Curvature, triangles, and maps

Thomas Leistner

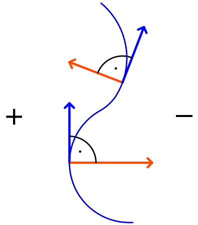
THE UNIVERSITY OF ADELAIDE AUSTRALIA

Undergraduate Seminar
University of Adelaide
September 8, 2010

How curved is a curve?

Consider a smooth curve in the plane

$$\mathbf{x} : \mathbb{R} \ni t \;\mapsto\; \mathbf{x}_t \in \mathbb{R}^2$$



with

- velocity vector $\mathbf{v}_t = \dot{\mathbf{x}}_t = \frac{d\mathbf{x}}{dt}$ and
- acceleration vector $\mathbf{a}_t = \dot{\mathbf{v}}_t = \frac{d^2\mathbf{x}}{dt^2}$.

Assume that the curve has constant speed, i.e. the length of the velocity vector is constant $\|\mathbf{v}_t\| := \sqrt{r_t^2 + s_t^2} \equiv 1$, if $\mathbf{v}_t = (r_t, s_t)$.
This means that the acceleration vector has no component in direction of the curve, only orthogonal to it.
It's lengths measures how curved the curve is, this is the curvature:

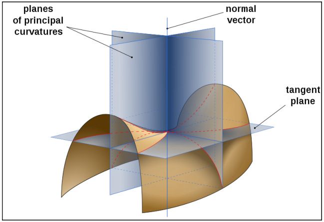
$$K_t = \pm\|\mathbf{a}_t\|$$

(We take it to be $+$ or minus – depending on whether the curve is bent to the left or right.)

How curved is a surface (in 3-dim space)?

At each point p of a surface S we have a plane tangent to S and a normal vector $\mathbf{n}$. For each tangent vector $\mathbf{v}$ of length 1 we consider the plane $P_{\mathbf{v}}$ that is spanned by the normal vector $\mathbf{n}$ and $\mathbf{v}$.

The intersection $P_{\mathbf{v}}$ with the surface S is a curve which has a certain curvature $K_{\mathbf{v}}$ at $t = 0$ with the sign depending on which side of the tangent plane the curve lies. It measures how curved the surface S is in direction of $\mathbf{v}$.



At each $p \in S$ we have a map

$$\begin{array}{rccc} K : & \text{circle in the tangent plane at point } p & \to & \mathbb{R} \\ & \mathbf{v} & \mapsto & K_{\mathbf{v}} \end{array}$$

Since the circle is compact, this map attains a

maximum K_1 and a minimum K_2.

These are the principal curvatures of the surface S at the point p.

Mean curvature and Gauss curvature of a surface

Given the two functions K_1 and K_2, we can form their mean and their product,

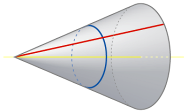
$$H := \frac{1}{2}(K_1 + K_2) \quad \text{and} \quad K := K_1 \cdot K_2$$

Mean curvature　　　　Gauss curvature

Examples

- Parts of a plane are flat, Gauss and mean curvature are zero.
- Cylinders and cones have Gauss curvature zero (since one principal curvature is zero).

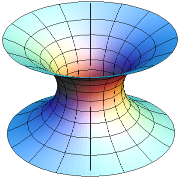
  The mean curvature of a cylinder of radius r is $1/2r$ and for the cone it decreases when moving away from the tip.

  
- Spheres of radius r have mean curvature $1/r$ and Gauss curvature $1/r^2$, because the great circles have curvature $1/r$.

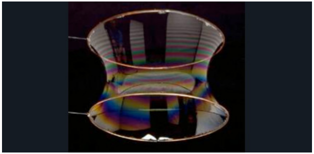
Minimal surfaces

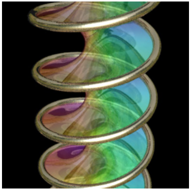
Minimal surfaces are surfaces of mean curvature zero.
[Sophie Germain, 1831, when studying elasticity]

The catenoid is an example. It is obtained by rotating the catenary $c(t) = r \cosh \frac{t}{r}$ around the z-axis.



Minimal surfaces minimise area and appear when soap is used:





But: Surfaces with the same mean curvature can be very different, *geometrically* (e.g. plane and catenoid).

Gauss curvature and *'intrinsic geometry'*

Theorem (Theorema egregium, C. F. Gauss, 1828)

If a surface S can be developed onto another surface $\hat{S}$, then their Gauss curvatures

$$K = K_1 \cdot K_2 \ \text{ and } \ \hat{K} = \hat{K}_1 \cdot \hat{K}_2$$

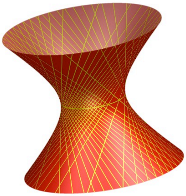
are the same.
Or, K does only depend on the intrinsic geometry of the surface.

What does this mean?

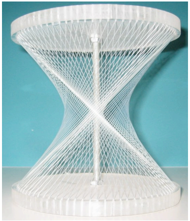
- developing $=$ bending without stretching
- intrinsic geometry: Two points on a surface S have a certain distance, defined by the length of the shortest curve in S joining them. If two surfaces can be mapped onto each other without changing the distances between points then hey have the same Gauss curvature.

Examples

- Cylinders and cones can be unrolled into the plane.
- Spheres of radius r have constant Gauss curvature $1/r^2$. They cannot be mapped onto the plane without distortion.
- The hyperboloid $H = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\right\}$ has negative Gauss curvature (depending on the z coordinate).



Note that the hyperboloid contains straight lines. However, it cannot be developed to the plane without stretching.



- Minimal surfaces have Gauss curvature ≤ 0.

Intrinsic geometry

- The intrinsic geometry of a surface is determined by the distance of two surface points when joined by a shortest curve in the surface. The length of a curve is given by its length as a curve in 3-dim space.
- We can consider this distance function regardless of being induced by the 3-dim ambient space. This is the intrinsic geometry.
- The Gauss curvature tells us when two surfaces have the same intrinsic geometry (e.g. plane and cylinder).
- Gauss' student B. Riemann ["Über die Hypothesen welche der Geometrie zugrunde liegen", 1868] generalised Gauss' idea of intrinsic geometry to arbitrary dimensions without using an external space $\rightsquigarrow$ Riemannian manifold.

How can we measure the intrinsic geometry without referring to an external space?

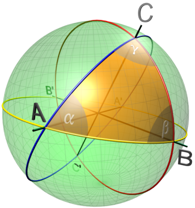
Triangles

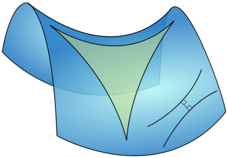
Theorem (Theorema elegantissimum, C. F. Gauss, 1828)

Let S be a (simply connected) surface with Gauss curvature K and let Δ be a geodesic *triangle in S with angles α, β and γ. Then*

$$\alpha + \beta + \gamma \; = \; \pi + \int_{\Delta} K \, dS.$$

It also holds $K(p) = \lim\limits_{\Delta \to p} \frac{\alpha+\beta+\gamma-\pi}{\mathrm{area}(\Delta)}$, with $p \in \Delta$.





E.g., for the sphere of radius one: $\alpha + \beta + \gamma - \pi = \mathrm{area}(\Delta)$ [T. Harriot, 1603].

Consequences

- Only by measuring distances and angles in triangles, without referring to a ‘surrounding space’ we are able to determine if we live on a sphere, a plane or something else. But we would not be able to decide whether we live on a cylinder or on the plane.
- This is particularly relevant in physics and cosmology as we have no knowledge of a ‘surrounding space’. But we are able to determine the geometry of the universe with the help of light rays. They run on shortest curves and thus determine the intrinsic geometry. For example, we could measure angles in cosmic triangles.
- Since the sphere has non zero Gauss curvature, there are no maps of the earth that show distances appropriately, i.e. without distortion.
  Which distortion is acceptable for which purpose is studied by map makers.

Map projections

Depending on the purpose parts of the the sphere can be mapped onto a part of the plane by

- preserving angle ('conformal map')
- preserving area

Both together is not possible:

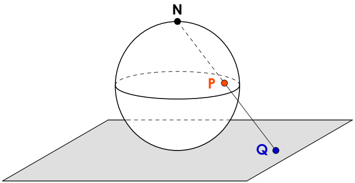
Proposition

A map between two surfaces preserves the distance of points if and only if it preserves the angles between curves and the area of pieces of the surface.

Conformal to the plane: Stereographic projection

- Fix a point N on the sphere (e.g. the north pole) and a plane tangent to the sphere at the opposite point (or through the equator).

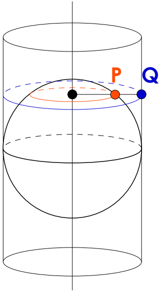
- Project each point $P \neq N$ on the sphere to the plane by intersecting the ray $\vec{NP}$ with the plane at Q



- This map is conformal (i.e., preserve angles)
- Meridians are mapped to straight lines and lines of the same latitude are mapped to circles (if projected from a pole).
- Regions closed to the pole are enlarged under this projection.

Area preserving to a cylinder: Archimedes projection

- Project from a line ℓ through two antipodes (e.g. the poles) to a cylinder with axis ℓ.

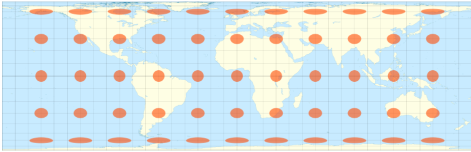


In formulae:

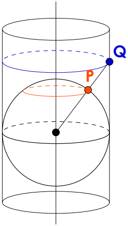
$$\begin{array}{ccccc} \textit{sphere} \setminus \textit{semi circle} & \to & \textit{cylinder} & \to & \textit{plane} \\ \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix} & \mapsto & \begin{pmatrix} \cos\lambda \\ \sin\lambda \\ \sin\phi \end{pmatrix} & \mapsto & \begin{pmatrix} \lambda \\ \sin\phi \end{pmatrix} \end{array}$$

$\lambda =$ longitude, $\phi =$ latidude.

- This map is area preserving.
- Meridians are mapped to parallel straight lines (not angle preserving).



Conformal to a cylinder: Mercator projection



- Project from the centre of the sphere to a cylinder ('gnomonic projection') followed by a logarithmic function in order to make it conformal.

$$\begin{array}{ccccccc}
\textit{sphere} \setminus s/c & \to & \textit{cylinder} & \to & \textit{plane} & \to & \textit{plane} \\
\begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix} & \mapsto & \begin{pmatrix} \cos\lambda \\ \sin\lambda \\ \tan\phi \end{pmatrix} & \mapsto & \begin{pmatrix} \lambda \\ \tan\phi \end{pmatrix} & \to & \begin{pmatrix} \lambda \\ \ln\tan(\frac{\pi}{4}+\frac{\phi}{2}) \end{pmatrix}
\end{array}$$

- The Mercator projection is conformal.
- Curves that have a constant angle with the meridians ('loxodromes') are mapped onto straight lines.
  This makes this projection useful for maritime navigation.