Tensor invariants of Legendrean contact geometry

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Adelaide Differential Geometry Seminar
September 24, 2021

Contact geometry

Definition (Contact manifold)

Smooth manifold M of dimension $2n+1$ equipped with $H \hookrightarrow TM$ such that $H = \ker \alpha$, where $\alpha \in \Gamma(\Lambda^1)$ with $\alpha \wedge d\alpha \wedge \ldots \wedge d\alpha$ non-vanishing.

- α is called a contact form. Can rescale $\implies$ Contact forms are the sections of a line bundle L.
- $d\alpha|_H$ non-vanishing skew-form on H $\implies$ H is non-integrable, $[H, H] = TM$.
- $d\alpha|_H$ is non-degenerate skew-form on H.
- Darboux theorem: There are no local invariants. $\exists$ local coordinates (t, p_i, q_i) with

$$\alpha = dt - p_i dq^i$$

Contact geometry

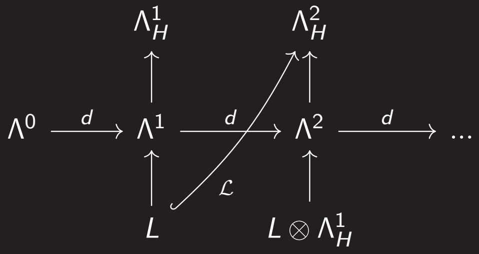
Define $\Lambda^k_H := \Lambda^k H^*$ and define $\Lambda^2_{H\perp}$ to be elements of Λ^2_H trace free with respect to $d\alpha|_H$.

In *abstract index notation* [PR84], we will write J_{ab} for the skew-form $d\alpha|_H$ and use this and its inverse J^{ab} to raise and lower indices.

e.g.

- Given $\omega_a \in \Lambda^1_H$, we define $\omega^a := J^{ab}\omega_b$.
- $\mu^a \nu_a$ is the natural pairing between μ^a and ν_b.

Contact geometry



We need the Rumin operator $d_\perp : \Lambda^1_H \to \Lambda^2_{H\perp}$:
Given $\omega \in \Lambda^1_H$, define $d_\perp \omega := (d\tilde{\omega}|_H)_\perp$ where $\tilde{\omega}$ is a lift of ω.
See more in [Rum90].

Legendrean contact geometry

Definition (Legendrean contact manifold)

Contact manifold with a decomposition $H = P \oplus V$ where P, V are of rank n, integrable and $d\alpha|_H$ vanishes on P, V.

- $d\alpha|_H$ is a perfect pairing between P and V.
- Canonical example:
  $F_{1,n+1}(\mathbb{R}^{n+2}) = \{\text{lines inside hyperplanes in } \mathbb{R}^{n+2}\}$.
  P consisting of velocities fixing the hyperplane,
  V consisting of velocities fixing the line.
- c.f. CR structures of hypersurface type.
  $\implies \omega_{\bar{\alpha}} \in P^*$ and $\mu_\alpha \in V^*$, while $J_{\bar{\alpha}\alpha} \in P^* \otimes V^*$.

Tensor invariants

"Fundamental gadgets" of Riemannian geometry?
(M, g) comes equipped with:

- TM - canonical vector bundle
- ∇^{LC} - canonical connection
- $R_{ab}{}^{c}{}_{d}$ - determines local flatness

Tensor invariants

Question

Can we find a canonical vector bundle $\mathbb{T}$ with canonical connection ∇ on $(M, H = P \oplus V)$ such that the geometry is locally isomorphic to the canonical 'flat' model $F_{1,2n-1}(\mathbb{R}^{2n})$ if and only if ∇ is flat?

- c.f. (M, g) with $\mathbb{T} = TM$ and $\nabla = \nabla^{LC}$
- Assume $n = 2$
- Isomorphism $\leftrightarrow$ a diffeomorphism which preserves H and the splitting $H = P \oplus V$

Tensor invariants

Issue

No canonical connection on H or TM
$\implies$ *Need to construct another vector bundle*

Partial connection

Let E be a vector bundle over a contact manifold.

Definition (Partial connection)

A partial connection is a map $\nabla : \Gamma(E) \to \Gamma(\Lambda^1_H \otimes E)$ *with:*

$$\nabla(fs) = df|_H \otimes s + f\nabla s$$

for a smooth function f *and section* s*.*

Recall the Rumin operator $d : \Lambda^1_H \to \Lambda^2_{H\perp}$.

Definition (Partial torsion)

Given an affine partial connection $\nabla : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$*, the partial torsion of* ∇ *is the homomorphism* $\Lambda^1_H \to \Lambda^2_{H\perp}$ *defined as* $d_\perp - \wedge_\perp \circ \nabla$*, where* $\wedge_\perp$ *is the projection* $\Lambda^1_H \otimes \Lambda^1_H \to \Lambda^2_{H\perp}$*.*

Partial connection

Definition (Partial curvature)

Given a partial connection $\nabla : E \to \Lambda^1_H \otimes E$ *for a vector bundle* $E \to M$ *define the partial curvature* $R_{ab}{}^{\mu}{}_{\nu} \in \Lambda^2_{H\perp} \otimes \mathrm{End}(E)$ *by*

$$R_{ab}{}^{\mu}{}_{\nu} s^{\nu} := \nabla_{[a} \nabla_{b]} s^{\mu} - \frac{1}{2n} J_{ab} \nabla_c \nabla^c s^{\mu}$$

Here we are using an auxiliary partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ *with vanishing partial torsion in order the above definition make sense.*

Partial connections

How do partial connections on contact manifolds relate to connections?

Theorem (Eastwood-Gover [EG11])

Let $\nabla : E \to \Lambda^1_H \otimes E$ be a partial connection with respect to a contact distribution $H \hookrightarrow TM$.
There exists a unique connection $\tilde{\nabla} : E \to \Lambda^1 \otimes E$ extending ∇ such that the 2-form part of $\tilde{\kappa}|_H$ is trace-free, where $\tilde{\kappa}$ is the curvature of $\tilde{\nabla}$.

Idea of proof

The trace part of $\tilde{\kappa}$ is contained in $L \otimes End(E)$ which is precisely the freedom in choosing $\tilde{\nabla}$ extending.

$$\underbrace{\alpha \otimes A}_{\text{Change in connection}} \quad \mapsto \quad \underbrace{d\alpha|_H \otimes A}_{\text{Change in } \kappa|_H}$$

Partial connections

Partial curvature is contained in a lower rank bundle than curvature so is in some sense is more efficient:

Theorem ([Moy21])

Let $\nabla : E \to \Lambda^1_H \otimes E$ be a partial connection with respect to a contact distribution $H \hookrightarrow TM$. Suppose that the partial curvature of ∇ vanishes, then the canonical extension from [EG11] is flat.

Proof

$$
\begin{array}{ccccc}
 & & \Lambda^2_H \otimes \mathrm{End}(E) & & \Lambda^3_H \otimes \mathrm{End}(E) \\
 & & \uparrow & & \uparrow \\
\cdots & \xrightarrow{d^{\tilde{\nabla}}} & \Lambda^2 \otimes \mathrm{End}(E) & \xrightarrow{d^{\tilde{\nabla}}} & \Lambda^3 \otimes \mathrm{End}(E) \\
 & & \uparrow & & \uparrow \\
 & & L \otimes \Lambda^1_H \otimes \mathrm{End}(E) & & L \otimes \Lambda^2_H \otimes \mathrm{End}(E)
\end{array}
$$

- *Check that the composition above giving* $L \otimes \Lambda^1_H \otimes \mathrm{End}(E) \to \Lambda^3_H \otimes \mathrm{End}(E)$ *is injective.*
- *That the partial curvature vanishes means the canonical extension has curvature in* $L \otimes \Lambda^1_H \otimes \mathrm{End}(E)$.
- *The Bianchi identity ensures the curvature is in the kernel of* $L \otimes \Lambda^1_H \otimes \mathrm{End}(E) \to \Lambda^3_H \otimes \mathrm{End}(E)$. □

A non-canonical connection

Recall for a Legendrean contact structure $d\alpha|_H$ gives a perfect pairing between P and V where $H = P^* \oplus V^*$. So a partial connection $V^* \to \Lambda^1_H \otimes V^*$ induces a partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$.
For a Legendrean contact structure we can write a partial connection $V^* \to \Lambda^1_H \otimes V^*$ like:

$$(\bar{\nabla}_{\bar{\alpha}}, \nabla_\alpha)\omega_\beta = (\bar{\nabla}_{\bar{\alpha}}\omega_\beta, \nabla_\alpha\omega_\beta)$$

Theorem ([Moy21])

Given a choice of contact form $\alpha \in L$ there is a unique partial connection $V^* \to \Lambda^1_H \otimes V^*$ such that the induced affine partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ has vanishing partial torsion.

A non-canonical connection

Idea of proof

The freedom in choosing a partial connection $V^* \to \Lambda^1_H \otimes V^*$ *is the bundle:*

$$\Lambda^1_H \otimes End(V^*) \qquad rank: \; 4 \times 2 \times 2 = 16$$

A priori the partial torsion lies in the bundle

$$Hom(\Lambda^1_H \otimes \Lambda^2_{H\perp}) \qquad rank: \; 4 \times 5 = 20$$

but integrability of P, V *further reduces the rank of the bundle in which the torsion lies to* 16.

A non-canonical connection

Writing $\Upsilon_\alpha = \nabla_\alpha \log \Omega$ and $\bar{\Upsilon}_{\bar{\alpha}} = \bar{\nabla}_{\bar{\alpha}} \log \Omega$, we $\hat{\alpha} = \Omega\alpha$ were get *change of connection* formulae for sections of V^*:

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_\beta, \hat{\nabla}_\alpha\omega_\beta) = (\bar{\nabla}_{\bar{\alpha}}\omega_\beta + J_{\bar{\alpha}\beta}\bar{\Upsilon}_{\bar{\gamma}}\omega^{\bar{\gamma}}, \nabla_\alpha\omega_\beta - 2\Upsilon_{(\alpha}\omega_{\beta)}),$$

for sections of P^*

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_{\bar{\beta}}, \hat{\nabla}_\alpha\omega_{\bar{\beta}}) = (\bar{\nabla}_{\bar{\alpha}}\omega_{\bar{\beta}} - 2\bar{\Upsilon}_{(\bar{\alpha}}\omega_{\bar{\beta})}, \nabla_\alpha\omega_{\bar{\beta}} + J_{\bar{\beta}\alpha}\Upsilon_\gamma\omega^\gamma)$$

for sections of L we have

$$(\hat{\bar{\nabla}}_{\bar{\alpha}} f, \hat{\nabla}_\alpha f) = (\bar{\nabla}_{\bar{\alpha}} f - \bar{\Upsilon}_{\bar{\alpha}} f, \nabla_\alpha f - \Upsilon_\alpha f)$$

and for sections of $\Lambda^2 P^*$,

$$(\hat{\bar{\nabla}}_{\bar{\alpha}} f, \hat{\nabla}_\alpha f) = (\bar{\nabla}_{\bar{\alpha}} f - 3\bar{\Upsilon}_{\bar{\alpha}} f, \nabla_\alpha f + \Upsilon_\alpha f).$$

An invariant differential operator

The point of all of this: Use the formulae to write down many differential operators *intrinsic* to the Legendrean contact structure. e.g.

$$\varepsilon\left(\frac{1}{4},-\frac{3}{4}\right) \to P^* \otimes \varepsilon\left(\frac{1}{4},-\frac{3}{4}\right) \quad \oplus \quad V^* \otimes V^* \otimes \varepsilon\left(\frac{1}{4},-\frac{3}{4}\right)$$

given by

$$f \mapsto \left(\bar{\nabla}_{\bar{\alpha}} f, \nabla_\alpha \nabla_\beta f - 2Z_{\alpha\beta} f\right).$$

$\varepsilon\left(p,q\right) = (\Lambda^2 P^*)^p \otimes L^q$ and where $Z_{\alpha\beta}$ is a particular component of the partial curvature.

An invariant differential operator

Definition (Jet bundle)

Given a vector bundle $\pi : E \to M$ define $J^r(E) \to M$ to be the rth-jet bundle of E, that is the bundle of r-jets of local sections of E. An r-jet at $x \in M$ is an equivalence class of local sections where $\sigma \sim \tau \iff \sigma$ and τ have partial derivatives up to order r at x. that agree in some chart.

Now consider the invariantl, linear, homogeneous *PDE*:

$$\left(\bar{\nabla}_{\bar{\alpha}} f, \nabla_\alpha \nabla_\beta f - 2Z_{\alpha\beta} f\right) = 0$$

This is a *linear equation* in fibres of the 2*nd*-jet bundle and so an invariantly defined subbundle $\mathbb{T} \leq J^2\left(\varepsilon\left(\frac{1}{4}, -\frac{3}{4}\right)\right)$.

An invariant differential operator

The next step is *prolongation*. f being a solution is equivalent to:

$$\bar{\nabla}_{\bar{\alpha}} f = 0$$
$$\nabla_{\alpha} f = \phi_{\alpha}$$

for $\phi_{\beta} \in V^* \otimes (\Lambda^2 P^*)^{1/4} \otimes L^{-3/4}$ satisfying some conditions:

$$\bar{\nabla}_{\bar{\alpha}} \phi_{\beta} = J_{\bar{\alpha}\beta} g - \frac{1}{2} Y_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} f$$
$$\nabla_{\alpha} \phi_{\beta} = 2 Z_{\alpha\beta} f$$

forg $\in (\Lambda^2 P^*)^{1/4} \otimes L^{1/4}$ satisfying some conditions:

$$\bar{\nabla}_{\bar{\alpha}} g = 2 X_{\bar{\alpha}}{}^{\beta} \phi_{\beta} + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f$$
$$\nabla_{\alpha} g = \frac{2}{3} \nabla_{\bar{\beta}} Z_{\alpha}{}^{\bar{\beta}} f - \frac{4}{3} Y_{\bar{\beta}\alpha}{}^{\bar{\beta}\gamma} \phi_{\gamma} + \frac{1}{6} Y_{\bar{\beta}\alpha} \phi^{\bar{\beta}} - \frac{1}{6} \nabla^{\bar{\beta}} Y_{\bar{\beta}\alpha} .$$

A canonical connection

The above invariant first order differential equations can be repackaged into an invariant connection. Directional derivative in P directions:

$$\bar{\nabla}_{\bar{\alpha}} \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \bar{\nabla}_{\bar{\alpha}} f \\ \bar{\nabla}_{\bar{\alpha}} \phi_\beta - J_{\bar{\alpha}\beta} g + \frac{1}{2} Y_{\bar{\alpha}\beta} f \\ \bar{\nabla}_{\bar{\alpha}} g - 2P_{\bar{\alpha}}{}^{\beta} \phi_\beta + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f \end{bmatrix} \tag{1}$$

Directional derivative in V directions:

$$\nabla_\alpha \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \nabla_\alpha f - \phi_\alpha \\ \nabla_\alpha \phi_\beta - K_{\alpha\beta} f \\ \Big\{ \nabla_\alpha g - \frac{1}{3} \bar{\nabla}_{\bar{\beta}} K_\alpha{}^{\bar{\beta}} f + \frac{4}{3} Y_{\bar{\beta}\alpha}{}^{\bar{\beta}\gamma} \phi_\gamma \\ + \frac{1}{6} (\nabla^{\bar{\gamma}} Y_{\bar{\gamma}\alpha}) f - \frac{1}{6} Y_{\bar{\gamma}\alpha} \phi^{\bar{\gamma}} \Big\} \end{bmatrix} \tag{2}$$

$K_{\alpha\beta}, P_{\bar{\alpha}\bar{\beta}}, Y_{\bar{\alpha}\beta}{}^{\bar{\gamma}\nu}, Y_{\bar{\alpha}\beta}$ are parts of the partial curvature.

Constructing the isomorphism

Theorem

Let $(M, H = P \oplus V)$ *be a 5-dimensional Legendrean contact structure.* $(M, H = P \oplus V)$ *is locally isomorphic to* $F_{1,3}(\mathbb{R}^4)$ *as a Legendrean contact structure if and only if the partial curvature of the invariant partial connection given by (1) and (2) vanishes.*

The main inspiration for the proof is the proof in the case of 2-dimensional projective structures [Eas17].

Constructing the isomorphism

Idea of proof

1. *Take some point* $x_0 \in U$ *and identify* $\mathbb{T}_{x_0}$ *with* $\mathbb{R}^4$*.*
2. *Map* $x \in U$ *to the flag* (L_x, U_x) *where* L_x *is the parallel translate of*

$$\begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$$

to the fibre above x_0 *of the subspace and* U_x *is the parallel translate to the fibre above* x_0 *of the subspace*

$$\begin{bmatrix} 0 \\ * \\ * \end{bmatrix}$$

3. *The special structure of the tractor connection ensures this is a diffeomorphism and preserves* $H = P \oplus V$*.*

Corollary

The Legendrean contact structure is flat if and only if

$$\bar{\bar{\nabla}}_{\bar{\alpha}} f = 0,$$
$$\nabla_\alpha \nabla_\beta f - 2Z_{\alpha\beta} f = 0$$

has a (locally) 4-dimensional solution space.

Context

This was an example of a tractor bundle for a geometry modelled on the homogeneous space G/P.
Here

$$G = GL(4, \mathbb{R}), \qquad P = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\} \tag{3}$$

Other prolongation procedures:

- $(\nabla_a \nabla_b + P_{ab})_\circ \sigma = 0$ (Conformal to Einstein equation) $\rightsquigarrow$ Conformal tractors [BEG94]
- $\nabla_{(ABC}\phi_{D)} = 0 \rightsquigarrow G_2/P$ standard tractors (thesis)

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