Special holonomy in Lorentzian geometry

Thomas Leistner

University of Adelaide

Workshop on New Currents in Geometry in Australia
Institute for Geometry and its Applications

Adelaide, November 16–18, 2009
Outline

1. Holonomy groups
   - The holonomy group of a linear connection
   - Classification problem and Berger algebras
   - Holonomy and geometric structure
   - Riemannian holonomy

2. Holonomy groups of Lorentzian manifolds
   - Special Lorentzian holonomy
   - Classification and Applications
   - Two ways of constructing metrics
   - Open problems
Holonomy group of a linear connection

Let $M$ be a smooth manifold and $\nabla$ a linear connection.

$\sim$ Parallel displacement along $\gamma : [0, 1] \to M$, piecewise smooth,

$$P_\gamma : T_{\gamma(0)}M \ni X_0 \sim \mapsto X(1) \in T_{\gamma(1)}M$$

where $X(t)$ is the solution to the ODE $\nabla_{\dot{\gamma}(t)}X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (Connected) Holonomy group

$$\text{Hol}_p^0(M, \nabla) := \left\{ P_\gamma \mid \gamma(0) = \gamma(1) = p, \gamma \sim \{p\} \right\} \subset \text{GL}(T_pM) \simeq \text{GL}(n, \mathbb{R})$$

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$.

- For $p, q \in M$:
  $$\text{Hol}_p(M, \nabla) \sim \text{Hol}_q(M, \nabla)$$

- If $\nabla = \text{LC}$ of a metric $g$ on $M$, then $\text{Hol}_p(M, g) \subset \text{O}(T_pM, g) = \text{O}(t, s)$. 

Conjugated in $\text{GL}(n, \mathbb{R})$.
Holonomy and curvature

- Recall that $\nabla$ and $\mathcal{P}_\gamma$ are related via
  \[
  \nabla_{\dot{\gamma}(0)}X|_p = \frac{d}{dt} \left[ \mathcal{P}^{-1}_\gamma|_{\gamma(t)} (X(\gamma(t))) \right]_{t=0}.
  \]
- This implies for the curvature $\mathcal{R}$ of $\nabla$: Let $X, Y \in T_p M$ and $\lambda_t$ the loop along the parallelogram at $p$ with sides $\sqrt{t}X$ and $\sqrt{t}Y$. Then
  \[
  \mathcal{R}(X, Y)|_p = \lim_{t \to 0} \frac{1}{t} \left( \mathcal{P}_{\lambda_t} - Id_{T_p M} \right).
  \]
  Hence, $\mathcal{R}(X, Y)|_p \in \mathfrak{hol}_p(M, \nabla)$ for all $X, Y \in T_p M$.
- One has to collect curvature all over $M$ to get all of $\mathfrak{hol}_p(M, \nabla)$:

**Theorem (Ambrose-Singer ’53)**

*If $M$ is connected, then $\mathfrak{hol}_p(M, \nabla)$ is spanned by

\[
\left\{ \mathcal{P}^{-1}_\gamma \circ \mathcal{R}(X, Y) \circ \mathcal{P}_\gamma \in \text{GL}(T_p M) \mid \gamma(0) = p \text{ and } X, Y \in T_{\gamma(1)} M \right\}
\]*
Classification: *Which groups occur as holonomy groups?*

- **Hano/Ozeki '56**: Any closed $G \subset \text{GL}(n, \mathbb{R})$! But $\nabla$ might have torsion.
- **Conditions on the torsion $T^\nabla$ of $\nabla$**, e.g. $T^\nabla = 0$ or $T^\nabla \in \Lambda^3 TM$
  - $\nabla$ algebraic constraints on the holonomy representation. *Why?*
  
  
  $$\mathcal{R}_\gamma := \mathcal{P}^{-1}_\gamma \circ \mathcal{R}\left(\mathcal{P}_\gamma(.), \mathcal{P}_\gamma(.)\right) \circ \mathcal{P}_\gamma \in \Lambda^2 (T^*_p M) \otimes \text{GL}(T_p M)$$

  Now, if $T^\nabla = 0$, then $\mathcal{R}$ and hence $\mathcal{R}_\gamma$ satisfy the Bianchi identity:

  $$\mathcal{R}_\gamma(X, Y)Z + \mathcal{R}_\gamma(Y, Z)X + \mathcal{R}_\gamma(Z, X)Y = 0.$$ 

  $\implies$ $\mathfrak{hol}_p(M, \nabla)$ is a Berger algebra: For $g \subset \mathfrak{gl}(n, \mathbb{R})$ define the $g$-module:

  $$\mathcal{K}(g) := \left\{ R \in \Lambda^2 \mathbb{R}^n^* \otimes g \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \right\}$$

  $g$ is a Berger algebra $\iff g = \text{span}\{R(x, y) \mid R \in \mathcal{K}(g), x, y \in \mathbb{R}^n\}.$

  $T^\nabla = 0$: Ambrose-Singer $\implies$ $\mathfrak{hol}_p(M, \nabla)$ is a Berger algebra.

Classification of **irreducible** Berger algebras

- $g \subset \mathfrak{so}(p, q)$ [Berger '55], $g \subset \mathfrak{gl}(n, \mathbb{R})$ [Schwachhöfer/Merkulov '99].
Let $V$ be a “geometric” vector bundle over $M$ and $\nabla$ a connection on $V$ “induced” by a connection $\nabla$ on $TM$ (e.g. tangent bundle, tensor bundles, spinor bundle). Then:

\[
\left\{ v \in V_p \mid Hol_p(M, \nabla)(v) = v \right\} \cong \left\{ \varphi \in \Gamma(V) \mid \nabla \varphi = 0 \right\}
\]

$v \mapsto \varphi := P_\gamma(v)$

independent of $\gamma$ with $\gamma(0) = p$

- $Hol_p(M, \nabla) \subset \text{SL}(n, \mathbb{R}) \iff \omega \in \Omega^n M : \nabla \omega = 0$.
- $Hol_p(M^{2k}, \nabla) \subset \text{GL}(k, \mathbb{C}) \iff J \in \text{End}(TM)$ with $J^2 = -\text{id}$: $\nabla J = 0$.
- $Hol_p(M, \nabla) \subset O(p, q) \iff$ metric $g \in \Gamma(\otimes^2 TM)$: $\nabla g = 0$. 
Holonomy and geometry 2: Parallel distributions

\[ \left\{ V \subset T_p M \mid Hol_p(M, \nabla)(V) \subset V \right\} \cong \left\{ \text{Distribution } V \subset TM \mid P_\gamma(V) \subset V \right\} \]

\[ V \mapsto V := P_\gamma(V) \]

\[ P_\gamma(V) \subset V \iff \nabla_X : \Gamma(V) \to \Gamma(V), \text{ in particular, } V \text{ is integrable.} \]

This leads to:

Decomposition of a semi-Riemannian manifold \((M, g)\)

If \(V \subset T_p M\) is \(Hol(M, g)\)-invariant, non-degenerate \((V \cap V^\perp = \{0\})\), i.e. \(T_p M = V \oplus V^\perp\) invariant decomposition, then

\[(M, g) \overset{\text{locally}}{\cong} (N, h) \times (N^{\perp}, h^{\perp})\]

with \(V^{\perp} \cong T_p N^{\perp}\) as \(Hol_p(M, g)\)-module.
De Rham–Wu decomposition

Complete decomposition of $T_pM$ into $Hol_p(M, g)$–modules:
$$T_pM = \bigoplus_{i=0}^{k} V_i, \text{ with } V_0 \text{ trivial and } V_i \text{ indecomposable for } i > 0$$
non-degenerate and only non-degenerate invariant subspaces

**Theorem (de Rham ’52, Wu ’64)**

Let $(M, g)$ be semi-Riemannian, complete and simply connected. Then there is a $k > 0$:
$$\begin{align*}
(M, g) &\overset{\text{globally}}{\cong} (M_1, g_1) \times \ldots \times (M_k, g_k) \\
(M_i, g_i) &\text{ complete and 1-connected,} \\
(M_i, g_i) &\text{ flat or with indecomposable holonomy representation,} \\
Hol_p(M, g) &\cong Hol_{p_1}(M_1, g_1) \times \ldots \times Hol_{p_k}(M_k, g_k).
\end{align*}$$

**Manifold of special holonomy:** Indecomposable holonomy $\nsubseteq SO(p, q)$. 
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible
\(\implies\) \(\text{Hol}_p(M, g) \simeq\) product of irreducible holonomy groups.

Berger’s list (’55)

Let \((M, g)\) be simply-connected, irreducible, non locally symmetric. Then

\[
\text{Hol}_p(M, g) \overset{O(n)}{\sim} \begin{array}{l}
\text{SO}(n) \\
\text{U}(\frac{n}{2}) \\
\text{SU}(\frac{n}{2}) \\
\text{Sp}(\frac{n}{4}) \\
\text{Sp}(1) \cdot \text{Sp}(\frac{n}{4}) \\
\text{G}_2 \\
\text{Spin}(7)
\end{array}
\]

generic    Kähler       hyper Kähler     quat. Kähler

<table>
<thead>
<tr>
<th>par. field</th>
<th>none</th>
<th>(J)</th>
<th>(J_1, J_2, J_3)</th>
<th>(\langle J_1, J_2, J_3 \rangle)</th>
<th>(\omega^3)</th>
<th>(\omega^4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ric</td>
<td>—</td>
<td>(\neq 0)</td>
<td>0</td>
<td>0</td>
<td>(c \cdot g)</td>
<td>0</td>
</tr>
<tr>
<td>(\text{dim}{\nabla \phi = 0})</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>(\frac{n}{4} + 1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>par. spinor</td>
<td>0</td>
<td>0</td>
<td>(\frac{n}{4} + 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Complete mf’s: Calabi (SU, Sp), LeBrun (qK), Bryant (G_2, Spin(7)).
- Compact mf’s: Yau (SU), Beauville, Mukai (Sp), LeBrun-Salamon (qK), Joyce (G_2, Spin(7)).
Special Lorentzian holonomy

Wu–Decomposition for a Lorentz manifold \((M, g)\)

Let \((M, g)\) be a complete, simply-connected Lorentzian manifold.

\[
(M, g) \cong \overline{\left( M, g \right)} \times \left( N_1, g_1 \right) \times \ldots \times \left( N_k, g_k \right)
\]

\[\uparrow\]

Riemannian, irreducible or flat

Lorentzian manifold which is either

1. \((\mathbb{R}, -dt^2)\), or
2. irreducible, i.e. \(\text{Hol}_p(\overline{M}, \overline{g}) = SO_0(1, n)\) [Olmos/Di Scala ’00], or
3. indecomposable, non-irreducible

I.e., an indecomposable Lorentzian manifold has special holonomy

\[\iff\]

its holonomy admits a degenerate invariant subspace

\[\iff\]

it admits a parallel null line.
Algebraic preliminaries

We have to consider $H \subset SO_0(1, n - 1)$ indecomposable, non-irreducible, i.e. $\exists V \subset \mathbb{R}^{1,n-1} : H(V) \subset V$ such that $L := V \cap V^\perp \neq \{0\}$ is a $H$-invariant, totally null line in $\mathbb{R}^{1,n-1}$

$$H \subset Iso_{SO_0(1,n-1)}(L) = (\mathbb{R}^+ \times SO(n - 2)) \ltimes \mathbb{R}^{n-2}$$

Change basis of $\mathbb{R}^{1,n-1}$: $\mathfrak{h} \subset \begin{cases} \begin{pmatrix} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{pmatrix} \bigg| \begin{aligned} a &\in \mathbb{R}, \\
v &\in \mathbb{R}^{n-2}, \\
A &\in \mathfrak{so}(n - 2) \end{aligned} \end{cases}$

The orthogonal part is reductive:

$$\mathfrak{g} := pr_{\mathfrak{so}(n-2)}\mathfrak{h} = \mathfrak{z} \bigoplus \mathfrak{g}' \quad (Levi - decomposition)$$

$\mathfrak{z}$ semisimple
Let $(M, g)$ be a Lorentzian manifold with
\[ H := Hol_p(M, g) \subset Iso(L) = (\mathbb{R}^+ \times SO(n-2)) \times \mathbb{R}^{n-2}. \]

- $L$ defines a filtration $\mathcal{L} \subset \mathcal{L}^\perp \subset TM$ with parallel null line $\mathcal{L}$ and parallel null hypersurface $\mathcal{L}^\perp$.
- If $H \subset SO(n-2) \rtimes \mathbb{R}^{n-2}$, then $\exists$ parallel null vector field (Brinkmann wave).
- What about $G := \text{pr}_{SO(n-2)} Hol_p(M, g) \subsetneq SO(n-2)$?

Proposition (TL ’03)

**The vector bundle (“screen bundle”)** $S = \mathcal{L}^\perp / \mathcal{L}$ with covariant derivative
\[ \nabla^S_U[V] := [\nabla_U V] \quad \text{satisfies} \quad \text{pr}_{SO(n-2)} Hol_p(M, g) = Hol_p(S, \nabla^S). \]

- Hence, algebraic structures for $G$ correspond to geometric structures on $S$, e.g. product structure, parallel complex structure etc.
- Which $G$’s can occur?
Classification 1: $\mathfrak{h} \subset \mathfrak{isO}_{\mathfrak{sO}(1,n-1)}(L)$ indecomposable

Theorem (Berard-Bergery/Ikemakhen ’96)

For $\mathfrak{h}$ with $\mathfrak{g} := \text{pr}_{\mathfrak{sO}(n-2)}\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ there are the following cases:

$\mathbb{R}^{n-2} \subset \mathfrak{h}$ — Type I: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$.

Type II: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$.

Type III: $\exists \varphi : \mathfrak{z} \to \mathbb{R}$: $\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(A) & \mathbf{v}^t & 0 \\ 0 & A + B & -\mathbf{v} \\ 0 & 0 & -\varphi(A) \end{pmatrix} \mid A \in \mathfrak{z}, B \in \mathfrak{g}', \mathbf{v} \in \mathbb{R}^{n-2} \right\}$

$\mathbb{R}^{n-2} \not\subset \mathfrak{h}$ — Type IV: $\exists \varphi : \mathfrak{z} \to \mathbb{R}^k$, for $0 < k < n - 2$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(A)^t & \mathbf{v}^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A + B & -\mathbf{v} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \begin{array}{l} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ \mathbf{v} \in \mathbb{R}^{n-2-k} \end{array} \right\}$$

Note: Groups of coupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen ’96.
Classification II: $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}}(1,n-1)(L)$ indecomposable

Theorem (TL ’03)

If $\mathfrak{h}$ is a Berger algebra (e.g. a Lorentzian holonomy algebra), then $\mathfrak{g} := \text{proj}_{\mathfrak{so}}(n-2)\mathfrak{h}$ is a Riemannian holonomy algebra (and hence known to be a product of algebras from Berger’s list).

Idea of the proof: For $\mathfrak{g} \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

$$\mathcal{B}(\mathfrak{g}) := \{Q \in \text{Hom}(\mathbb{R}^n, \mathfrak{g}) \mid \langle Q(x)y, z \rangle + \langle Q(y)z, x \rangle + \langle Q(z)x, y \rangle = 0\}.$$ 

$\mathfrak{g}$ is a weak Berger algebra $\iff \mathfrak{g} = \text{span}\{Q(x) \mid Q \in \mathcal{B}(\mathfrak{g}), x \in \mathbb{R}^n\}$

[TL ’02] If $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}}(1,n-1)(L)$ is an indecomposable Berger algebra, then $\mathfrak{g} := \text{proj}_{\mathfrak{so}}(n-2)(\mathfrak{h})$ is a weak Berger algebra. Classify them $\implies$ result. □

Theorem (Berard-Bergery-Ikemakhen ’96, Boubel ’00, TL ’03, Galaev ’05)

If $\mathfrak{g} := \text{proj}_{\mathfrak{so}}(n-2)\mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric $h$ with $\text{hol}_{\mathfrak{p}}(h) = \mathfrak{h}$. 
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

Let \(\left(\Sigma, \nabla^\Sigma\right)\) be the spinor bundle over \((M, g)\).

Assume: \(\exists \varphi \in \Gamma(\Sigma)\) with \(\nabla^\Sigma \varphi = 0\) a parallel spinor field.

\[\implies \exists \text{ causal vector field } X_\varphi \in \Gamma(TM): \nabla X_\varphi = 0.\]

Two cases:

\[g(X_\varphi, X_\varphi) < 0 : (M, g) = (\mathbb{R}, -dt^2) \] 
\[\times \text{ Riemannian mf.} \]

\[g(X_\varphi, X_\varphi) = 0 : (M, g) = (\overline{M}, \overline{g}) \] 
\[\uparrow \text{ indecomposable with parallel spinor} \]

Theorem (TL '03)

\((M, g)\) indecomposable Lorentzian spin with parallel spinor. Then
\[\text{Hol}_p(M, g) = G \times \mathbb{R}^{n-2}\] where \(G\) is a product of the following groups:

\[\{1\}, \text{ SU}(p), \text{ Sp}(q), \text{ G}_2, \text{ Spin}(7)\]

\[\text{dim}\{\nabla \varphi = 0\} : 2^{[k/2]} \quad 2 \quad q + 1 \quad 1 \quad 1\]

This generalizes the result for \(n \leq 11\) in [Bryant '99].
Lorentzian Einstein manifolds

Theorem (Galaev-TL ’06)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

\[
\text{Hol}^0_p(M, g) = \begin{cases} 
\left( \mathbb{R}^+ \times G \right) \ltimes \mathbb{R}^{n-2}, & \text{or} \\
G \ltimes \mathbb{R}^{n-2} 
\end{cases}
\]

with a Riemannian holonomy group \(G\). Furthermore:

- If \(\text{Hol}^0_p(M, g) = G \ltimes \mathbb{R}^{n-2}\), then \(\text{Ric} = 0\) and \(G = \text{Holonomy of Ricci-flat Riemannian manifold, i.e. } G = \text{product of } \text{SO}(n), \text{SU}(p), \text{Sp}(q), G_2, \text{and Spin}(7)\).
Coordinates

Theorem (Brinkmann’25, Walker’49)

For a Lorentzian manifold \((M, h)\) with parallel null line \(L\) there are coordinates \((x, y_1, \ldots, y_{n-2}, z)\): \(\frac{\partial}{\partial x}\) spans \(L\), \(\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-2}}\right)\) span \(L^\perp\), and

\[
\begin{align*}
    h &= 2 \, dx \, dz + \sum_{i=1}^{n-2} u_i \, dy_i \, dz + f \, dz^2 + \sum_{i,j=1}^{n-2} g_{ij} \, dy_i \, dy_j, \\
    &= \phi_z \quad \text{family of 1-forms} \\
    &= g_z \quad \text{family of Riem. metrics}
\end{align*}
\]

\[\text{with } \frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0, \; f \in C^\infty(M).\]

\[\exists \text{ parallel null vector field } \iff \frac{\partial f}{\partial x} = 0.\]

Note: \(\text{Hol}_p(g_z) \subset \text{pr}_{SO(n)}\text{Hol}_p(h)\), but in general \(\neq\) (see Galaev’s examples on next slides)
Manifolds of uncoupled holonomy type

Construction method for the uncoupled types

Let \((N^{n-2}, g)\) be a Riemannian manifold and \(f \in C^\infty(\mathbb{R}^2 \times N)\) “sufficiently generic”. Then \(M = \mathbb{R}^2 \times N\) with the metric \(h := 2dx dz + f dz^2 + g\) is indecomposable, non irreducible with holonomy

\[
(\mathbb{R}^+ \times Hol(N, g)) \ltimes \mathbb{R}^{n-2} \quad \text{or} \quad Hol(N, g) \ltimes \mathbb{R}^{n-2}, \quad \text{if} \quad \frac{\partial f}{\partial x} = 0
\]

Example: \((M, h)\) pp-wave :\(\iff\) \(g \equiv \) flat metric.

[TL ’01]: An indecomposable Lorentzian mfd. has Abelian holonomy \(\mathbb{R}^{n-2}\) \(\iff\) it is a pp-wave.

- E.g. Symmetric spaces (Cahen-Wallach spaces) \(\iff\) \(f\) is a quadratic polynomial in the \(y_i\)’s.
- Plane waves: \(f\) is a quadratic polynomial in the \(y_i\)’s with coefficients depending on \(z\) \[Hull-Figueroa O’Farrill-Papadopoulos ’02\].
Coupled types — Proof of Theorem [Galaev ’05]

For a Riemannian holonomy algebra \( g \), fix \( Q_1, \ldots, Q_N \), a basis of \( \mathcal{B}(g) \), and define polynomials on \( \mathbb{R}^{n-1} \):

\[
u_i(y_1, \ldots, y_{n-2}, z) := \sum_{A=1}^{N} \sum_{k,l=1}^{n-2} \frac{1}{(A-1)!} \left\langle Q_A(e_k)e_l, e_i \right\rangle y_k y_l z^A.
\]

Theorem (Galaev ’05)

For any indecomposable \( \mathfrak{h} \subset \mathfrak{so}(1, n-1)_L \), for which \( g = \text{proj}_{\mathfrak{so}(n-2)}(\mathfrak{h}) \) is a Riemannian holonomy algebra, there exists an analytic \( f : \mathbb{R}^n \to \mathbb{R} \) such that the following Lorentzian metric has holonomy \( \mathfrak{h} \):

\[
h = 2dx dz + f dz^2 + 2 \sum_{i=1}^{n-2} u_i dy_i \quad dz + \sum_{k=1}^{n-2} dy_k^2,
\]

family of 1-forms on \( \mathbb{R}^n \)  
flat metric
Open Problems

Study *Lorentzian manifolds with special holonomy*!

1. Find global examples of metrics with prescribed holonomy, which are globally hyperbolic with complete or compact Cauchy surface (cylinder constructions in [Bär-Gauduchon-Moroianu ’05] and [Baum-Müller ’06]).

2. Describe the geometric structures corresponding to the coupled types III and IV.

3. Describe indecomposable, non-irreducible *Lorentzian homogeneous spaces* and their holonomy.

4. Study further spinor field equations for these manifolds (Killing spinors, generalised Killing spinors).

Thank you!