Pseudo-Riemannian cones and their holonomy

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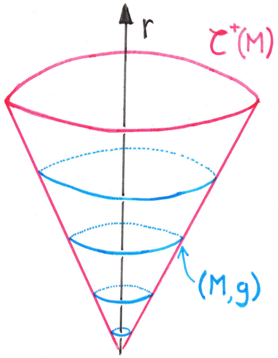
Outline

What is a metric cone?



Base: (M, g) a (pseudo-)Riemannian manifold.

Cone: Cartesian product:

$$\hat{M} = \mathbb{R}_+ \times M \ni (r, x)$$

with the (pseudo-)Riemannian cone metric

$$\hat{g} = \pm dr^2 + r^2 g,$$

denoted by $\mathcal{C}^{\pm}(M)$.

Example

$\mathcal{C}^+(S^n_{round}) = \mathbb{R}^{n+1}$ and $\mathcal{C}^-(H^n_{hyp}) = \mathbb{R}^{1,n}$, i.e. flat (Minkowski-) space.

Gallot's Theorem

The converse is also true:

If $\mathcal{C}^{\pm}(M)$ is flat, then (M, g) has constant sectional curvature ± 1.

Proof: Curvature of the cone is given by $\partial_r \lrcorner \hat{R} = 0$, and

$$\hat{R}(X, Y)Z = R(X, Y)Z \mp \left[g(Y, Z)X - g(X, Z)Y\right] \;\; \forall\, X, Y, Z \in TM$$

Theorem (S. Gallot, 1979)

Let (M, g) be a Riemannian manifold that is geodesically complete. If $\mathcal{C}^{+}(M)$ is a Riemannian product, then it is flat and (M, g) has constant sectional curvature 1. Moreover, if M is 1-connected, then $M = S^n$.

Cones and geometric structures

Base (M, g)		Cone $C^{\pm}(M)$
constant curvature ± 1	$\Leftrightarrow$	*flat*
Einstein, $Ric = \pm(n-1)g$	$\Leftrightarrow$	*Ricci-flat*, $\hat{Ric} = 0$
$\frac{\text{Riemannian}}{\text{Lorentzian}}$ *Sasaki* *Killing vector field* ξ *of lengths* ± 1 *s. th.* $\Phi := \nabla\xi$ *satisfies:* • $\Phi^2 = -\mathrm{Id} \pm g(\xi, .)\xi$, • $\pm(\nabla_X \Phi)(Y) = g(X, Y)\xi - g(Y, \xi)X$.	$\Leftrightarrow$	$C^{\pm}(M)$ *is Kähler* *Complex structure* J *s. th.* $\hat{\nabla} J = 0$ *and* $J^*\hat{g} = \hat{g}$. $\xi \mapsto J$ *defined by* • $J\partial_r = \xi$ *and* $J\xi = -\partial_r$, • $J\vert_{\xi^\perp \cap \partial_r^\perp} = -\Phi\vert_{\xi^\perp \cap \partial_r^\perp}$

etc.

Killing spinors

Definition

A spinor field φ on a pseudo-Riemannian spin manifold is a Killing spinor *to the* Killing number $\lambda \in \mathbb{C}$ *if* $\nabla_X \varphi = \lambda X \cdot \varphi$.

A manifold with Killing spinor is Einstein with $Ric = 4\lambda^2(n-1)g$.

(M, g) has a Killing spinor to the Killing no. $\lambda = \sqrt{\pm 1}/2$ $\Leftrightarrow$ *$\mathcal{C}^{\pm}(M)$ admits a parallel spinor φ, i.e. $\hat{\nabla}_X \varphi = 0$.*

Theorem (C. Bär '93, "Bär-correspondence")

Let M be a complete Riemannian spin manifold with a real Killing spinor. Then M is S^n, or a compact Einstein space with one of the following structures: Sasaki, 3-Sasaki, 6-dim. nearly-Kähler, or nearly parallel G_2.

Manifolds with real Killing spinors

Proof.

M has real Killing spinor $\Longrightarrow$ $\mathcal{C}^+(M)$ has a parallel spinor.

Gallot's Thm. $\Longrightarrow$ $\mathcal{C}^+(M)$ is flat (and $M = S^n$), or not a product , i.e. the cone has an irreducible holonomy group.

Berger's classification of irreducible holonomy groups $\Longrightarrow$

$Hol(\mathcal{C}^+(M))$	$\mathcal{C}^+(M)$	M
SU(n/2)	Calabi-Yau	Einstein-Sasaki
Sp(n/4)	hyper-Kähler	3-Sasaki
G_2	parallel stable 3-form	6-dim nearly-Kähler
Spin(7)	parallel stable 4-form	nearly parallel G_2

□

Holonomy groups of pseudo-Riemannian manifolds

$Hol_p(M,g) :=$ *group of parallel transports along loops starting in* $p \in M$.

$Hol_p \sim Hol_q$, $Hol_p(M,g)$ is a Lie group in $O(T_pM) \simeq O(r,s)$, represented on $T_pM \simeq \mathbb{R}^n$

- decomposable $\iff$ $\exists$ a non-degenerate invariant subspace V. In this case $\mathbb{R}^n = V \oplus V^\perp$ invariant decomposition. Or
- indecomposable $\iff$ no non-degenerate invariant subspace, but possibly a degenerate one. No decomposition, as $L \cap L^\perp \neq \{0\}$.

In fact,

$$\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$$

with V_i trivial or indecomposable representations of *Hol*.

Example

- (M,g) *flat* $\Rightarrow$ $Hol(M,g) = \Pi_1(M)$.
- $Hol(S^n) = Hol(H^n) = SO(n)$.

Holonomy principle and decomposition

$$Hol\,(M_1 \times M_2, g_1 \oplus g_2) \;=\; Hol(M_1,g_1) \times Hol(M_2,g_2) \text{ on } T_{p_1}M_1 \oplus T_{p_2}M_2$$

Holonomy principle

$$\begin{array}{ccc} \{hol - inv.\ subspaces\ in\ T_pM\} & \simeq & \{distributions\ \mathcal{V} \subset TM \text{ with } \nabla\mathcal{V} \subset \mathcal{V}\} \\ V & \mapsto & \mathcal{V} := \mathcal{P}_\gamma(V) \end{array}$$

Theorem (de Rham '52, Wu '64)

Let (M,g) be pseudo-Riemannian, complete and 1-connected.

Then there is a $k > 0$: $(M,g) \overset{globally}{\simeq} (M_1,g_1) \times \ldots \times (M_k,g_k)$ with

- (M_i,g_i) complete and 1-connected,
- (M_i,g_i) *flat or with indecomposable holonomy representation,*
- $Hol_p(M,g) \simeq Hol_{p_1}(M_1,g_1) \times \ldots \times Hol_{p_k}(M_k,g_k)$.

Cones and holonomy

For Riemannian manifolds: irreducible = indecomposable.

Gallot's Theorem (rephrased)

If the holonomy group of the cone $\mathcal{C}^+(M)$ over a complete Riemannian manifold M admits an invariant subspace, then the cone is flat, etc ...

[Alekseevsky, Cortés, Galaev, L: arXiv:0707.3063, to appear in *Crelle's Journal*]: For (M, g) with indefinite metric we give generalisations of this in the following situations:

1. *Hol*$(\mathcal{C}^+(M)$ is decomposable, i.e. has a non-degenerate invariant subspace, i.e. the cone is a product.
2. *Hol*$(\mathcal{C}^+(M))$ has a degenerate invariant subspace V with an invariant complement V^*. In this case, $\mathcal{C}^+(M)$ has neutral signature (n, n) and V and V^* are totally null $\Leftrightarrow$ para-Kähler structure.
3. $\mathcal{C}^+(M)$ is Lorentzian with a hol-invariant subspace.

Example: Gallot's Thm is false without completeness

Consider Riemannian manifolds (M_1, g_1) and (M_2, g_2),
build another Riemannian manifold "doubly warped product")

$$\left(M := \left(0, \frac{\pi}{2}\right) \times M_1 \times M_2, g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2\right).$$

This metric is not complete, its cone is isometric to a product of cones

$$\begin{array}{ccc} \mathcal{C}^+(M_1) \times \mathcal{C}^+(M_2) & \overset{\text{isometric}}{\simeq} & \mathcal{C}^+(M_1 \times M_2) \\ \big((r_1, x), (r_2, y)\big) & \mapsto & \left(r := \sqrt{r_1^2 + r_2^2}, s := \arctan \frac{r_2}{r_1}, x, y\right) \end{array}$$

Hence, the cone is decomposable but not flat (provided that the M_i's do not have constant curvature 1).

Example: Gallot's Thm is false for indefinite metrics

Consider a complete Riemannian manifold (N, h), not of constant sectional curvature 1. Form a Lorentzian manifold

$$\big(M = \mathbb{R} \times N, g = -ds^2 + \cosh^2(s)h, \big)$$

which is still complete. Consider the cone $\mathcal{C}^+(M)$ and the vector field

$$V := \cosh^2(s)\partial_r - \frac{1}{r}\sinh(s)\cosh(s)\partial_s$$

of length $\cosh^2(s) > 0$. Then $\mathbb{R} \cdot V \oplus TN \subset T\hat{M}$ defines a non-degenerate holonomy-invariant subspace, i.e. the cone is a product but not flat.

$$\hat{\nabla}_{\partial_r} V = 0, \quad \hat{\nabla}_{\partial_s} V = \tanh(s)V, \quad \hat{\nabla}_X V = \frac{1}{r}X$$
$$\hat{\nabla}_{\partial_r} Y = \frac{1}{r}Y, \quad \hat{\nabla}_{\partial_s} Y = \tanh(s)Y \quad \hat{\nabla}_X Y = \nabla^h_X Y - rh(X,Y)V$$

Local description of decomposable cones

Theorem 1

Let (M, g) be a pseudo-Riemannian manifold of signature (r, s), not necessarily complete, with decomposable cone $\mathcal{C}^+(M)$. Then

1. $Hol^0(M, g) = SO_0(r, s)$ *(as for the sphere).*
2. $\exists$ *an open end dense subset* $M^0 \subset M$ *that is locally isometric to* $U = (a, b) \times N_1 \times N_2$ *with either*

$$(a,b) \subset \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad g|_U = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2 \text{ or}$$

$$(a,b) \subset \mathbb{R}_+ \quad \text{and} \quad g|_U = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2$$

where (N_1, g_1) and (N_2, g_2) are pseudo-Riemannian manifolds of appropriate signatures.

Sketch of proof

$\mathcal{C}^+(M)$ decomposable $\Rightarrow$ $\exists \alpha \in C^\infty(\hat{M})$ and $S \in \Gamma(TM)$:

$$\begin{array}{ccccccc} T\hat{M} & = & V^1 & \oplus^\perp & V^2 & & \\ \partial_r & = & X_1 & \oplus & X_2 & = & \left[\alpha\partial_r + S\right] \oplus \left[(1-\alpha)\partial_r - S\right] \end{array}$$

Then $\langle S, S\rangle = \alpha(1-\alpha)$, $\langle X_1, X_1\rangle = \alpha$, $\langle X_2, X_2\rangle = (1-\alpha)$
Over $M^0 := \{\alpha \neq 0, 1\}$, $T\hat{M}$ decomposes into involutive non-degenerate distributions,

$$T\hat{M} = \mathbb{R}\cdot\partial_r \oplus^\perp \mathbb{R}\cdot S \oplus^\perp X_1^\perp \oplus^\perp X_2^\perp.$$

$\nabla r\partial_r = \mathrm{Id} \;\Rightarrow\; \{\partial_r \notin V^\pm\}$ dense $\;\Rightarrow\; M^0$ dense in M.
Finally, α is a function only of the flow parameter of S, say $\alpha = \alpha(s)$ and

$$\begin{array}{rccccc} \text{for } 0 < \alpha < 1 & : & \alpha' = -2\sqrt{\alpha - \alpha^2} & \Rightarrow & \alpha(s) = \cos^2(s+c) \\ \text{for } 1 < \alpha & : & \alpha' = 2\sqrt{\alpha^2 - \alpha} & \Rightarrow & \alpha(s) = \cosh^2(s+c) \end{array}$$

Decomposable cones over a complete base

Theorem 2

Let (M, g) be a complete pseudo-Riemannian manifold with decomposable cone. Then $\exists$ a dense $M^0 \subset M$ such that every connected component of M^0 is isometric to either

1. *a manifold with constant sectional curvature 1 (i.e. the cone is flat), or*
2. $N = \mathbb{R}_+ \times N_1 \times N_2$ *with metric*

$$h = -ds^2 + \cosh^2(s)g_1 + \sinh^2(s)g_2,$$

where (N_i, g_i) are pseudo-Riemannian manifolds and (N_2, g_2) has constant curvature -1 or $\dim \leq 2$. Here the cone is a product of cones.

Idea of proof: Use previous theorem and study geodesics. Completeness excludes the cos / sin-case.

Decomposable cones over a complete and compact base

Theorem 3

Let (M, g) be a complete and compact pseudo-Riemannian manifold with decomposable cone. Then the cone is flat, and hence, (M, g) has constant sectional curvature 1.

Idea of proof: M compact and complete $\Rightarrow \alpha \in [0, 1]$, i.e. S space-like. Use original proof of Gallot.

Corollary

The cone over a simply-connected, compact manifold with complete indefinite metric is indecomposable.

(Indefinite space forms are never compact.)

Para-Kähler cones

Assume now that that the cone admits an invariant degenerate distribution and an invariant complement, i.e $T\hat{M} = V^+ \oplus^* V^-$ with V^+, V^- invariant and totally null.

This corresponds to a para-Kähler structure J on $(\hat{M}, \hat{g})$:= like Kähler, $\hat{\nabla} J = 0$, but with $J^2 = \mathrm{Id}$ and $J^* \hat{g} = -\hat{g}$.

$\hat{g}$ has neutral signature (n, n) and $V^{\pm} = Eig_{\pm 1}(J)$.

Theorem 4

Let (M, g) be a pseudo-Riemannian manifold of signature $(n, n-1)$. $C^+(M)$ admits a para-Kähler structure $\iff$ (M, g) admits a para-Sasaki structure.

Definition

A para-Sasaki structure on a pseudo-Riemannian manifold of signature $(n, n-1)$ is a time-like, geodesic Killing vector field ξ such that $\Phi = \nabla \xi$ satisfies $\Phi^2 = \mathrm{Id} + g(., \xi)\xi$ and $(\nabla_X \Phi) Y = -g(X, Y)\xi + g(\xi, Y)X$.

Lorentzian cones

For Lorentzian cones, only one situation remains:
indecomposable, but with invariant degenerate subspace.
Here, the cone admits a bundle of parallel null lines in $T\hat{M}$.
For cones, this implies that, locally, the cone has a parallel null vector field.

Theorem 5

Let $\mathcal{C}^+(M)$ be a Lorentzian cone over a manifold M with either

- *a negative definite metric g, or*
- *a Lorentzian metric of signature $(+\dots+-)$,*

and assume that $\mathcal{C}^+(M)$ admits a parallel null vector field. Then there is a dense $M^0 \subset M$ that is locally isometric to

$$-ds^2 + \mathrm{e}^{-2s}h$$

for a definite metric h.

Application: Lorentzian manifolds with real Killing spinors

Corollary (Bohle '99, Baum '99)

Let (M, g) be a geodesically complete Lorentzian manifold with real Killing spinor to the Killing number $1/2$. Then (M, g) is either

- *of constant sectional curvature 1, or*
- *isometric to $(\mathbb{R} \times N, -ds^2 + \cosh^2(s)h)$ with a Riemannian manifold (N, h) with real Killing spinors,*

or $M\backslash$(hyper surface) is isometric to a union of warped products of the form $(\mathbb{R} \times N, -ds^2 + e^{2s}h)$ with a Riemannian (N, h) with parallel spinor.

Proof with our results: M has real Killing spinor $\Rightarrow \hat{M}$ has parallel spinor φ and hence a parallel vector field V_φ with $\langle V_\varphi, V_\varphi\rangle \leq 0$.
$\langle V_\varphi, V_\varphi\rangle < 0$: apply Theorem 2.
$\langle V_\varphi, V_\varphi\rangle = 0$: apply Theorem 5 and see that $Hol(M, g) = Hol(N, h) \ltimes \mathbb{R}^n$ which implies that (N, h) has a parallel spinor.

Thank you very much!