Half flat structures and special holonomy

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THE UNIVERSITY OF ADELAIDE AUSTRALIA

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Outline

1. Linear algebra of stable forms
2. Half flat structures and their evolution under the Hitchin flow
3. Examples

Joint work with V. Cortés, F. Schulte-Hengesbach (Hamburg), and L. Schäfer (Hannover) [Proc. London Math. Soc., 2010]

Stable forms

Definition

Let V be a real vector space of dimension n. A k-form $\omega \in \Lambda^k V^* =: \Lambda^k$, with $1 < k < n-1$, is called *stable* if and ω has an open orbit under the action of $\mathrm{GL}_n\mathbb{R}$.

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- Stable forms come in pairs: There is a $\mathrm{GL}_n^+\mathbb{R}$-invariant map $\phi : \Lambda^k \to \Lambda^n$ which sends stable forms to volume forms which and defines a *dual stable form* $\hat{\omega}$ by

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$$\phi(\omega) = \tfrac{1}{m!}\omega^m\ , \quad \hat{\omega} = \tfrac{1}{(m-1)!}\omega^{m-1}\ , \quad \text{and} \quad Stab_{\mathrm{GL}_n}(\omega) = \mathrm{Sp}_m\mathbb{R}.$$

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$$3\hat{\varphi} = *\varphi\,, \;\; 7\phi(\varphi) = \varphi \wedge *\varphi\,, \;\; Stab_{\mathrm{GL}_7}(\varphi) = \begin{cases} \mathrm{G}_2 \subset \mathrm{SO}(7)\,, & g_\varphi > 0 \\ \mathrm{G}_2^* \subset \mathrm{SO}(3,4)\,, & \text{else.} \end{cases}$$

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- Conversely, fix $v \in V^7$ with $g_\varphi(v, v) = -\epsilon$, then $\omega := (v \lrcorner \varphi)|_{v^\perp}$ and $\rho := \varphi|_{v^\perp}$ are compatible stable forms on $v^\perp$.

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  It is parallel if $\nabla^{LC}\varphi = 0$, or equivalently, if $d\varphi = 0$ and $d{*}\varphi = 0$.

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Theorem (Hitchin '01 for M compact & $H = \mathrm{SU}(3)$, CLSS in gen.)

Let (ρ, ω) be a 1-parameter family of stable forms on M^6 satisfying the Hitchin flow eq's. If $(\rho_{t_0}, \omega_{t_0})$ is half flat for a $t_0 \in I$, then (ρ, ω) is a family of half flat H-structures.

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In particular, the three-form $\varphi = \omega \wedge dt + \rho$ defines a parallel $\mathrm{G}_2^{(*)}$-structure on $M \times I$ with induced metric $g_\varphi = g_t - \epsilon dt^2$.

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In general, the $\mathrm{G}_2^{(*)}$-metrics obtained in this way will only be geodesically complete if $I = \mathbb{R}$. But they can be conformally changed to a complete metric.

- CY 3-manifolds are half flat, $d\omega = 0$ and $d\rho = d\hat{\rho} = 0$. The resulting $\mathrm{G}_2^{(*)}$-metric is a direct product: $g_\varphi = -\epsilon dt^2 + g_0$.

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$$\omega_t = t^2\omega_0 \ , \quad \rho_t = t^3\rho_0 \ \text{ defining the metric } g_t = t^2 g_0$$

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- Conversely, $\mathrm{G}_2^{(*)}$-cone-metrics define nearly-ϵ-Kähler metrics on the base.

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$\rightsquigarrow$ algebraic problem as $d\alpha(X, Y) = \alpha([X, Y])$ for $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$.

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for a basis $(e^1, e^2, e^3, f^1, f^2, f^3)$ be a basis of $H_3 \times H_3$ with commutator $de^3 = e^{12}$ and $df^3 = f^{12}$ and $\beta \neq -1$ a parameter.

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- Compatible stable closed 3-forms ρ are given by a linear 8-parameter family $\rho_i = \rho_i(a^1, \ldots, a^8)$ subject to a quartic non-degeneracy condition.

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- In order to evolve these structures we define $\kappa : I \to \mathbb{R}$:

$$
\begin{array}{rcll}
\mathrm{SU}(3) & : & \kappa(x) = (x - \sqrt{2})^3 (x + \sqrt{2}) & , \quad I = (-\sqrt{2}, \sqrt{2}) \\
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\mathrm{SL}_3\mathbb{R} & : & \kappa(x) = (2 + x)^2 & , \quad I = \mathbb{R}
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- The Hitchin flow is defined for all times and defines a G_2^*-metric on $G \times \mathbb{R}$.

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- The pseudo-Riemannian manifold $(H_3 \times H_3, g)$ is either flat or isometric to the product of
  - a two-dimensional flat factor and
  - the unique 4-dimensional simply connected para-hyper Kähler symmetric space (N, g_N) with 1-dimensional holonomy group [Alekseevsky et al '05]

  In particular, the metric g is Ricci-flat.
- The Hitchin flow is defined for all times and defines a G_2^*-metric on $G \times \mathbb{R}$.
- However, this metric is either flat or isometric to a product of (N, g_N) and a 3-dim flat factor, so its holonomy is at most one-dimensional.