Holonomy groups

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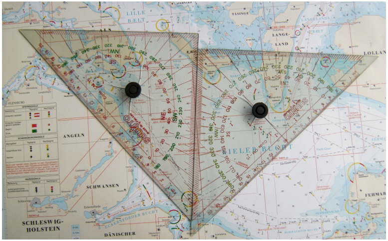
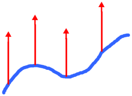
The notion of holonomy groups is based on

Parallel translation



Let $\gamma : [0,1] \to \mathbb{R}^2$ be a curve, and $X : [0,1] \to \mathbb{R}^2$ a vector field along this curve, then $X(t)$ is parallel translated along γ if

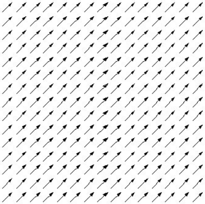
$$X' := \frac{\mathrm{d}X}{\mathrm{d}t} \equiv 0.$$



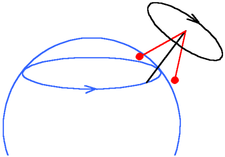
If we parallel translate a vector to every point in $\mathbb{R}^2$ we obtain a vector field Y on $\mathbb{R}^2$ that is constant, i.e

$$D_V Y \equiv 0, \quad \text{for all vectors } V,$$

$DY = \left(\partial_i Y^j\right)$ denotes the Jacobian matrix of Y and $D_V Y$ the derivative of Y in direction of the vector V.



On curved surfaces parallel transport is more tricky: Travelling along a curve on the sphere, the oscillation plane of the pendulum is parallel transported along the curve.



Let S be a surface (e.g. the sphere), $\gamma : [0,1] \to S$ a curve and $X : [0,1] \to \mathbb{R}^3$ a vector field along γ tangential to S.

X is parallel translated along γ if

(1) X is tangential to S: $\langle X, \gamma \rangle \equiv 0$

(2) X changes only in directions orthogonal to the surface, i.e. the projection of $X' := \frac{\mathrm{d}X}{\mathrm{d}t}$ onto the tangent plane $T_{\gamma(t)}S$ vanishes:

$$X' - \langle X', \gamma \rangle \gamma \equiv 0$$

This implies: X is parallel transported along γ iff X satisfies the system of linear ODE's:

$$(*) \qquad X' + \langle X, \gamma' \rangle \gamma \equiv 0$$

The parallel transport is an isomorphism of the tangent spaces,

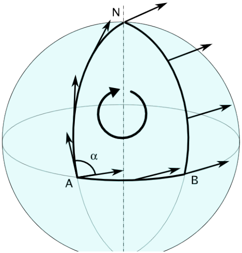
$$P_\gamma : T_{\gamma(0)}S \to T_{\gamma(1)}S\,, \quad P_\gamma(X_0) = X(1),$$

where $X(t)$ is the solution to $(*)$ with initial condition $X(0) = X_0$.
Define the holonomy group of S at a point $x \in S$:

$$Hol_x^0 := \{P_\gamma \mid \gamma \text{ a loop with } \gamma(0) = \gamma(1) = x\}, \gamma \text{ contractible}\}$$

- Hol_x is contained in the group of rotations of the tangent plane at x. Hol_x and Hol_y are conjugated to each other in $\mathrm{O}(2)$.
- Hol^0 is connected and $\pi_1(S) \twoheadrightarrow Hol/Hol^0$.

For the sphere $Hol = \mathrm{SO}(2)$:



Note: In contrast to the flat case on slide 1, if $Hol \neq \{\mathbf{1}\}$, transporting a vector parallel to any point on S does not define a constant vector field Y with $D_V Y \equiv 0$, because parallel transport depends on the chosen path.

All this can be generalised to

- n-dimensional surfaces: $Hol \subset \mathrm{SO}(n)$ (in an obvious way)
- n-dimensional manifolds that carry a Riemannian metric g_{ij}.
  - Parallel transport defined using the Levi-Civita connection ∇,

$$\nabla_{\gamma'} X|_t \equiv 0, \quad \text{or} \quad (\xi^k)' + \Gamma^k_{ij}(\gamma^i)'\xi^j \equiv 0 \ \text{ if } X = \xi^k \frac{\partial}{\partial x^k}.$$

  - $Hol(M, g)$ is a Lie group with Lie algebra $\mathfrak{hol}(M, g)$.

Classification problem:
What are the possible holonomy groups of Riemannian manifolds?

Why do we want to know this?

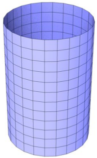
- Holonomy groups encode geometric information
- Information about solutions to "geometric" differential equations can be obtained by algebraic means.

For example: covariantly constant vector fields

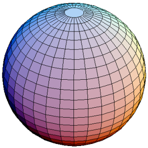
$$\left\{ y \in \mathbb{R}^n \text{ with } Hol(y) = y \right\} \simeq \left\{ \begin{array}{l} \text{constant vector fields } Y \\ \nabla_V Y \equiv 0 \text{ for all } V \end{array} \right\}$$

- Get Y from y by parallel transport (y inv. $\Rightarrow$ indep. of path).
- The same applies to other geometric vector bundles

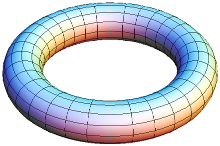
Product manifolds: The manifold is a Cartesian product $M = M_1 \times M_2$ and metric is a sum $g = g_1 + g_2$ of metrics on M_1 and M_2.



The cylinder is a product of the circle S^1 and $\mathbb{R}$.



The sphere is not a product.



The torus is a product of two circles S^1, BUT the metric is not a sum.

- Product structure of a Riemannian manifold can be detected by the holonomy group.

Decomopsition Theorem [G. De Rham, *Math. Helv.*, '52]

A Riemannian manifold (complete and simpy connected) is a product of Riemannian manifolds $M = M_1 \times \ldots \times M_k \iff$ *its holonomy group is a product group* $H_1 \times \ldots \times H_k$ *acting block diagonal*

$$Hol = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_k \end{pmatrix}$$

In this case $H_i \subset \mathrm{SO}(\dim M_i)$ *is the holonomy of* M_i*.*

Holonomy and curvature

- $\nabla_{\dot\gamma(0)}X|_p = \frac{d}{dt}\left[\mathcal{P}^{-1}_{\gamma|_{[0,t]}}(X(\gamma(t)))\right]|_{t=0}$.
- curvature $\mathcal{R}$ of ∇: $X, Y \in T_pM$, extended such that $[X, Y] = 0$ and λ_t parallelogram of flows of X and Y of length $\sqrt{t} \Rightarrow$

$$\mathcal{R}(X, Y)|_p = \lim_{t\to 0}\frac{1}{t}\left(\mathcal{P}_{\lambda_t} - Id_{T_pM}\right).$$

Hence, $\mathcal{R}(X, Y)|_p \in \mathfrak{hol}_p(M, g)$ $\forall X, Y \in T_pM$.

Theorem (Ambrose & Singer [*Trans. AMS* '53])

If M is connected, then $\mathfrak{hol}_p(M, g)$ is spanned by

$$\left\{ \mathcal{P}_\gamma^{-1} \circ \mathcal{R}(X, Y) \circ \mathcal{P}_\gamma \in \mathrm{GL}(T_pM) \mid \gamma(0) = p \text{ und } X, Y \in T_{\gamma(1)}M \right\}$$

Bianchi-Identity for $\mathcal{R} \implies \mathfrak{hol}_p(M, g)$ is a Berger algebra, i.e.,

$$\mathfrak{hol} = \mathrm{span}\left\{R(x, y) \mid R \in \mathcal{K}(\mathfrak{hol}), x, y \in \mathbb{R}^n\right\},$$

with

$$\mathcal{K}(\mathfrak{g}) := \left\{ R \in \Lambda^2\mathbb{R}^{n^*} \otimes \mathfrak{g} \mid R(x, y)z + R(y, z)x + R(z, x)y = 0\right\}.$$

Berger's list [Berger, *Bull. Soc. Math. France*, '55]

The holonomy group of a (simply connected) n-dim'l Riemannian manifold that is not a locally product is isomorphic to one of

$\mathrm{SO}(n)$	group of rotations of $\mathbb{R}^n$	generically curved
$\mathrm{U}(n/2)$	unitary matrices of $\mathbb{C}^{n/2} = \mathbb{R}^n$	$\mathbb{C}$ manifold, i.e. holomorphic coordinates, "Kähler"
$\mathrm{SU}(n/2)$	unitary matrices of determinant 1	∃ holomorphic volume form, "special Kähler"
$\mathrm{Sp}(n/4)$	quaternionic unitary matrices of $\mathbb{H}^{n/4} = \mathbb{R}^n$	∃ quaternionic structure, "hyper-Kähler" Kähler + parallel holomorphic symplectic form
$\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$	unit quaternions times $\mathrm{Sp}(n/4)$	"quaternionic Kähler"
$\mathrm{G}_2 \subset \mathrm{SO}(7)$	exceptional Lie group	∃ octonionic structure, "G_2-manifold"
$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	universal cover of $\mathrm{SO}(7)$ with spin representation	"Spin(7)-manifold"

That's all! (... apart from symmetric spaces ...)

It tells us a lot about the possible geometry of the manifold, e.g. if *Hol* $\subset \mathrm{U}(n/2)$ then the manifold is complex (∃ holomorphic coordinates), or if the manifold admits constant spinor fields

Question: $\exists$? Riemannian metrics for all groups G on Berger's list?

Symmetric examples:

- sphere, hyperbolic space: $Hol = \mathrm{SO}(n)$
- complex projective space $\mathbb{CP}^n$: $Hol = \mathrm{U}(n)$
- quaternionic projective space $\mathbb{HP}^n$: $Hol = \mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$.

Local existence: Construct metric with $Hol = G$ on open set in $\mathbb{R}^n$.

- $\mathrm{SO}(n)$: generic Riemannian manifold
- $\mathrm{U}(n)$: Kähler metric given by generic Kähler potential on complex manifold
- $\mathrm{SU}(n)$: elliptic equation on the Kähler potential in order to get holomorphic volume form
- $\mathrm{Sp}(n)$: hyper-Kähler metric on $T^*\mathbb{CP}^n$ [Calabi, *Ann. ENS* '79]
- G_2 and $\mathrm{Spin}(7)$: metrics exist [Bryant, *Ann. Math* '87], general method to describe space of local metrics as solutions to system of overdetermined PDE's

Compact examples:

- $\mathrm{SU}(n)$: Yau's solution of the Calabi conjecture [Com. Pure Appl. Math 1978]:

$$\left\{ \begin{array}{l} \text{compact Kähler mf's with} \\ \text{trivial canonical bundle} \end{array} \right\} = \left\{ \begin{array}{l} \text{compact complex mf's} \\ \text{admitting an } \mathrm{SU}(n)\text{-metric} \end{array} \right\}$$

  Problem: non constructive proof, only very few explicit "Calabi-Yau metrics" known.

- $\mathrm{Sp}(n)$: examples by Fujiki, Mukai, and Beauville.
- $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$: symmetric spaces are the only known examples of compact manifolds with this holonomy
- Construction of compact G_2 and $\mathrm{Spin}(7)$ manifolds by Joyce [J. Diff. Geom. & Inv. Math. 1996]

Relevance for physics

Let $M^{1,n-1}$ be a space time, i.e. manifold with $(- + \cdots +)$ metric g_{ij}

- General relativity: $n = 4$ and the metric g_{ij} satisfies

$$R_{ij} - \frac{1}{2} R g_{ij} = T_{ij} \quad \text{(Einstein eq's)}$$

- String theory (and the like): $n = 10, 11, 12$, Einsteins equations and spinor field equations of the form

$$\nabla_X \psi = F(X) \cdot \psi \quad \text{("preserved supersymmetry")}$$

Simplified versions

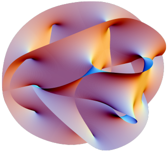
$$\begin{aligned} R_{ij} &= 0 \quad \text{(Ricci flat)} \\ \nabla \psi &= 0 \quad \text{(constant spinor)} \end{aligned}$$

Ansatz: Product structure of the space time

$$M^{1,n-1} = \mathbb{R}^{1,3} \times X^k,$$

X^k compact Riemannian manifold, k=6,7,8, with constant spinor ψ. This implies $Hol(X)\psi = \psi$ and hence (by Berger's list)

$k = 6$	$Hol(X) \subset \mathrm{SU}(3)$	Calabi-Yau 3-manifold
$k = 7$	$Hol(X) \subset \mathrm{G}_2$	G_2-manifold
$k = 8$	$Hol(X) \subset \mathrm{Spin}(7)$	$\mathrm{Spin}(7)$-manifold
or	$Hol(X) \subset \mathrm{SU}(4)$	Calabi-Yau 4-manifold



More general: $M^{1,n-1}$ is not a product

Question: What are holonomy groups of n-dim'l space-times (with constant spinors)?

Holonomy groups of Lorentzian manifolds (= space-times)

Let H be the holonomy group of a space time of dimension $n+2$ that is not locally a product. Then

- either H is the full Lorentz group $\mathrm{SO}(1, n+1)$ [Berger '55] or $H \subset (\mathbb{R}^+ \times \mathrm{SO}(n)) \ltimes \mathbb{R}^n \; =$ stabiliser of a null line,
- $G := pr_{SO(n)}(H)$ is a Riemannian holonomy group [TL, *J. Differential Geom.*, '07]
- H=$G \ltimes \mathbb{R}^n$ or $H = (\mathbb{R} \times G) \ltimes \mathbb{R}^n$, or $(L \times S) \ltimes \mathbb{R}^{n-k}$, where S is the semisimple part of G and $L \subset \mathbb{R}^+ \times Z(G)$ or $L \subset \mathbb{R}^k \times Z(G)$ [Bérard-Bergery & Ikemakhen, *Proc. Symp. Pure Math.* '93].

If the space time admits a constant spinor, then

$$G \ltimes \mathbb{R}^{n-2}$$

where G is a product of $\mathrm{SU}(k)$, $\mathrm{Sp}(l)$, G_2 or $\mathrm{Spin}(7)$ [TL '07].
For all possible groups there exist examples [... , Galaev '06].

Further applications to spinor field equations

Results from holonomy theory can be applied to more general spinor field equations, such as the Killing spinor equation:
A spinor field φ is a *Killing spinor* to the *Killing number* $\lambda \in \mathbb{C}$ if

$$\nabla_X \varphi = \lambda X \cdot \varphi.$$

Manifolds with Killing spinor are Einstein .

A manifold M has a Killing spinor with real Killing no.	$\Longleftrightarrow$	The cone over M admits a constant spinor.

"Bär-correspondence" [Bär, *Comm. Math. Phys.* '93]

Let M be a complete Riemannian spin manifold with a real Killing spinor. Then M is S^n, or a compact Einstein space with one of the following structures: Sasaki, 3-Sasaki, 6-dim. nearly-Kähler, or nearly parallel G_2.

The Bär correspondence is based on

Theorem (Gallot [*Ann. Sci. Ec. Norm. Sup. '79*])

Let M be a Riemannian manifold that is geodesically complete. If the cone over M is a Riemannian product, then the cone is flat and M has constant sectional curvature 1*.*

Corollary: If M^{n-1} has a real Killing spinor, then cone is flat or has holonomy $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$, G_2 or $\mathrm{Spin}(7)$.
This implies Bär's correspondence.

Generalisation to manifolds with indefinite metrics
[Alekseevsky, Cortés, Galaev & TL, *Crelle's Journal* '09]:

- Same result, under the additional assumption: *M* compact.
- Detailed description in cases when one of the assumptions fails and for Lorentzian and para-Kähler cones.

Work in progress:
Apply these results to Killing spinor equation for indefinite metrics.

Construction of manifolds with exceptional holonomy

- Long history for Riemannian manifolds: Calabi, Yau, Le Brun, Bryant, Salomon, Joyce ...
- Only few attempts for indefinite metrics.
- Method that can be generalised to indefinite metrics: Hitchin flow

Half flat structures

Let M be a 6-manifold. Two stable forms $\rho \in \Lambda^3 M$ and $\omega \in \Lambda^2 M$ such that

$$\omega \wedge \rho = 0, \quad \mathrm{d}\rho = 0, \quad \mathrm{d}(\omega \wedge \omega) = 0,$$

are called "half-flat structure".
Note: ω and ρ define a (non-integrable) complex strucure J on M.

Hitchin flow for half-flat structures

[Hitchin, *J. Differential Geom.* '00 (M compact Riemannian), Cortés, Schäfer, Schulte-Hengesbach & TL, *Proc. LMS* '10]

Let M be a 6-manifold with a half-flat structure (ρ, ω). Then there is a one-parameter family ω_t and ρ_t satisfying the Hitchin flow equations

$$\partial_t \rho = \mathrm{d}\omega \,, \quad \partial_t(\omega \wedge \omega) = \mathrm{d}(J^* \rho)$$

with initial conditions $\omega_0 = \omega$ and $\rho_0 = \rho$. This family defines a parallel G_2-structure on $M \times [a, b]$ via $\varphi = \omega \wedge \mathrm{d}t + \rho$.

- Construction of explicit examples with $Hol = \mathrm{G}_2$ starting from homogeneous half-flat structures on 6-dimensional solvable Lie groups.

Thank you!