarrow{\nabla} \Omega^{1}_{\mathcal{P}_{\pm}}(M,E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{m}_{\mathcal{P}_{\pm}}(M,E) \to 0,$$

  where

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Proposition

- $H^{q}_{\mathcal{P}_{-}}(M,E) := H^{q}(\Omega^{\bullet}_{\mathcal{P}_{-}}(M,E),\nabla) \cong \operatorname{Ker}\mathcal{B}^{2}_{q,\mathcal{P}_{-}} = \operatorname{Ker}\mathcal{B}^{2}_{q,\mathrm{rel}} \cong H^{q}(M,Y,E)$,
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Refined analytic torsion on manifolds with boundary

Definition & Theorem (— Y. Lee)

Under above assumptions. The refined analytic torsion defined by

$$\rho_{\mathrm{an},\mathcal{P}_-}(\nabla) := \rho_{\Gamma,\widetilde{\mathcal{P}}_0}(\nabla, g^M) \cdot \mathrm{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathcal{P}_-}) \cdot e^{i\pi\cdot \mathrm{rk}\, E\cdot \eta_{\mathrm{trivial},\mathcal{P}_-}(g^M)},$$

is independent of the choice of g^M in the interior of M.

- If ∇ is acyclic and Hermitian, then

$$\log \mathrm{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathcal{P}_-}) + \log \mathrm{Det}_{\mathrm{gr}}(-\mathcal{B}_{\mathrm{even},\mathcal{P}_+})$$

$$= \left(\log \rho^{\mathrm{RS}}_{\mathrm{rel}}(\nabla) + \log \rho^{\mathrm{RS}}_{\mathrm{abs}}(\nabla)\right) - i\pi \left(\eta\left(\mathcal{B}_{\mathrm{even},\mathcal{P}_-}\right) - \eta\left(\mathcal{B}_{\mathrm{even},\mathcal{P}_+}\right)\right)$$

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Comparison theorem for refined analytic torsions

- $\widehat{\rho}_{\mathrm{an},\mathcal{P}_+}(\nabla)$ refined analytic torsion defined by $-\Gamma$ instead of Γ.
- The fusion isomorphism

$$\mu : \det H^{\bullet}_{\widetilde{\mathcal{P}}_0}(M,E) \otimes \det H^{\bullet}_{\widetilde{\mathcal{P}}_1}(M,E) \to \det(H^{\bullet}_{\mathrm{rel}}(M,E) \oplus H^{\bullet}_{\mathrm{abs}}(M,E))$$

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Thank you!