Special holonomy in Lorentzian geometry

Thomas Leistner

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Outline

1. Holonomy groups
   - The holonomy group of a linear connection
   - Classification problem and Berger algebras
   - Holonomy and geometric structure
   - Riemannian holonomy

2. Holonomy groups of Lorentzian manifolds
   - Special Lorentzian holonomy
   - Classification and Applications
   - Constructing metrics
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3. References
Holonomy group of a linear connection

- Let $M$ be a smooth manifold and $\nabla$ a linear connection.
  - $\sim$ Parallel transport along $\gamma : [0, 1] \rightarrow M$, piecewise smooth,
    
    $$\mathcal{P}_\gamma : T_{\gamma(0)}M \ni X_0 \mapsto \sim X(1) \in T_{\gamma(1)}M$$

    where $X(t)$ is the solution to the ODE $\nabla_{\dot{\gamma}(t)} X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (Connected) Holonomy group

$$\text{Hol}_p^0(M, \nabla) := \left\{ \mathcal{P}_\gamma | \gamma(0) = \gamma(1) = p, \gamma \sim \{p\} \right\} \subset \text{GL}(T_pM) \cong \text{GL}(n, \mathbb{R})$$

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$.

- For $p, q \in M$:
  - $\text{Hol}_p(M, \nabla) \sim \text{Hol}_q(M, \nabla)$
  - If $\nabla = \text{LC}$ of a metric $g$ on $M$, then $\text{Hol}_p(M, g) \subset O(T_pM, g) = O(t, s)$. 
Holonomy and curvature

- Recall that $\nabla$ and $\mathcal{P}_\gamma$ are related via
  \[
  \nabla_{\dot{\gamma}(0)} X|_p = \frac{d}{dt} \left[ \mathcal{P}^{-1}_{\gamma|[0,t]} (X(\gamma(t))) \right]|_{t=0}.
  \]

- This implies for the curvature $\mathcal{R}$ of $\nabla$: Let $X, Y \in T_p M$ and $\lambda_t$ the loop along the parallelogram at $p$ with sides $\sqrt{t}X$ and $\sqrt{t}Y$. Then
  \[
  \mathcal{R}(X, Y)|_p = \lim_{t \to 0} \frac{1}{t} \left( \mathcal{P}_{\lambda_t} - Id_{T_p M} \right).
  \]

  Hence, $\mathcal{R}(X, Y)|_p \in \mathfrak{hol}_p(M, \nabla)$ for all $X, Y \in T_p M$.

- One has to collect curvature all over $M$ to get all of $\mathfrak{hol}_p(M, \nabla)$:

Theorem (Ambrose-Singer '53)

*If $M$ is connected, then $\mathfrak{hol}_p(M, \nabla)$ is spanned by*

\[
\left\{ \mathcal{P}^{-1}_\gamma \circ \mathcal{R}(X, Y) \circ \mathcal{P}_\gamma \in \text{GL}(T_p M) \mid \gamma(0) = p \text{ and } X, Y \in T_{\gamma(1)} M \right\}
\]
Classification: Which groups occur as holonomy groups?

- **Hano/Ozeki ’56:** Any closed $G \subset \text{GL}(n, \mathbb{R})$. But $\nabla$ might have torsion.
- Conditions on the torsion $T^\nabla$ of $\nabla$, e.g. $T^\nabla = 0$ or $T^\nabla \in \Lambda^3 TM$
  $\leadsto$ algebraic constraints on the holonomy representation:
  \[ R_\gamma := P_\gamma^{-1} \circ R(P_\gamma(\cdot), P_\gamma(\cdot)) \circ P_\gamma \in \Lambda^2(T_p^* M) \otimes \text{GL}(T_p M) \]

Now, if $T^\nabla = 0$, then $\mathcal{R}$ and hence $\mathcal{R}_\gamma$ satisfy the Bianchi identity:

\[ \mathcal{R}_\gamma(X, Y)Z + \mathcal{R}_\gamma(Y, Z)X + \mathcal{R}_\gamma(Z, X)Y = 0. \]

$\implies$ $\mathfrak{hol}_p(M, \nabla)$ is a Berger algebra: For $g \subset \mathfrak{gl}(n, \mathbb{R})$ define the $g$-module:

\[ \mathcal{K}(g) := \left\{ R \in \Lambda^2 \mathbb{R}^{n^*} \otimes g \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \right\} \]

$g$ is a Berger algebra $\iff$ $g = \text{span} \{ R(x, y) \mid R \in \mathcal{K}(g), x, y \in \mathbb{R}^n \}$.

$T^\nabla = 0$: Ambrose-Singer $\implies$ $\mathfrak{hol}_p(M, \nabla)$ is a Berger algebra.

Classification of irreducible Berger algebras

- $g \subset \mathfrak{so}(t, s)$ [Berger ’55], $g \subset \mathfrak{gl}(n, \mathbb{R})$ [Schwachhöfer/Merkulov ’99].
Holonomy and geometry

Let $V$ be a “geometric” vector bundle over $M$ with connection $\nabla$. Then:

$$\{ v \in V_p \mid Hol_p(M, \nabla)(v) = v \} \ni v \simeq \varphi := P_\gamma(v) \in \{ \varphi \in \Gamma(V) \mid \nabla \varphi = 0 \}$$

$$\{ V \subset T_p M \mid Hol_p(M, \nabla)(V) \subset V \} \simeq \{ \text{Distrib. } \mathcal{V} \subset TM \mid \mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \}$$

$\mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \iff \nabla_X : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$, in particular, $\mathcal{V}$ is integrable.

Decomposition of a semi-Riemannian manifold $(M, g)$:

If $V \subset T_p M$ is $Hol(M, g)$-invariant, non-degenerate (i.e. $V \cap V^\perp = \{0\}$), i.e. $T_p M = V \oplus V^\perp$ invariant decomposition, then

$$(M, g) \overset{\text{locally}}{\simeq} (N, h) \times (N^\perp, h^\perp)$$

with $V(\perp) \simeq T_p N(\perp)$ as $Hol_p(M, g)$–module.
De Rham–Wu decomposition

Complete decomposition of $T_p M$ into $\text{Hol}_p(M, g)$–modules:

$$T_p M = \bigoplus_{i=0}^{k} V_i,$$

with $V_0$ trivial and $V_i$ indecomposable for $i > 0$

non-degenerate and only degenerate invariant subspaces

Theorem (de Rham ‘52, Wu ‘64)

Let $(M, g)$ be semi-Riemannian, complete and simply connected. Then there is a $k > 0$:

$$(M, g) \, \text{globally} \simeq (M_1, g_1) \times \ldots \times (M_k, g_k)$$

- $(M_i, g_i)$ complete and 1-connected,
- $(M_i, g_i)$ flat or with indecomposable holonomy representation,
- $\text{Hol}_p(M, g) \simeq \text{Hol}_{p_1}(M_1, g_1) \times \ldots \times \text{Hol}_{p_k}(M_k, g_k)$.

Manifold of special holonomy: Indecomposable holonomy $\not\subset \text{SO}(t, s)$. 
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible \(\implies\) \(\text{Hol}_p(M, g) \simeq\) product of irreducible holonomy groups.

Berger’s list (’55)

Let \((M, g)\) be 1-connected, irreducible, not locally symmetric. Then \(\text{Hol}_p(M, g) \overset{O(n)}{\sim}\)

<table>
<thead>
<tr>
<th>(\text{par. field})</th>
<th>(\text{SO}(n))</th>
<th>(\text{U}(\frac{n}{2}))</th>
<th>(\text{SU}(\frac{n}{2}))</th>
<th>(\text{Sp}(\frac{n}{4}))</th>
<th>(\text{Sp}(1) \cdot \text{Sp}(\frac{n}{4}))</th>
<th>(G_2)</th>
<th>(\text{Spin}(7))</th>
</tr>
</thead>
<tbody>
<tr>
<td>generic</td>
<td>Kähler</td>
<td>hyper Kähler</td>
<td>quat. Kähler</td>
<td>(\langle J_1, J_2, J_3 \rangle)</td>
<td>(\omega^3)</td>
<td>(\omega^4)</td>
<td></td>
</tr>
<tr>
<td>(\text{Ric})</td>
<td>(-)</td>
<td>(\neq 0)</td>
<td>0</td>
<td>0</td>
<td>(\Lambda \cdot g)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\text{dim}{\nabla \varphi = 0})</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>(\frac{n}{4} + 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Complete mf’s: Calabi (SU, Sp), LeBrun (qK), Bryant (\(G_2, \text{Spin}(7)\)).
- Compact mf’s: Yau (SU), Beauville, Mukai (Sp), LeBrun-Salamon (qK), Joyce (\(G_2, \text{Spin}(7)\)).
Yau ’93: “Berger has classified holonomy groups for Riemannian manifolds. If the metric is not Riemannian but allows different signature, the corresponding theorem of Berger should exist. ... In particular, classify complete Lorentzian manifolds with parallel spinors.”

Wu–Decomposition for a Lorentz manifold \((M, g)\)

Let \((M, g)\) be a complete, simply-connected Lorentzian manifold \(\implies\)

\[
(M, g) \cong (\overline{M}, \overline{g}) \times (N_1, g_1) \times \ldots \times (N_k, g_k)
\]

\[\uparrow\]

Riemannian, irreducible or flat

Lorentzian manifold which is either

1. \((\mathbb{R}, -dt^2)\), or
2. irreducible, i.e. \(\text{Hol}_p(\overline{M}, \overline{g}) = \text{SO}_0(1, n)\), or
3. indecomposable, non-irreducible
Algebraic preliminaries

Hence, an indecomposable Lorentzian manifold has special holonomy if and only if its holonomy admits a degenerate invariant subspace or it admits a parallel null line.

We have to consider $H \subset \text{SO}_0(1, n-1)$ indecomposable, non-irreducible, i.e. $\exists V \subset \mathbb{R}^{1,n-1} : H(V) \subset V$ such that

$L := V \cap V^\perp \neq \{0\}$ is a $H$-invariant, totally null line in $\mathbb{R}^{1,n-1}$

$$\Rightarrow H \subset \text{Iso}_{\text{SO}_0(1,n-1)}(L) = (\mathbb{R}^+ \times \text{SO}(n-2)) \ltimes \mathbb{R}^{n-2}$$

Change basis of $\mathbb{R}^{1,n-1}$: $\mathfrak{h} \subset \left\{ \begin{pmatrix} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{pmatrix} \right\}_{a \in \mathbb{R}, \ v \in \mathbb{R}^{n-2}, \ A \in so(n-2)}$

The orthogonal part is reductive:

$$\mathfrak{g} := pr_{so(n-2)}\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{g}' \quad (\text{Levi – decomposition})$$

$$\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}] \text{ semisimple}$$
Parallel null line and screen bundle

Let $(M, g)$ be a Lorentzian manifold with

\[ H := \text{Hol}_p(M, g) \subset \text{Iso}(L) = (\mathbb{R}^+ \times \text{SO}(n-2)) \times \mathbb{R}^{n-2}. \]

- $L$ defines a filtration $L \subset L^\perp \subset TM$ with parallel null line $\mathcal{L}$ and parallel null hypersurface $\mathcal{L}^\perp$.
- If $H \subset \text{SO}(n-2) \times \mathbb{R}^{n-2}$, then $\exists$ parallel null vector field (Brinkmann wave).
- What about $G := \text{pr}_{\text{SO}(n-2)} \text{Hol}_p(M, g) \subset \text{SO}(n-2)$?

**Proposition (TL ’05)**

The vector bundle ("screen bundle") $S = L^\perp / L$ with covariant derivative

\[ \nabla^S_U[V] := [\nabla_U V] \text{ satisfies } \text{pr}_{\text{SO}(n-2)} \text{Hol}_p(M, g) = \text{Hol}_p(S, \nabla^S). \]

- Hence, algebraic structures for $G$ correspond to geometric structures on $S$, e.g. product structure, parallel complex structure etc.
- Which $G$’s can occur?
Classification 1: $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ indecomposable

Theorem (Berard-Bergery/Ikemakhen '96)

For $\mathfrak{h}$ with $\mathfrak{g} := \text{pr}_{\mathfrak{so}(n-2)}\mathfrak{h} = \mathfrak{3} \oplus [\mathfrak{g}, \mathfrak{g}]$ there are the following cases:

1. Type I: $\mathbb{R}^{n-2} \subset \mathfrak{h}$
   - $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$.

2. Type II: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$.

3. Type III: $\exists \varphi : \mathfrak{3} \rightarrow \mathbb{R}$: $\mathfrak{h} = \begin{cases} \varphi(A) & v^t & 0 \\ 0 & A + B & -v \\ 0 & 0 & -\varphi(A) \end{cases} \begin{array}{l} A \in \mathfrak{3} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2} \end{array}$

4. Type IV: $\mathbb{R}^{n-2} \nsubseteq \mathfrak{h}$
   - $\exists \varphi : \mathfrak{3} \rightarrow \mathbb{R}^k$, for $0 < k < n - 2$:
     $\mathfrak{h} = \begin{cases} 0 & \psi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A + B & -v \\ 0 & 0 & 0 & 0 \end{cases} \begin{array}{l} A \in \mathfrak{3} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2-k} \end{array}$

Note: Groups of coupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen '96
Classification II: Let $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{s}0}(1,n-1)(L)$ be indecomposable.

**Theorem (TL ’03)**

If $\mathfrak{h}$ is a Berger algebra (e.g. a Lorentzian holonomy algebra), then $\mathfrak{g} := \text{proj}_{\mathfrak{s}0}(n-2)\mathfrak{h}$ is a Riemannian holonomy algebra (and hence known to be a product of algebras from Berger’s list).

**Idea of the proof:** For $\mathfrak{g} \subset \mathfrak{s}0(n)$ define weak curvature endomorphisms:

$$\mathcal{B}(\mathfrak{g}) := \{ Q \in \text{Hom}(\mathbb{R}^n, \mathfrak{g}) \mid \langle Q(x)y, z \rangle + \langle Q(y)z, x \rangle + \langle Q(z)v, y \rangle = 0 \}.$$ 

$\mathfrak{g}$ is a weak Berger algebra $\iff \mathfrak{g} = \text{span}\{ Q(x) \mid Q \in \mathcal{B}(\mathfrak{g}), x \in \mathbb{R}^n \}$.

[TL ’02] If $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{s}0}(1,n-1)(L)$ is an indecomposable Berger algebra, then $\mathfrak{g} := \text{proj}_{\mathfrak{s}0}(n-2)(\mathfrak{h})$ is a weak Berger algebra. Classify them $\implies$ result. □

**Theorem (Berard-Bergery-Ikemakhen ’96, Boubel ’00, TL ’03, Galaev ’05)**

If $\mathfrak{g} := \text{proj}_{\mathfrak{s}0}(n-2)\mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric $h$ with $\text{hol}_p(h) = \mathfrak{h}$. 

Thomas Leistner (Adelaide)
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

Let \((\Sigma, \nabla^\Sigma)\) be the spinor bundle over \((M, g)\).

Assume: \(\exists \varphi \in \Gamma(\Sigma)\) with \(\nabla^\Sigma \varphi = 0\) a parallel spinor field.

\[\implies \exists \text{ causal vector field } V_\varphi \in \Gamma(TM) : \nabla V_\varphi = 0.\]

Two cases:

\[g(V_\varphi, V_\varphi) < 0 : \quad (M, g) = (\mathbb{R}, -dt^2) \quad \text{Riemannian mf.}\]

\[g(V_\varphi, V_\varphi) = 0 : \quad (M, g) = (\overline{M}, \overline{g}) \quad \text{with parallel spinor indecomposable with parallel spinor}\]

**Theorem (TL ’03)**

\((M, g)\) indecomposable Lorentzian spin with parallel spinor. Then \(\text{Hol}_p(M, g) = G \times \mathbb{R}^{n-2}\) where \(G\) is a product of the following groups:

\[\{1\}, \quad \text{SU}(p), \quad \text{Sp}(q), \quad G_2, \quad \text{Spin}(7)\]

\[\text{dim}\{\nabla \varphi = 0\} : \quad 2^{[k/2]} \quad 2 \quad q + 1 \quad 1 \quad 1\]

This generalizes the result for \(n \leq 11\) in [Bryant ’99].
Theorem (Galaev/TL ’06)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

\[
\text{Hol}_p^0(M, g) = \begin{cases} 
  \left( \mathbb{R}^+ \times G \right) \rtimes \mathbb{R}^{n-2}, & \text{or} \\
  G \rtimes \mathbb{R}^{n-2} 
\end{cases}
\]

with a Riemannian holonomy group $G$. Furthermore:

- If $\text{Hol}_p^0(M, g) = G \rtimes \mathbb{R}^{n-2}$, then $\text{Ric} = 0$ and $G = \text{Holonomy of Ricci-flat Riemannian manifold}$, i.e. $G = \text{product of } \text{SO}(n), \text{SU}(p), \text{Sp}(q), G_2, \text{and } \text{Spin}(7)$. 
Coordinates

Theorem (Brinkmann’25, Walker’49)

For a Lorentzian manifold \((M, h)\) with parallel null line \(L\) there are coordinates \((v, x^1, \ldots, x^{n-2}, u)\): \(\frac{\partial}{\partial v}\) spans \(L\), \(\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-2}}\right)\) span \(L^\perp\), and

\[
h = 2\, dv\, du + \sum_{i=1}^{n-2} A_i dx^i \, du + f du^2 + \sum_{i,j=1}^{n-2} g_{ij} dx^i \, dx^j,
\]

where \(A_i = f = 0\), \(f \in C^\infty(M)\).

- \(\exists\) parallel null vector field \(\iff\) \(\frac{\partial f}{\partial v} = 0\). [Schimming ’74]: \(A_i = f = 0\).
- [Galaev/TL ’09]: \(\exists\) coordinates such that \(A_i = 0\).

Note: \(Hol_p(g_u) \subset pr_{SO(n-2)}Hol_p(h)\), but in general \(\neq\) (see Galaev’s examples on next slides).
Manifolds of uncoupled holonomy type

Construction method for the uncoupled types

Let \((N^{n-2}, g)\) be a Riemannian manifold and \(f \in C^\infty(\mathbb{R}^2 \times N)\) “sufficiently generic”. Then \(M = \mathbb{R}^2 \times N\) with the metric \(h := 2dvdu + fdu^2 + g\) is indecomposable, non irreducible with holonomy

\[(\mathbb{R}^+ \times \text{Hol}(N, g)) \ltimes \mathbb{R}^{n-2}\text{ or } \text{Hol}(N, g) \ltimes \mathbb{R}^{n-2}, \text{ if } \frac{\partial f}{\partial v} = 0\]

Example: \((M, h)\) pp-wave :\(\iff\) \(g \equiv \text{flat metric.}\)

[TL ’01]: An indecomposable Lorentzian mfd. has Abelian holonomy \(\mathbb{R}^{n-2}\) \(\iff\) it is a pp-wave.

- E.g. Symmetric spaces (Cahen-Wallach spaces) \(\iff\) \(f\) is a quadratic polynomial in the \(x^i\)’s.
- Plane waves: \(f\) is a quadratic polynomial in the \(x^i\)’s with coefficients depending on \(u\) [Hull-Figueroa O’Farrill-Papadopoulos ’02].
Coupled types — Proof of Theorem [Galaev ’05]

For a Riemannian holonomy algebra \( \mathfrak{g} \), fix \( Q_1, \ldots, Q_N \), a basis of \( \mathcal{B}(\mathfrak{g}) \), and define polynomials on \( \mathbb{R}^{n-1} \):

\[
A_i(x^1, \ldots, x^{n-2}, u) := \sum_{A=1}^{N} \sum_{k,l=1}^{n-2} \frac{1}{(A - 1)!} \left( Q_A(e_k)e_l, e_i \right) x^k x^l u^A.
\]

**Theorem (Galaev ’05)**

*For any indecomposable \( \mathfrak{h} \subset \mathfrak{so}(1, n - 1)_L \), for which \( \mathfrak{g} = \text{proj}_{\mathfrak{so}(n-2)}(\mathfrak{h}) \) is a Riemannian holonomy algebra, there exists an analytic \( f : \mathbb{R}^n \to \mathbb{R} \) such that the following Lorentzian metric has holonomy \( \mathfrak{h} \):

\[
h = 2dvdu + fdu^2 + 2 \sum_{i=1}^{n-2} A_idx^i du + \sum_{k=1}^{n-2} (dx^k)^2,
\]

*family of 1-forms on \( \mathbb{R}^n \)  flat metric*
Coordinates for Einstein manifolds

[Gibbons/Pope ’08]: Lorentzian Einstein mf with parallel null line and Einstein constant $\Lambda$ written in Walker coordinates $\Rightarrow$

$$f = \Lambda v^2 + vf_1 + f_0 \quad \text{with} \quad f_{0/1} = f_{0/1}(x^1, \ldots, x^{n-2}, u).$$

Theorem (Galaev/TL ’09)

Let $(M, g)$ be a Lorentzian Einstein mf. with parallel null line with $\Lambda \neq 0$. Then $\exists$ coordinates $(v, x^1, \ldots, x^{n-2}, u)$ such that the metric is given as

$$g = 2dvdu + (\Lambda v^2 + f_0)du^2 + g_{kl}dx^kdx^l$$

with $f_1 = f_1(x^i, u)$ and $g_{kl} = g_{kl}(x^i, u)$ a family of Riemannian Einstein metrics for $\Lambda$ satisfying

$$\Delta f_0 = -\frac{1}{2}g^{ij}\ddot{g}_{ij}, \quad \nabla^j\dot{g}_{ij} = 0, \quad g^{ij}\dot{g}_{ij} = 0.$$

Conversely, any such Lorentzian metric is Einstein with constant $\Lambda.$
Coordinates for Ricci-flat manifolds

Theorem (Galaev/TL ’09)

Let \((M, g)\) be a Lorentzian Ricci-flat mf. with parallel null line. Then \(\exists\) coordinates \((v, x^1, \ldots, x^{n-2}, u)\) such that the metric is given as

\[
g = 2dvdu + vf_1 du^2 + g_{kl}dx^k dx^l
\]

with \(f_1 = f_1(x^i, u)\) a harmonic function w.r.t. the \(x^i\)'s and \(g_{kl} = g_{kl}(x^i, u)\) a family of Ricci-flat Riemannian metrics satisfying

\[
\frac{1}{2} \ddot{g}^{ij} \dot{g}_{ij} + g^{ij} \dddot{g}_{ij} + \frac{1}{2} g^{ij} \ddot{g}_{ij} f_1 = 0 \quad \text{and} \quad \partial_i f_1 = \partial_i (g^{jk} \dot{g}_{jk}) - \nabla^j \dot{g}_{ij}.
\]

Conversely, any such Lorentzian metric is Ricci flat.
Open Problems

Study *Lorentzian manifolds with special holonomy*!

1. Find global examples of metrics with prescribed holonomy, which are globally hyperbolic with complete or compact Cauchy hypersurface (cylinder constructions in [Bär-Gauduchon-Moroianu ’05] and [Baum-Müller ’06]).

2. Describe the geometric structures corresponding to the coupled types III and IV.

3. Describe indecomposable, non-irreducible Lorentzian homogeneous spaces and their holonomy.

4. Study further spinor field equations for these manifolds (Killing spinors, generalised Killing spinors).

Thank you!