Holonomy groups

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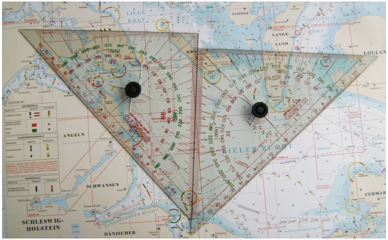
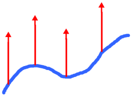
The notion of holonomy groups is based on

Parallel translation



Let $\gamma : [0,1] \to \mathbb{R}^2$ be a curve, and $X : [0,1] \to \mathbb{R}^2$ a vector field along this curve, then $X(t)$ is parallel translated along γ if

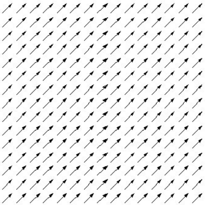
$$X' := \frac{\mathrm{d}X}{\mathrm{d}t} \equiv 0.$$



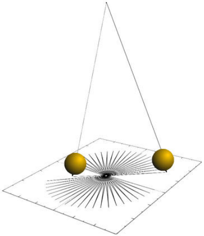
If we parallel translate a vector to every point in $\mathbb{R}^2$ we obtain a vector field Y on $\mathbb{R}^2$ that is constant, i.e

$$D_V Y \equiv 0, \quad \text{for all vectors } V,$$

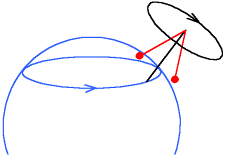
$DY = \left(\partial_i Y^j\right)$ denotes the Jacobian matrix of Y and $D_V Y$ its multiplication with the vector V.



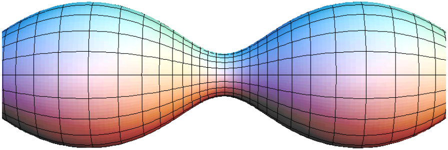
But: We live on something curved and thus cannot use rulers.



Instead, use a pendulum:
If we travel along a curve on the sphere, the oscillation plane of the pendulum is parallel transported along the curve.

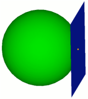


This works on every curved surface if we assume that there is a force *orthogonal to the surface* that pulls the pendulum down.



Let S be a surface (e.g. the sphere), $\gamma : [0,1] \to S$ a curve and $X : [0,1] \to \mathbb{R}^3$ a vector field along γ tangential to S.

X is parallel translated along γ if



(1) X is tangential to S: $\langle X, \gamma \rangle \equiv 0$

(2) because of (1), X cannot be constant but it changes only in directions orthogonal to the surface, i.e. the projection of $X' := \frac{\mathrm{d}X}{\mathrm{d}t}$ onto the tangent plane $T_{\gamma(t)}S$ vanishes:

$$X' - \langle X', \gamma \rangle \gamma \equiv 0$$

Now (1) implies that

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}t} \langle X, \gamma \rangle = \langle X', \gamma \rangle + \langle X, \gamma' \rangle$$

and (2) then becomes

$$(*) \qquad X' + \langle X, \gamma' \rangle \gamma \equiv 0$$

Hence, X is parallel transported along γ iff X satisfies the system $(*)$ of linear ODE's.

The parallel transport is a linear isomorphism between the tangent spaces

$$\begin{array}{rcl} P_\gamma \;:\; T_{\gamma(0)}S & \to & T_{\gamma(1)}S \\ X_0 & \to & X(1) \end{array}$$

where $X(t)$ is the solution to $(*)$ with initial condition $X(0) = X_0$.

P_γ is a linear isometry, because for X, Y parallel along γ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X, Y\rangle = \langle X', Y\rangle + \langle X, Y'\rangle \overset{(*)}{=} -\langle X, \gamma'\rangle\langle Y, \gamma\rangle - \langle Y, \gamma'\rangle\langle X, \gamma\rangle \overset{(1)}{=} 0,$$

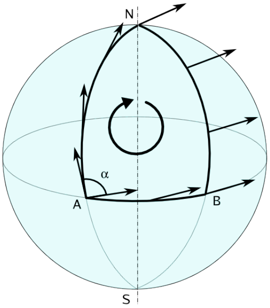
and thus $\langle X(t), Y(t)\rangle \equiv \langle X_0, Y_0\rangle$ for all t, in particular for $t = 1$.

Define the holonomy group of S at a point $x \in S$:

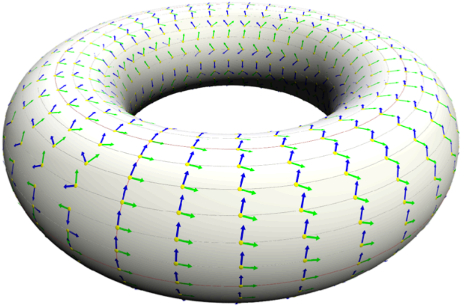
$$Hol_x := \{P_\gamma \mid \gamma \text{ a loop with } \gamma(0) = \gamma(1) = x\} \subset GL(T_x S)$$

Note that Hol_x does not depend on x.
As the P_γ's are isometries (and assuming that S is oriented), Hol is a subgroup of the group of rotations of the tangent plane at x, which is $\mathrm{SO}(2)$.

For the sphere we get $Hol = \mathrm{SO}(2)$:



The same for the torus:



In the same way we generalise the derivative of smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ to surfaces obtaining a linear connection ∇ by projecting $D_V Y$ to the tangent plane:

$$\nabla_V Y|_x := \mathrm{proj}_{T_x S}\left(D_V Y|_x\right).$$

We say that a vector field Y on S is (covariantly) constant if

$$\nabla_V Y \equiv 0 \quad \forall \text{ vectors } V.$$

But: If $Hol \neq \{Id\}$, transporting a vector parallel to any point on S does not define a constant vector field Y with $\nabla_V Y \equiv 0$, because parallel transport depends on the chosen path.

All this can be generalised to

- n-dimensional surfaces: $Hol \subset \mathrm{SO}(n)$ (in an obvious way)
- n-dimensional manifolds that carry a Riemannian metric

$$g_{ij} = \text{ family of scalar products on the tangent spaces.}$$

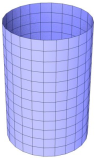
What are the possible holonomy groups of Riemannian manifolds?

Why do we want to know this?

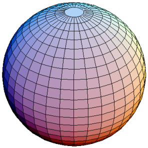
Holonomy groups are algebraic objects that encode geometric information about the Riemannian manifold, e.g.

Product manifolds:

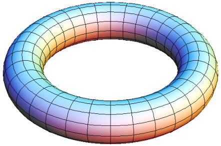
The manifold is a Cartesian product $M = M_1 \times M_2$ and metric is a sum $g = g_1 + g_2$ of metrics on M_1 and M_2. E.g.



The cylinder is a product of the circle S^1 and $\mathbb{R}$.



The sphere is not a product.



The torus is a product of two circles S^1, BUT the metric is not a sum.

All this is detected by the holonomy group.

Decomopsition Theorem [G. de Rham, Math. Helv. 1952]

A Riemannian manifold (that is complete and simpy connected) is a product of Riemannian manifolds $M = M_1 \times \ldots \times M_k$ if and only if its holonomy group is block diagonal

$$Hol = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_k \end{pmatrix}$$

In this case $H_i \subset \mathrm{SO}(\dim M_i)$ is the holonomy of M_i.

For example:

$$\left\{ y \in \mathbb{R}^n \text{ with } Hol(y) = y \right\} \quad \simeq \quad \left\{ \begin{array}{l} \text{constant vector fields } Y \\ \nabla_V Y \equiv 0 \text{ for all } V \end{array} \right\}$$

Y is obtained from y by parallel transport, which, for y, is independent of the chosen path.

Berger's list [M. Berger, *Bull. Soc. Math. France*, 1955]

The holonomy group of a (complete, simply connected) n-dim'l Riemannian manifold that is not a product is isomorphic to one of

$\mathrm{SO}(n)$	group of rotations of $\mathbb{R}^n$	generically curved
$\cup$		
$\mathrm{U}(n/2)$	unitary matrices of $\mathbb{C}^{n/2} = \mathbb{R}^n$	$\mathbb{C}$-manifold, i.e. holomorphic coordinates, "Kähler"
$\cup$		
$\mathrm{SU}(n/2)$	unitary matrices of determinant 1	$\exists$ holomorphic volume form, "special Kähler"
$\cup$		
$\mathrm{Sp}(n/4)$	quaternionic unitary matrices of $\mathbb{H}^{n/4} = \mathbb{R}^n$	$\exists$ quaternionic structure, "hyper-Kähler"
$\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$	unit quaternions times $\mathrm{Sp}(n/4)$	"quaternionic-Kähler"
$\mathrm{G}_2 \subset \mathrm{SO}(7)$	exceptional Lie group	$\exists$ octonionic structure, "G_2-manifold"
$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	universal cover of $\mathrm{SO}(7)$ with spin representation	"Spin(7)-manifold"

That's all! (... apart from symmetric spaces ...)

It tells us a lot about the possible geometry of the manifold, e.g. if $Hol \subset \mathrm{U}(n/2)$ then the manifold is complex ($\exists$ holomorphic coordinates).

Question: $\exists$? Riemannian metrics for all groups G on Berger's list?

"Obvious" examples:

- sphere, hyperbolic space: $Hol = \mathrm{SO}(n)$
- complex projective space $\mathbb{CP}^n$: $Hol = \mathrm{U}(n)$
- quaternionic projective space $\mathbb{HP}^n$: $Hol = \mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$.

Local existence: Construct metric with $Hol = G$ on small open set.

- $\mathrm{SO}(n)$: generic Riemannian manifold
- $\mathrm{U}(n)$: Kähler metric given by generic Kähler potential on complex manifold
- $\mathrm{SU}(n)$: elliptic equation on the Kähler potential in order to get holomorphic volume form
- $\mathrm{Sp}(n)$: hyper-Kähler metric on $T^*\mathbb{CP}^n$ [Calabi, *Ann. ENS* '79]
- G_2 and $\mathrm{Spin}(7)$: metrics exist [Bryant, *Ann. Math* '87], general method to describe space of local metrics as solutions to system of overdetermined PDE's

Compact examples:

- $\mathrm{U}(n)$: compact smooth varieties.
- $\mathrm{SU}(n)$: Yau's solution of the Calabi conjecture [Com. Pure Appl. Math 1978]:

$$\left\{ \begin{array}{l} \text{compact Kähler mf's with} \\ \text{trivial canonical bundle} \end{array} \right\} = \left\{ \begin{array}{l} \text{compact complex mf's} \\ \text{admitting an } \mathrm{SU}(n)\text{-metric} \end{array} \right\}$$

  Problem: non constructive proof, only very few explicit "Calabi-Yau metrics" known.

- Construction of compact G_2 and $\mathrm{Spin}(7)$ manifolds by Joyce [J. Diff. Geom. & Inv. Math. 1996]

Relevance for physics

Let $M^{1,n-1}$ be a space time, i.e. manifold with $(-+\cdots+)$ metric g_{ij}

- General relativity: $n = 4$ and the metric g_{ij} satisfies

$$R_{ij} - \tfrac{1}{2} R g_{ij} = T_{ij} \quad \text{(Einstein eq's)}$$

- String theory (and the like): $n = 10, 11, 12$, Einsteins equations and spinor field equations of the form

$$\nabla_X \psi = F(X) \cdot \psi \quad \text{(“preserved supersymmetry”)}$$

  Simplified versions (vacua)

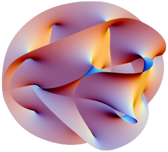
$$\begin{aligned} R_{ij} &= 0 \quad \text{(Ricci flat)} \\ \nabla \psi &= 0 \quad \text{(constant spinor)} \end{aligned}$$

Ansatz: Product structure of the space time

$$M^{1,n-1} = \mathbb{R}^{1,3} \times X^k,$$

X^k compact Riemannian manifold, k=6,7,8, with constant spinor ψ.
This implies $Hol(X)\psi = \psi$ and hence (by Berger's list)

$k = 6$	$Hol(X) \subset \mathrm{SU}(3)$	Calabi-Yau 3-manifold
$k = 7$	$Hol(X) \subset \mathrm{G}_2$	G_2-manifold
$k = 8$	$Hol(X) \subset \mathrm{Spin}(7)$	$\mathrm{Spin}(7)$-manifold
or	$Hol(X) \subset \mathrm{SU}(4)$	Calabi-Yau 4-manifold



More general: $M^{1,n-1}$ is not a product
Question: What are holonomy groups of n-dim'l space times (with constant spinors)?

Holonomy groups of space times (Lorentzian manifolds)

The holonomy group of a space time of dim n that is not a product and carries constant spinors is equal to

$$G \ltimes \mathbb{R}^{n-2} \quad (*)$$

where G is a product of $\mathrm{SU}(k)$, $\mathrm{Sp}(l)$, G_2 or $\mathrm{Spin}(7)$ [TL '03].
If it does not admit constant spinors then it is equal to

- ... something similar to $(*)$ ... [Berard-Bergery '96, TL '03], or
- the full Lorentz group $\mathrm{SO}(1, n-1)$ [Berger '55].

For all possible groups there exist examples.

Thank you!