Special holonomy in Lorentzian geometry

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Outline

Holonomy group of a linear connection

- Let M be a smooth manifold and ∇ a linear connection.

  $\rightsquigarrow$ Parallel transport along $\gamma : [0,1] \to M$, piecewise smooth,

$$\mathcal{P}_\gamma \quad : \quad T_{\gamma(0)}M \ni X_0 \quad \overset{\sim}{\longmapsto} \quad X(1) \in T_{\gamma(1)}M$$

  where $X(t)$ is the solution to the ODE $\nabla_{\dot\gamma(t)}X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (Connected) Holonomy group

$$Hol_p{}^0(M,\nabla) := \left\{\mathcal{P}_\gamma \mid \gamma(0) = \gamma(1) = p, \gamma \sim \{p\}\right\} \subset \mathrm{GL}(T_pM) \simeq \mathrm{GL}(n,\mathbb{R})$$

and its Lie algebra $\mathfrak{hol}_p(M,\nabla)$.

- For $p, q \in M$: $Hol_p(M,\nabla) \overset{\overset{\text{conjugated in } \mathrm{GL}(n,\mathbb{R})}{\downarrow}}{\sim} Hol_q(M,\nabla)$
- If $\nabla =$ LC of a metric g on M, then $Hol_p(M,g) \subset \mathrm{O}(T_pM, g) = \mathrm{O}(t,s)$.

Holonomy and curvature

- Recall that ∇ and $\mathcal{P}_\gamma$ are related via

$$\nabla_{\dot\gamma(0)}X|_p = \frac{d}{dt}\left[\mathcal{P}^{-1}_{\gamma|_{[0,t]}}(X(\gamma(t)))\right]|_{t=0}.$$

- This implies for the curvature $\mathcal{R}$ of ∇: Let $X, Y \in T_pM$ and λ_t the loop along the parallelogram at p with sides $\sqrt{t}X$ and $\sqrt{t}Y$. Then

$$\mathcal{R}(X,Y)|_p = \lim_{t\to 0}\frac{1}{t}\left(\mathcal{P}_{\lambda_t} - Id_{T_pM}\right).$$

  Hence, $\mathcal{R}(X,Y)|_p \in \mathfrak{hol}_p(M,\nabla)$ for all $X, Y \in T_pM$.
- One has to collect curvature all over M to get all of $\mathfrak{hol}_p(M,\nabla)$:

Theorem (Ambrose-Singer '53)

If M is connected, then $\mathfrak{hol}_p(M,\nabla)$ is spanned by

$$\left\{\mathcal{P}^{-1}_\gamma \circ \mathcal{R}(X,Y)\circ\mathcal{P}_\gamma \in \mathrm{GL}(T_pM) \mid \gamma(0) = p \text{ and } X, Y \in T_{\gamma(1)}M\right\}$$

Classification: *Which groups occur as holonomy groups?*

- Hano/Ozeki '56: Any closed $G \subset \mathrm{GL}(n,\mathbb{R})$! But ∇ might have torsion.
- Conditions on the torsion T^∇ of ∇, e.g. $T^\nabla = 0$ or $T^\nabla \in \Lambda^3 TM$
  $\rightsquigarrow$ algebraic constraints on the holonomy representation:
  $$\mathcal{R}_\gamma \ := \ \mathcal{P}_\gamma^{-1} \circ \mathcal{R}\Big(\mathcal{P}_\gamma(.), \mathcal{P}_\gamma(.)\Big) \circ \mathcal{P}_\gamma \ \in \ \Lambda^2(T_p^*M) \otimes \mathrm{GL}(T_pM)$$

Now, if $T^\nabla = 0$, then $\mathcal{R}$ and hence $\mathcal{R}_\gamma$ satisfy the Bianchi identity:

$$\mathcal{R}_\gamma(X,Y)Z + \mathcal{R}_\gamma(Y,Z)X + \mathcal{R}_\gamma(Z,X)Y = 0.$$

$\Longrightarrow$ $\mathfrak{hol}_p(M,\nabla)$ is a Berger algebra: For $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ define the $\mathfrak{g}$-module:

$$\mathcal{K}(\mathfrak{g}) := \Big\{ R \in \Lambda^2 {\mathbb{R}^n}^* \otimes \mathfrak{g} \mid \ R(x,y)z + R(y,z)x + R(z,x)y = 0 \Big\}$$

$\mathfrak{g}$ is a Berger algebra $:\Longleftrightarrow$ $\mathfrak{g} = \mathrm{span}\{R(x,y) \mid R \in \mathcal{K}(\mathfrak{g}), x, y \in \mathbb{R}^n\}$.

$T^\nabla = 0$: Ambrose-Singer $\Longrightarrow$ $\mathfrak{hol}_p(M,\nabla)$ is a Berger algebra.

Classification of irreducible Berger algebras

- $\mathfrak{g} \subset \mathfrak{so}(t,s)$ [Berger '55], $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ [Schwachhöfer/Merkulov '99] .

Holonomy and geometry

Let V be a "geometric" vector bundle over M with connection ∇. Then:

$$\left\{ v \in V_p \mid Hol_p(M,\nabla)(v) = v \right\} \ni v \quad \simeq \quad \varphi := \mathcal{P}_\gamma(v) \in \left\{ \varphi \in \Gamma(V) \mid \nabla\varphi = 0 \right\}$$

$$\left\{ V \subset T_pM \mid Hol_p(M,\nabla)(V) \subset V \right\} \simeq \left\{ \text{Distrib. } \mathcal{V} \subset TM \mid \mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \right\}$$

$\mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \iff \nabla_X : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$, in particular, $\mathcal{V}$ is integrable.

Decomposition of a semi-Riemannian manifold (M,g):

If $V \subset T_pM$ is $Hol(M,g)$-invariant, non-degenerate (i.e. $V \cap V^\perp = \{0\}$), i.e. $T_pM = V \oplus V^\perp$ invariant decomposition, then

$$(M,g) \overset{locally}{\simeq} (N,h) \times (N^\perp, h^\perp)$$

with $V^{(\perp)} \simeq T_pN^{(\perp)}$ as $Hol_p(M,g)$–module.

De Rham–Wu decomposition

Complete decomposition of T_pM into $Hol_p(M,g)$–modules:

$$T_pM = \oplus_{i=0}^{k} V_k, \text{ with } V_0 \text{ trivial and } V_i \underbrace{\text{indecomposable}}_{\text{non-degenerate and only degenerate invariant subspaces}} \text{ for } i > 0$$

Theorem (de Rham '52, Wu '64)

Let (M,g) be semi-Riemannian, complete and simply connected. Then there is a $k > 0$:

$$(M,g) \stackrel{globally}{\simeq} (M_1,g_1) \times \ldots \times (M_k,g_k)$$

- (M_i,g_i) *complete and 1-connected,*
- (M_i,g_i) *flat or with indecomposable holonomy representation,*
- $Hol_p(M,g) \simeq Hol_{p_1}(M_1,g_1) \times \ldots \times Hol_{p_k}(M_k,g_k)$.

Manifold of special holonomy: Indecomposable holonomy $\subsetneq \mathrm{SO}(t,s)$.

Holonomy of Riemannian manifolds (M, g)

Positive definite metric $\Longrightarrow$ indecomposable $=$ irreducible
$\Longrightarrow$ $Hol_p(M, g) \simeq$ product of irreducible holonomy groups.

Berger's list ('55)

Let (M, g) be 1-connected, irred., not loc. symm. Then $Hol_p(M, g) \overset{O(n)}{\sim}$

	$\mathrm{SO}(n)$	$\mathrm{U}(\frac{n}{2})$	$\mathrm{SU}(\frac{n}{2})$	$\mathrm{Sp}(\frac{n}{4})$	$\mathrm{Sp}(1) \cdot \mathrm{Sp}(\frac{n}{4})$	G_2	$\mathrm{Spin}(7)$
	generic	*Kähler*		*hyper Kähler*	*quat. Kähler*		
par. field	*none*	J		J_1, J_2, J_3	$\langle J_1, J_2, J_3 \rangle$	ω^3	ω^4
Ric	—	$\neq 0$	0	0	$\Lambda \cdot g$	0	0
$dim\{\nabla \varphi = 0\}$	0	0	2	$\frac{n}{4} + 1$	0	1	1

- Complete mf's: Calabi (SU, Sp), LeBrun (qK), Bryant (G_2, $\mathrm{Spin}(7)$).
- Compact mf's: Yau (SU), Beauville, Mukai (Sp), LeBrun-Salamon (qK), Joyce (G_2, $\mathrm{Spin}(7)$).

Special Lorentzian holonomy

Yau '93: *"Berger has classified holonomy groups for Riemannian manifolds. If the metric is not Riemannian but allows different signature, the corresponding theorem of Berger should exist. ... In particular, classify complete Lorentzian manifolds with parallel spinors."*

Wu–Decomposition for a Lorentz manifold (M, g)

Let (M, g) be a complete, simply-connected Lorentzian manifold $\Longrightarrow$

$$(M, g) \quad \simeq \quad \underset{\uparrow}{(\overline{M}, \overline{g})} \times \underbrace{(N_1, g_1) \times \ldots \times (N_k, g_k)}_{\text{Riemannian, irreducible or flat}}$$

Lorentzian manifold which is either

1. $(\mathbb{R}, -dt^2)$, or
2. irreducible, i.e. $Hol_p(\overline{M}, \overline{g}) = \mathrm{SO}_0(1, n)$, or
3. indecomposable, non-irreducible

Algebraic preliminaries

Hence, an indecomposable Lorentzian manifold has special holonomy
$\iff$ its holonomy admits a degenerate invariant subspace
$\iff$ it admits a parallel null line.

We have to consider $H \subset \mathrm{SO}_0(1, n-1)$ indecomposable, non-irreducible, i.e. $\exists\ V \subset \mathbb{R}^{1,n-1} : H(V) \subset V$ such that

$L := V \cap V^{\perp} \neq \{0\}$ is a H-invariant, totally null line in $\mathbb{R}^{1,n-1}$

$$\Rightarrow \quad H \quad \subset \quad Iso_{\mathrm{SO}_0(1,n-1)}(L) \quad = \quad (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \ltimes \mathbb{R}^{n-2}$$

Change basis of $\mathbb{R}^{1,n-1}$: $\quad \mathfrak{h} \quad \subset \quad \left\{ \left(\begin{array}{ccc} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{array} \right) \middle| \begin{array}{l} a \in \mathbb{R}, \\ v \in \mathbb{R}^{n-2}, \\ A \in \mathfrak{so}(n-2) \end{array} \right\}$

The orthogonal part is reductive:

$$\mathfrak{g} := pr_{\mathfrak{so}(n-2)} \mathfrak{h} = \underbrace{\mathfrak{z}}_{\textit{centre}} \oplus \underbrace{\mathfrak{g}'}_{= [\mathfrak{g}, \mathfrak{g}] \text{ semisimple}} \qquad (\textit{Levi} - \textit{decomposition})$$

Parallel null line and screen bundle

Let (M, g) be a Lorentzian manifold with
$H := Hol_p(M, g) \subset Iso(L) = (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \times \mathbb{R}^{n-2}$.

- L defines a filtration $L \subset L^\perp \subset TM$ with parallel null line $\mathcal{L}$ and parallel null hypersurface $\mathcal{L}^\perp$.
- If $H \subset \mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$, then $\exists$ parallel null vector field (Brinkmann wave).
- What about $G := \mathrm{pr}_{\mathrm{SO}(n-2)} Hol_p(M, g) \subsetneq \mathrm{SO}(n-2)$?

Proposition (TL '05)

The vector bundle ("screen bundle") $\mathcal{S} = L^\perp / L$ with covarant derivative $\nabla^{\mathcal{S}}_U[V] := [\nabla_U V]$ satisfies $\mathrm{pr}_{\mathrm{SO}(n-2)} Hol_p(M, g) = Hol_p(\mathcal{S}, \nabla^{\mathcal{S}})$.

- Hence, algebraic structures for G correspond to geometric structures on $\mathcal{S}$, e.g. product structure, parallel complex structure etc.
- Which G's can occur?

Classification 1: $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ indecomposable

Theorem (Berard-Bergery/Ikemakhen '96)

For $\mathfrak{h}$ with $\mathfrak{g} := pr_{\mathfrak{so}(n-2)}\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ there are the following cases:

$\mathbb{R}^{n-2} \subset \mathfrak{h}$ – *Type I:* $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$.

Type II: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$.

Type III: $\exists\, \varphi : \mathfrak{z} \twoheadrightarrow \mathbb{R}$: $\mathfrak{h} = \left\{ \left(\begin{array}{ccc} \varphi(A) & v^t & 0 \\ 0 & A + B & -v \\ 0 & 0 & -\varphi(A) \end{array} \right) \middle| \begin{array}{l} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2} \end{array} \right\}$

$\mathbb{R}^{n-2} \not\subset \mathfrak{h}$ – *Type IV:* $\exists\, \varphi : \mathfrak{z} \twoheadrightarrow \mathbb{R}^k$, *for* $0 < k < n-2$:

$$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & \psi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A + B & -v \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \begin{array}{l} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^{n-2-k} \end{array} \right\}$$

Note: Groups of coupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen '96

Classification II: Let $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ be indecomposable

Theorem (TL '03)

If $\mathfrak{h}$ is a Berger algebra (e.g. a Lorentzian holonomy algebra), then $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}\mathfrak{h}$ is a Riemannian holonomy algebra (and hence known to be a product of algebras from Berger's list).

Idea of the proof: For $\mathfrak{g} \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

$$\mathcal{B}(\mathfrak{g}) := \left\{Q \in Hom(\mathbb{R}^n, \mathfrak{g}) \mid \langle Q(x)y, z\rangle + \langle Q(y)z, x\rangle + \langle Q(z)v, y\rangle = 0\right\}.$$

$\mathfrak{g}$ is a weak Berger algebra $:\Longleftrightarrow \mathfrak{g} = \mathrm{span}\left\{Q(x) \mid Q \in \mathcal{B}(\mathfrak{g}), x \in \mathbb{R}^n\right\}$

[TL '02] If $\mathfrak{h} \subset \mathfrak{iso}_{\mathfrak{so}(1,n-1)}(L)$ is an indecomposable Berger algebra, then $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}(\mathfrak{h})$ is a weak Berger algebra. Classify them $\Longrightarrow$ result. □

Theorem (Berard-Bergery-Ikemakhen '96, Boubel '00, TL '03, Galaev '05)

If $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}\mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric h with $\mathfrak{hol}_p(h) = \mathfrak{h}$.

Parallel spinors on a Lorentzian spin manifold (M, g)

Let $\left(\Sigma, \nabla^{\Sigma}\right)$ be the spinor bundle over (M, g).

Assume: $\exists \varphi \in \Gamma(\Sigma)$ with $\nabla^{\Sigma}\varphi = 0$ a parallel spinor field.
$\Longrightarrow$ $\exists$ causal vector field $V_\varphi \in \Gamma(TM)$: $\nabla V_\varphi = 0$. Two cases:

$$\begin{array}{lcccc}
g(V_\varphi, V_\varphi) < 0 & : & (M,g) & = & (\mathbb{R}, -dt^2) \\
g(V_\varphi, V_\varphi) = 0 & : & (M,g) & = & (\underset{\substack{\uparrow \\ \text{indecomposable with parallel spinor}}}{\overline{M}, \overline{g}})
\end{array} \times \begin{array}{l}\text{Riemannian mf.} \\ \text{with parallel spinor}\end{array}$$

Theorem (TL '03)

(M, g) indecomposable Lorentzian spin with parallel spinor. Then $Hol_p(M, g) = G \ltimes \mathbb{R}^{n-2}$ where G is a product of the following groups:

	$\{1\}$,	$\mathrm{SU}(p)$,	$\mathrm{Sp}(q)$,	G_2,	$\mathrm{Spin}(7)$
$dim\{\nabla\varphi = 0\}$:	$2^{[k/2]}$	2	$q+1$	1	1

This generalizes the result for $n \leq 11$ in [Bryant '99].

Lorentzian Einstein manifolds

Theorem (Galaev/TL '06)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

$$Hol_p^0(M,g) = \begin{cases} (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^{n-2}, & \text{or} \\ G \ltimes \mathbb{R}^{n-2} & \end{cases}$$

with a Riemannian holonomy group G. Furthermore:

- *If $Hol_p^0(M,g) = G \ltimes \mathbb{R}^{n-2}$, then $Ric = 0$ and G = Holonomy of Ricci-flat Riemannian manifold, i.e. G = product of $\mathrm{SO}(n)$, $\mathrm{SU}(p)$, $\mathrm{Sp}(q)$, G_2, and $\mathrm{Spin}(7)$.*

Coordinates

Theorem (Brinkmann'25, Walker'49)

For a Lorentzian manifold (M, h) with parallel null line $\mathcal{L}$ there are coordinates $(v, x^1, \ldots, x^{n-2}, u)$: $\frac{\partial}{\partial v}$ spans $\mathcal{L}$, $\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-2}}\right)$ span $\mathcal{L}^\perp$, and

- $$h = 2\, dvdu + \underbrace{\sum_{i=1}^{n-2} A_i dx^i\, du}_{\substack{= \phi_u \\ \text{family of 1-forms}}} + fdu^2 + \underbrace{\sum_{i,j=1}^{n-2} g_{ij} dx^i\, dx^j}_{\substack{= g_u \\ \text{family of Riem. metrics}}} \quad ,$$

  with $\frac{\partial g_{ij}}{\partial v} = \frac{\partial A_i}{\partial v} = 0$, $f \in C^\infty(M)$.
- $\exists$ *parallel null vector field* $\iff$ $\frac{\partial f}{\partial v} = 0$. *[Schimming '74]:* $A_i = f = 0$.
- *[Galaev/TL '09]:* $\exists$ *coordinates such that* $A_i = 0$.

Note: $Hol_p(g_u) \subset pr_{SO(n-2)} Hol_p(h)$, but in general $\neq$ (see Galaev's examples on next slides)

Manifolds of uncoupled holonomy type

Construction method for the uncoupled types

Let (N^{n-2}, g) be a Riemannian manifold and $f \in C^\infty(\mathbb{R}^2 \times N)$ "sufficiently generic". Then $M = \mathbb{R}^2 \times N$ with the metric $h := 2dvdu + fdu^2 + g$ is indecomposable, non irreducible with holonomy

$$(\mathbb{R}^+ \times Hol(N,g)) \ltimes \mathbb{R}^{n-2} \quad or \quad Hol(N,g) \ltimes \mathbb{R}^{n-2}, \ \ if \ \frac{\partial f}{\partial v} = 0$$

Example: (M, h) pp-wave $:\Longleftrightarrow g \equiv$ flat metric.
[TL '01]: An indecomposable Lorentzian mfd. has Abelian holonomy $\mathbb{R}^{n-2}$ $\Longleftrightarrow$ it is a pp-wave.

- E.g. Symmetric spaces (Cahen-Wallach spaces) $\Longleftrightarrow$ f is a quadratic polynomial in the x^i's.
- Plane waves: f is a quadratic polynomial in the x^i's with coefficients depending on u [Hull-Figueroa O'Farrill-Papadopoulos '02].

Coupled types — Proof of Theorem [Galaev '05]

For a Riemannian holonomy algebra $\mathfrak{g}$, fix $Q_1, \ldots, Q_N$, a basis of $\mathcal{B}(\mathfrak{g})$, and define polynomials on $\mathbb{R}^{n-1}$:

$$A_i(x^1, \ldots, x^{n-2}, u) \; := \; \sum_{A=1}^{N} \sum_{k,l=1}^{n-2} \frac{1}{(A-1)!} \Big\langle Q_A(e_k) e_l, e_i \Big\rangle x^k x^l u^A .$$

Theorem (Galaev '05)

For any indecomposable $\mathfrak{h} \subset \mathfrak{so}(1, n-1)_L$, for which $\mathfrak{g} = proj_{\mathfrak{so}(n-2)}(\mathfrak{h})$ is a Riemannian holonomy algebra, there exists an analytic $f : \mathbb{R}^n \to \mathbb{R}$ such that the following Lorentzian metric has holonomy $\mathfrak{h}$:

$$h \; = \; 2dvdu + fdu^2 + 2 \underbrace{\sum_{i=1}^{n-2} A_i dx^i \, du}_{\textit{family of 1-forms on } \mathbb{R}^n} + \underbrace{\sum_{k=1}^{n-2} (dx^k)^2}_{\textit{flat metric}} ,$$

Coordinates for Einstein manifolds

[Gibbons/Pope '08]: Lorentzian Einstein mf with parallel null line and Einstein constant Λ written in Walker coordinates $\Rightarrow$

$$f = \Lambda v^2 + v f_1 + f_0 \quad \text{with } f_{0/1} = f_{0/1}(x^1, \ldots, x^{n-2}, u).$$

Theorem (Galaev/TL '09)

Let (M, g) be a Lorentzian Einstein mf. with parallel null line with $\Lambda \neq 0$. Then $\exists$ coordinates $\left(v, x^1, \ldots, x^{n-2}, u\right)$ such that the metric is given as

$$g = 2dvdu + \left(\Lambda v^2 + f_0\right) du^2 + g_{kl} dx^k dx^l$$

with $f_1 = f_1(x^i, u)$ and $g_{kl} = g_{kl}(x^i, u)$ a family of Riemannian Einstein metrics for Λ satisfying

$$\Delta f_0 = -\frac{1}{2} g^{ij} \ddot{g}_{ij}, \quad \nabla^j \dot{g}_{ij} = 0, \quad g^{ij} \dot{g}_{ij} = 0.$$

Conversely, any such Lorentzian metric is Einstein with constant Λ.

Coordinates for Ricci-flat manifolds

Theorem (Galaev/TL '09)

Let (M,g) be a Lorentzian Ricci-flat mf. with parallel null line. Then $\exists$ coordinates $\left(v, x^1, \ldots, x^{n-2}, u\right)$ such that the metric is given as

$$g = 2dvdu + vf_1 du^2 + g_{kl} dx^k dx^l$$

with $f_1 = f_1(x^i, u)$ a harmonic function w.r.t. the x^i's and $g_{kl} = g_{kl}(x^i, u)$ a family of Ricci-flat Riemannian metrics satisfying

$$\frac{1}{2}\dot{g}^{ij}\dot{g}_{ij} + g^{ij}\ddot{g}_{ij} + \frac{1}{2}g^{ij}\dot{g}_{ij}f_1 = 0 \ \text{ and } \ \partial_i f_1 = \partial_i(g^{jk}\dot{g}_{jk}) - \nabla^j \dot{g}_{ij}.$$

Conversely, any such Lorentzian metric is Ricci flat.

Open Problems

Study *Lorentzian manifolds with special holonomy* !

1. Find global examples of metrics with prescribed holonomy, which are globally hyperbolic with complete or compact Cauchy hypersurface (cylinder constructions in [Bär-Gauduchon-Moroianu '05] and [Baum-Müller '06])
2. Describe the geometric structures corresponding to the coupled types III and IV.
3. Describe indecomposable, non-irreducible Lorentzian homogeneous spaces and their holonomy.
4. Study further spinor field equations for these manifolds (Killing spinors, generalised Killing spinors).

Thank you!

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