# Special holonomy in Lorentzian geometry

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# Outline

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# Holonomy group of a linear connection

- Let $M$ be a smooth manifold and $\nabla$ a linear connection.

  $\rightsquigarrow$ Parallel displacement along $\gamma : [0,1] \to M$, piecewise smooth,

  $$\mathcal{P}_\gamma \quad : \quad T_{\gamma(0)}M \ni X_0 \quad \overset{\sim}{\longmapsto} \quad X(1) \in T_{\gamma(1)}M$$

  where $X(t)$ is the solution to the ODE $\nabla_{\dot\gamma(t)}X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (Connected) Holonomy group

$$Hol_p{}^0(M,\nabla) := \left\{ \mathcal{P}_\gamma \mid \gamma(0) = \gamma(1) = p, \gamma \sim \{p\} \right\} \subset \mathrm{GL}(T_pM) \simeq \mathrm{GL}(n,\mathbb{R})$$

and its Lie algebra $\mathfrak{hol}_p(M,\nabla)$.

- For $p, q \in M$: $$Hol_p(M,\nabla) \quad \overset{\substack{\text{conjugated in } \mathrm{GL}(n,\mathbb{R}) \\ \downarrow}}{\sim} \quad Hol_q(M,\nabla)$$
- If $\nabla =$ LC of a metric $g$ on $M$, then $Hol_p(M,g) \subset \mathrm{O}(T_pM, g) = \mathrm{O}(t,s)$.

# Holonomy and curvature

- Recall that $\nabla$ and $\mathcal{P}_\gamma$ are related via

$$\nabla_{\dot\gamma(0)}X|_p = \frac{d}{dt}\left[\mathcal{P}^{-1}_{\gamma|_{[0,t]}}(X(\gamma(t)))\right]|_{t=0}.$$

- This implies for the curvature $\mathcal{R}$ of $\nabla$: Let $X, Y \in T_pM$ and $\lambda_t$ the loop along the parallelogram at $p$ with sides $\sqrt{t}X$ and $\sqrt{t}Y$. Then

$$\mathcal{R}(X,Y)|_p = \lim_{t\to 0}\frac{1}{t}\left(\mathcal{P}_{\lambda_t} - Id_{T_pM}\right).$$

  Hence, $\mathcal{R}(X,Y)|_p \in \mathfrak{hol}_p(M,\nabla)$ for all $X, Y \in T_pM$.
- One has to collect curvature all over $M$ to get all of $\mathfrak{hol}_p(M,\nabla)$:

**Theorem (Ambrose-Singer ’53)**

*If $M$ is connected, then $\mathfrak{hol}_p(M,\nabla)$ is spanned by*

$$\left\{\mathcal{P}^{-1}_\gamma \circ \mathcal{R}(X,Y)\circ\mathcal{P}_\gamma \in \mathrm{GL}(T_pM) \mid \gamma(0)=p \text{ and } X, Y \in T_{\gamma(1)}M\right\}$$

# Classification: *Which groups occur as holonomy groups?*

- Hano/Ozeki '56: Any closed $G \subset \mathrm{GL}(n,\mathbb{R})$! But $\nabla$ might have torsion.
- Conditions on the torsion $T^\nabla$ of $\nabla$, e.g. $T^\nabla = 0$ or $T^\nabla \in \Lambda^3 TM$

  $\rightsquigarrow$ algebraic constraints on the holonomy representation. Why?

$$\mathcal{R}_\gamma \ := \ \mathcal{P}_\gamma^{-1} \circ \mathcal{R}\Big(\mathcal{P}_\gamma(.), \mathcal{P}_\gamma(.)\Big) \circ \mathcal{P}_\gamma \ \in \ \Lambda^2(T_p^* M) \otimes \mathrm{GL}(T_p M)$$

Now, if $T^\nabla = 0$, then $\mathcal{R}$ and hence $\mathcal{R}_\gamma$ satisify the Bianchi identity:

$$\mathcal{R}_\gamma(X,Y)Z + \mathcal{R}_\gamma(Y,Z)X + \mathcal{R}_\gamma(Z,X)Y = 0.$$

$\Longrightarrow$ $\mathfrak{hol}_p(M,\nabla)$ is a Berger algebra: For $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ define the $\mathfrak{g}$-module:

$$\mathcal{K}(\mathfrak{g}) := \Big\{ R \in \Lambda^2 \mathbb{R}^{n^*} \otimes \mathfrak{g} \mid \ R(x,y)z + R(y,z)x + R(z,x)y = 0 \Big\}$$

$\mathfrak{g}$ is a Berger algebra $:\Longleftrightarrow$ $\mathfrak{g} = \mathrm{span}\,\{R(x,y) \mid R \in \mathcal{K}(\mathfrak{g}), x, y \in \mathbb{R}^n\}$.

$T^\nabla = 0$: Ambrose-Singer $\Longrightarrow$ $\mathfrak{hol}_p(M,\nabla)$ is a Berger algebra.

Classification of irreducible Berger algebras

- $\mathfrak{g} \subset \mathfrak{so}(p,q)$ [Berger '55], $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ [Schwachhöfer/Merkulov '99] .

# Holonomy and geometry 1: Parallel sections

Let $V$ be a "geometric" vector bundle over $M$ and $\nabla$ a connection on $V$ "induced" by a connection $\nabla$ on $TM$ (e.g. tangent bunde, tensor bundles, spinor bundle) . Then:

$$\begin{array}{rcl} \left\{ v \in V_p \mid Hol_p(M,\nabla)(v) = v \right\} & \simeq & \left\{ \varphi \in \Gamma(V) \mid \nabla\varphi = 0 \right\} \\ v & \mapsto & \varphi := \mathcal{P}_\gamma(v) \\ & & \text{independent of } \gamma \text{ with } \gamma(0) = p \end{array}$$

- $Hol_p(M,\nabla) \subset \mathrm{SL}(n,\mathbb{R}) \Leftrightarrow \omega \in \Omega^n M : \ \nabla\omega = 0.$
- $Hol_p(M^{2k},\nabla) \subset \mathrm{GL}(k,\mathbb{C}) \Leftrightarrow J \in End(TM)$ with $J^2 = -id$: $\nabla J = 0.$
- $Hol_p(M,\nabla) \subset \mathrm{O}(p,q) \Leftrightarrow$ metric $g \in \Gamma(\odot^2 TM)$: $\nabla g = 0.$

# Holonomy and geometry 2: Parallel distributions

$$\begin{array}{rcl}\left\{V \subset T_pM \mid Hol_p(M,\nabla)(V) \subset V\right\} & \simeq & \left\{\text{Distribution } \mathcal{V} \subset TM \mid \mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V}\right\} \\ V & \mapsto & \mathcal{V} := \mathcal{P}_\gamma(V)\end{array}$$

$\mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \iff \nabla_X : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}),$ in particular, $\mathcal{V}$ is integrable.

This leads to:

**Decomposition of a semi-Riemannian manifold $(M,g)$**

If $V \subset T_pM$ is $Hol(M,g)$-invariant, non-degenerate ($V \cap V^\perp = \{0\}$), i.e. $T_pM = V \oplus V^\perp$ invariant decomposition, then

$$(M,g) \overset{locally}{\simeq} (N,h) \times (N^\perp, h^\perp)$$

with $V^{(\perp)} \simeq T_pN^{(\perp)}$ as $Hol_p(M,g)$–module.

# De Rham–Wu decomposition

Complete decomposition of $T_pM$ into $Hol_p(M,g)$–modules:

$$T_pM = \oplus_{i=0}^{k} V_k\text{, with } V_0 \text{ trivial and } V_i \underbrace{\text{indecomposable}}_{\text{non-degenerate and only degenerate invariant subspaces}} \text{ for } i > 0$$

Theorem (de Rham ’52, Wu ’64)

*Let $(M,g)$ be semi-Riemannian, complete and simply connected. Then there is a $k > 0$:*

$$(M,g) \overset{\text{globally}}{\simeq} (M_1,g_1) \times \ldots \times (M_k,g_k)$$

- $(M_i,g_i)$ *complete and 1-connected,*
- $(M_i,g_i)$ *flat or with indecomposable holonomy representation,*
- $Hol_p(M,g) \simeq Hol_{p_1}(M_1,g_1) \times \ldots \times Hol_{p_k}(M_k,g_k)$.

Manifold of special holonomy: Indecomposable holonomy $\subsetneq \mathrm{SO}(p,q)$.

# Holonomy of Riemannian manifolds $(M, g)$

Positive definite metric $\Longrightarrow$ indecomposable = irreducible
$\Longrightarrow$ $Hol_p(M, g) \simeq$ product of irreducible holonomy groups.

## Berger's list ('55)

*Let $(M, g)$ be simply-connected, irreducible, non locally symmetric. Then*

$Hol_p(M, g) \overset{O(n)}{\sim}$

| | $\mathrm{SO}(n)$ | $\mathrm{U}(\frac{n}{2})$ | $\mathrm{SU}(\frac{n}{2})$ | $\mathrm{Sp}(\frac{n}{4})$ | $\mathrm{Sp}(1)\cdot\mathrm{Sp}(\frac{n}{4})$ | $\mathrm{G}_2$ | $\mathrm{Spin}(7)$ |
|---|---|---|---|---|---|---|---|
| | *generic* | *Kähler* | | *hyper Kähler* | *quat. Kähler* | | |
| *par. field* | *none* | $J$ | | $J_1, J_2, J_3$ | $\langle J_1, J_2, J_3\rangle$ | $\omega^3$ | $\omega^4$ |
| *Ric* | — | $\neq 0$ | 0 | 0 | $c \cdot g$ | 0 | 0 |
| *dim*$\{\nabla\varphi = 0\}$<br>$\uparrow$<br>*par. spinor* | 0 | 0 | 2 | $\frac{n}{4} + 1$ | *0* | 1 | 1 |

- Complete mf's: Calabi ($\mathrm{SU}$, $\mathrm{Sp}$), LeBrun (qK), Bryant ($\mathrm{G}_2$, $\mathrm{Spin}(7)$).
- Compact mf's: Yau ($\mathrm{SU}$), Beauville, Mukai ($\mathrm{Sp}$), LeBrun-Salamon (qK), Joyce ($\mathrm{G}_2$, $\mathrm{Spin}(7)$).

# Special Lorentzian holonomy

## Wu–Decomposition for a Lorentz manifold $(M, g)$

Let $(M, g)$ be a complete, simply-connected Lorentzian manifold.

$$(M,g) \quad \simeq \quad \underset{\uparrow}{(\overline{M},\overline{g})} \times \underbrace{(N_1,g_1)\times\ldots\times(N_k,g_k)}_{\text{Riemannian, irreducible or flat}}$$

Lorentzian manifold which is either

1. $(\mathbb{R}, -dt^2)$, or
2. irreducible, i.e. $Hol_p(\overline{M},\overline{g}) = \mathrm{SO}_0(1,n)$ [Olmos/Di Scala '00], or
3. $\boxed{\text{indecomposbable, non-irreducible}}$

I.e., an indecomposable Lorentzian manifold has special holonomy
$\Longleftrightarrow$ its holonomy admits a degenerate invariant subspace
$\Longleftrightarrow$ it admits a parallel null line.

# Algebraic preliminaries

We have to consider $H \subset \mathrm{SO}_0(1, n-1)$ indecomposable, non-irreducible, i.e. $\exists\ V \subset \mathbb{R}^{1,n-1} :\ H(V) \subset V$ such that

$L := V \cap V^{\perp} \neq \{0\}$ is a $H$-invariant, totally null line in $\mathbb{R}^{1,n-1}$

$$\Rightarrow \quad H \quad \subset \quad Iso_{\mathrm{SO}_0(1,n-1)}(L) \quad = \quad (\mathbb{R}^{+} \times \mathrm{SO}(n-2)) \ltimes \mathbb{R}^{n-2}$$

Change basis of $\mathbb{R}^{1,n-1}$: $\mathfrak{h} \quad \subset \quad \left\{ \left( \begin{array}{ccc} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{array} \right) \middle| \begin{array}{l} a \in \mathbb{R}, \\ v \in \mathbb{R}^{n-2}, \\ A \in \mathfrak{so}(n-2) \end{array} \right\}$

The orthogonal part is reductive:

$$\mathfrak{g} := pr_{\mathfrak{so}(n-2)}\mathfrak{h} = \underbrace{\mathfrak{z}}_{centre} \oplus \underbrace{\mathfrak{g}'}_{= [\mathfrak{g}, \mathfrak{g}] \text{ semisimple}} \qquad (Levi - decomposition)$$

# Parallel null line and screen bundle

Let $(M,g)$ be a Lorentzian manifold with
$H := Hol_p(M,g) \subset Iso(L) = (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \times \mathbb{R}^{n-2}$.

- $L$ defines a filtration $\mathcal{L} \subset \mathcal{L}^\perp \subset TM$ with parallel null line $\mathcal{L}$ and parallel null hypersurface $\mathcal{L}^\perp$.
- If $H \subset \mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$, then $\exists$ parallel null vector field (Brinkmann wave).
- What about $G := \mathrm{pr}_{\mathrm{SO}(n-2)} Hol_p(M,g) \subsetneq \mathrm{SO}(n-2)$?

**Proposition (TL '03)**

*The vector bundle ("screen bundle")* $\mathcal{S} = \mathcal{L}^\perp / \mathcal{L}$ *with covarant derivative* $\nabla^{\mathcal{S}}_U[V] := [\nabla_U V]$ *satsifies* $\mathrm{pr}_{\mathrm{SO}(n-2)} Hol_p(M,g) = Hol_p(\mathcal{S}, \nabla^{\mathcal{S}})$ *.*

- Hence, algebraic structures for $G$ corrspond to geometric structures on $\mathcal{S}$, e.g. product structure, parallel complex structure etc.
- Which $G$'s can occur?

# Classification 1: $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ indecomposable

**Theorem (Berard-Bergery/Ikemakhen '96)**

*For* $\mathfrak{h}$ *with* $\mathfrak{g} := pr_{\mathfrak{so}(n-2)}\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$ *there are the following cases:*

$\mathbb{R}^{n-2} \subset \mathfrak{h}$ – *Type I:* $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$.

*Type II:* $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$.

*Type III:* $\exists\, \varphi : \mathfrak{z} \twoheadrightarrow \mathbb{R}$:

$$\mathfrak{h} = \left\{ \left( \begin{array}{ccc} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{array} \right) \middle| \begin{array}{l} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2} \end{array} \right\}$$

$\mathbb{R}^{n-2} \not\subset \mathfrak{h}$ – *Type IV:* $\exists\, \varphi : \mathfrak{z} \twoheadrightarrow \mathbb{R}^k$, *for* $0 < k < n-2$*:*

$$\mathfrak{h} = \left\{ \left( \begin{array}{cccc} 0 & \psi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \begin{array}{l} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2-k} \end{array} \right\}$$

Note: Groups of coupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen '96

# Classification II: $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ indecomposable

**Theorem (TL '03)**

*If $\mathfrak{h}$ is a Berger algebra (e.g. a Lorentzian holonomy algebra), then $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}\mathfrak{h}$ is a Riemannian holonomy algebra (and hence known to be a product of algebras from Berger's list).*

*Idea of the proof:* For $\mathfrak{g} \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

$$\mathcal{B}(\mathfrak{g}) := \left\{ Q \in Hom(\mathbb{R}^n, \mathfrak{g}) \mid \langle Q(x)y, z\rangle + \langle Q(y)z, x\rangle + \langle Q(z)x, y\rangle = 0 \right\}.$$

$\mathfrak{g}$ is a weak Berger algebra $:\Longleftrightarrow \mathfrak{g} = \mathrm{span}\left\{ Q(x) \mid Q \in \mathcal{B}(\mathfrak{g}), x \in \mathbb{R}^n \right\}$

[TL '02] If $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ is an indecomposable Berger algebra, then $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}(\mathfrak{h})$ is a weak Berger algebra. Classify them $\Longrightarrow$ result. □

**Theorem (Berard-Bergery-Ikemakhen '96, Boubel '00, TL '03, Galaev '05)**

*If $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}\mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric $h$ with $\mathfrak{hol}_p(h) = \mathfrak{h}$.*

# Parallel spinors on a Lorentzian spin manifold $(M,g)$

Let $\left(\Sigma,\nabla^{\Sigma}\right)$ be the spinor bundle over $(M,g)$.

Assume: $\exists\varphi\in\Gamma(\Sigma)$ with $\nabla^{\Sigma}\varphi=0$ a parallel spinor field.
$\Longrightarrow$ $\exists$ causal vector field $X_\varphi\in\Gamma(TM)$ : $\nabla X_\varphi=0$. Two cases:

$$\begin{array}{lcccl} g(X_\varphi,X_\varphi)<0 & : & (M,g) & = & (\mathbb{R},-dt^2) \\ g(X_\varphi,X_\varphi)=0 & : & (M,g) & = & \underset{\substack{\uparrow\\ \text{indecomposable with parallel spinor}}}{(\overline{M},\overline{g})} \end{array} \times \begin{array}{l}\text{Riemannian mf.}\\ \text{with parallel spinor}\end{array}$$

**Theorem (TL ’03)**

*$(M,g)$ indecomposable Lorentzian spin with parallel spinor. Then $Hol_p(M,g)=G\ltimes\mathbb{R}^{n-2}$ where $G$ is a product of the following groups:*

$$\begin{array}{rccccc} & \{1\}, & \mathrm{SU}(p), & \mathrm{Sp}(q), & \mathrm{G}_2, & \mathrm{Spin}(7) \\ dim\{\nabla\varphi=0\}: & 2^{[k/2]} & 2 & q+1 & 1 & 1 \end{array}$$

This generalizes the result for $n\leq 11$ in [Bryant ’99].

# Lorentzian Einstein manifolds

**Theorem (Galaev-TL ’06)**

*The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.*

$$Hol_p^0(M,g) = \begin{cases} (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^{n-2}, \text{ or} \\ G \ltimes \mathbb{R}^{n-2} \end{cases}$$

*with a Riemannian holonomy group $G$. Furthermore:*

- *If $Hol_p^0(M,g) = G \ltimes \mathbb{R}^{n-2}$, then $Ric = 0$ and $G =$ Holonomy of Ricci-flat Riemannian manifold, i.e. $G =$ product of $\mathrm{SO}(n)$, $\mathrm{SU}(p)$, $\mathrm{Sp}(q)$, $\mathrm{G}_2$, and $\mathrm{Spin}(7)$.*

# Coordinates

**Theorem (Brinkmann'25, Walker'49)**

*For a Lorentzian manifold* $(M, h)$ *with parallel null line* $\mathcal{L}$ *there are coordinates* $(x, y_1, \ldots, y_{n-2}, z)$*:* $\frac{\partial}{\partial x}$ *spans* $\mathcal{L}$ *,* $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-2}}\right)$ *span* $\mathcal{L}^\perp$*, and*

- $$h = 2\, dxdz + \underbrace{\sum_{i=1}^{n-2} u_i dy_i}_{\substack{=\phi_z \\ \text{family of 1-forms}}} \; dz + fdz^2 + \underbrace{\sum_{i,j=1}^{n-2} g_{ij} dy_i\, dy_j}_{\substack{=g_z \\ \text{family of Riem. metrics}}} \quad ,$$

  *with* $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$*,* $f \in C^\infty(M)$*.*
- $\exists$ *parallel null vector field* $\iff$ $\frac{\partial f}{\partial x} = 0$*.*

Note: $Hol_p(g_z) \ \subset \ pr_{SO(n)} Hol_p(h)$, but in general $\neq$ (see Galaev's examples on next slides)

# Manifolds of uncoupled holonomy type

## Construction method for the uncoupled types

*Let* $(N^{n-2}, g)$ *be a Riemannian manifold and* $f \in C^\infty(\mathbb{R}^2 \times N)$ *"sufficiently generic". Then* $M = \mathbb{R}^2 \times N$ *with the metric* $h := 2dxdz + fdz^2 + g$ *is indecomposable, non irreducible with holonomy*

$$(\mathbb{R}^+ \times Hol(N,g)) \ltimes \mathbb{R}^{n-2} \quad or \quad Hol(N,g) \ltimes \mathbb{R}^{n-2}, \;\; if \; \frac{\partial f}{\partial x} = 0$$

Example: $(M,h)$ pp-wave $:\Longleftrightarrow$ $g \equiv$ flat metric.
[TL '01]: An indecomposable Lorentzian mfd. has Abelian holonomy $\mathbb{R}^{n-2}$ $\Longleftrightarrow$ it is a pp-wave.

- E.g. Symmetric spaces (Cahen-Wallach spaces) $\Longleftrightarrow$ $f$ is a quadratic polynomial in the $y_i$'s.
- Plane waves: $f$ is a quadratic polynomial in the $y_i$'s with coefficients depending on $z$ [Hull-Figueroa O'Farrill-Papadopoulos '02].

# Coupled types — Proof of Theorem [Galaev '05]

For a Riemannian holonomy algebra $\mathfrak{g}$, fix $Q_1, \dots, Q_N$, a basis of $\mathcal{B}(\mathfrak{g})$, and define polynomials on $\mathbb{R}^{n-1}$:

$$u_i(y_1, \dots, y_{n-2}, z) \ := \ \sum_{A=1}^{N} \sum_{k,l=1}^{n-2} \frac{1}{(A-1)!} \Big\langle Q_A(e_k)e_l, e_i \Big\rangle y_k y_l z^A.$$

**Theorem (Galaev '05)**

*For any indecomposable $\mathfrak{h} \subset \mathfrak{so}(1, n-1)_L$, for which $\mathfrak{g} = proj_{\mathfrak{so}(n-2)}(\mathfrak{h})$ is a Riemannian holonomy algebra, there exists an analytic $f : \mathbb{R}^n \to \mathbb{R}$ such that the following Lorentzian metric has holonomy $\mathfrak{h}$:*

$$h \ = \ 2dxdz + fdz^2 + 2 \underbrace{\sum_{i=1}^{n-2} u_i dy_i \ dz}_{\text{family of 1-forms on } \mathbb{R}^n} + \underbrace{\sum_{k=1}^{n-2} dy_k^2}_{\text{flat metric}},$$

# Open Problems

Study *Lorentzian manifolds with special holonomy* !

1. Find global examples of metrics with prescribed holonomy, which are globally hyperbolic with complete or compact Cauchy surface (cylinder constructions in [Bär-Gauduchon-Moroianu '05] and [Baum-Müller '06])
2. Describe the geometric structures corresponding to the coupled types III and IV.
3. Describe indecomposable, non-irreducible Lorentzian homogeneous spaces and their holonomy.
4. Study further spinor field equations for these manifolds (Killing spinors, generalised Killing spinors).

Thank you!