Conformal holonomy, symmetric spaces, and skew symmetric torsion

Thomas Leistner

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Joint with Jesse Alt (University of the Witwatersrand) and Antonio J. Di Scala (Politecnico di Torino)
Outline

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2. Holonomy reductions and skew symmetric torsion
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   - Nearly para-Kähler structures

3. Fefferman-Graham ambient metric and conformal holonomy
   - The ambient metric construction
   - Tractor extensions

[arXiv:1208.2191]
Conformal geometry

Conformal manifold: \((M, [g]), [g] = \text{class of conformally equivalent semi-Riemannian metrics}, \dim(M) = p + q.\)

- Flat model: \(\mathbb{S}^{p,q} := \mathcal{N}/\mathbb{R}^+ = G/P\) with
  - \(\mathcal{N}\) null cone in \(\mathbb{R}^{p+1,q+1}\) with \(\mathbb{R}^+\)-action,
  - \(G := \text{SO}^0(p + 1, q + 1), P := \text{Stab}_G(\text{null line in } \mathbb{R}^{p+1,q+1}).\)

The curved version is described by

- A \(P\)-bundle \(\mathcal{G}\) (conformal Cartan bundle)

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{G_+} & \mathcal{G}^0 = \{\text{conformal frames}\} \\
\uparrow & & \xrightarrow{G_0 = \text{CO}(p,q)} \downarrow \\
\text{horizontal subspaces in } T\mathcal{G}^0, \text{kernel of some } \omega^g, \ g \in [g]
\end{array}
\]

- Normal conformal Cartan connection \(\omega \in T^*\mathcal{G} \otimes g,\)
  - \(\omega : T\mathcal{G} \to g\) parallelism, \(R_p^*\omega = \text{Ad}(p^{-1})\omega, \omega(\tilde{X}) = X \in p,\)
  - \(\Omega(X, Y) \in p\) (“torsion-free”) and curvature condition.

- E.g., flat model: \(\mathcal{G} := G \to G/P\) and \(\omega = \text{Maurer-Cartan form of } G.\)
What is conformal holonomy?

ω does not give horizontal subspaces and no parallel transport.

- ω defines connection \( \hat{\omega} \) on G-bundle \( \hat{G} = G \times_P G \) by \( \hat{\omega}|_{\hat{G}} = \omega \).
- tractor connection \( \hat{\nabla} \) on (standard) tractor bundle

\[
\mathcal{T} = \hat{G} \times_G \mathbb{R}^{p+1,q+1} = G \times_P \mathbb{R}^{p+1,q+1}.
\]

Conformal holonomy: \( \text{Hol}_x(M, [g]) := \text{Hol}_x(\mathcal{T}, \hat{\nabla}) \cong \text{Hol}_p(\hat{G}, \hat{\omega}) \subset G \).

1. Which groups can occur?
2. Are they holonomy groups of semi-Riemannian metrics?
3. Which structures correspond to holonomy reductions?

Obstacles:

- No obvious algebraic criterion for holonomy algebra.
- \( \text{Hol} \) is defined up to conjugation in \( G \), not only in \( P \).
- Reduction to subgroup \( H \) might not define a Cartan connection on \( M \), as we could have \( \text{dim}(H/H \cap P) \neq \text{dim}(M) \).
Let $\mathfrak{h}$ be the holonomy algebra of a semi-Riemannian manifold. Ambrose-Singer holonomy theorem,

\[ \mathfrak{h} = \text{span}\left\{ \mathcal{P}_\gamma^{-1} \circ \mathcal{R}(X, Y) \circ \mathcal{P}_\gamma \in SO(T_p M) \mid \gamma(0) = p, X, Y \in T_{\gamma(1)} M \right\} \]

and 1st Bianchi-identity for $\mathcal{R}$ imply

\[ (B) \quad \mathfrak{h} = \text{span}\left\{ R(x, y) \mid R \in \mathcal{K}(\mathfrak{h}), x, y \in \mathbb{R}^n \right\}, \]

with $\mathcal{K}(\mathfrak{h}) := \left\{ R \in \Lambda^2 \mathbb{R}^n \otimes \mathfrak{h} \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \right\}$. For $\mathfrak{h} \subset \mathfrak{so}(p, q)$ irreducible, (B) yields a classification (Berger '55).

No such algebraic criterion known for conformal holonomy.
Q: When can we rescale a given metric $g$ to an Einstein metric?

~ Overdetermined PDE system: $\tilde{g} = \sigma^{-2} \cdot g$ is Einstein $\iff$

$$Hess^g(\sigma) + P^g \cdot \sigma = \lambda \cdot g$$

for some function $\lambda$. $P^g = \frac{1}{n-2}(Ric^g - \frac{S}{2n-2})g$ is the Schouten tensor of $g$.

Let $g^\Lambda \in [g]$ be an Einstein metric, i.e. $Ric = (n - 1)\Lambda \cdot g^\Lambda$. Then

1. $\mathcal{T}$ admits a parallel section $\eta$ with $\hat{g}(\eta, \eta) = -\Lambda$ and hence, $\text{Hol}(M, [g])$ admits an invariant vector.

2. the **Fefferman-Graham ambient** metric $\tilde{g}$ is given as
   - $\Lambda \neq 0$: $\tilde{g} = -\frac{1}{\Lambda}ds^2 + \frac{1}{\Lambda}dr^2 + r^2 g^\Lambda$
   - $\Lambda = 0$: $\tilde{g} = -du dt + t^2 g^\Lambda$

   and $\text{Hol}(\tilde{M}, \tilde{g}) = \text{Hol}(M, [g])$, i.e., the conformal holonomy is a semi-Riemannian holonomy.

Conversely, if $\text{Hol}(M, [g])$ admits an invariant line, then on an open dense subset $M_0$ of $M$, locally, there exist an Einstein metric $g^\Lambda \in [g|_{M_0}]$, and all of the above holds for $\text{Hol}(M_0, [g|_{M_0}])$. 

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Conformally Einstein metrics and parallel tractors
Tractor bundle and its constant sections

[Bailey/Eastwood/Gover]

Let $P = \text{Stab}_G(L)$, $L = \text{null line}$.

- Filtration $L \subset L^\perp \subset \mathbb{R}^{p+1,q+1}$ gives $\mathcal{L} \subset \mathcal{L}^\perp \subset \mathcal{T}$.
- Projection $\mathcal{L}^\perp = \mathcal{G} \times_P L^\perp \to \mathcal{L}^\perp / \mathcal{L} \simeq TM \simeq G^0 \times_{CO_0(p,q)} (L^\perp / L)$.

Every $g \in [g]$ splits $\mathcal{T} = \mathcal{L}^\perp \oplus \mathbb{R} = \mathbb{R} \oplus TM \oplus \mathbb{R}$ with

$$\hat{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\nabla}_X \begin{pmatrix} \tau \\ Y \\ \sigma \end{pmatrix} = \begin{pmatrix} d\tau(X) - P(X, Y) \\ \nabla_X Y + \tau X + \sigma(\chi \downarrow P)\# \\ d\sigma(X) - g(X, Y) \end{pmatrix},$$

$$\hat{X} = \begin{pmatrix} \rho \\ \chi \\ \sigma \end{pmatrix} \text{ with } \hat{\nabla} \hat{X} = 0 \iff \sigma^{-2}g \text{ is Einstein metric on the open and dense complement of } zero(\sigma).$$
Irreducible case: Riemannian conf. structures

Theorem (Berger '55, Di Scala/Olmos '00)

If $H \subset SO^0(1, n + 1)$ acts irreducibly, then $H = SO^0(1, n + 1)$.

$\Rightarrow$ A Riemannian conformal manifold has generic conformal holonomy unless: On an open dense subset of $M$

- $[g]$ contains an Einstein metric or
- a certain product of Einstein metrics:

  Decomposition thm by S. Armstrong '04: $\text{Hol}(M, [g])$ has invariant subspace of dim $k > 1$ $\iff$ locally, $[g]$ contains product of Einstein metrics $g_1$ and $g_2$ of dim $(k - 1)$ and $(n - k + 1)$ with

  $$\frac{n-k+1}{k-1} \Lambda_1 = -\frac{k-1}{n-k+1} \Lambda_2$$

  and the conformal holonomy is given by the holonomy of the products of cones. (cf. Leitner '04, Leitner/Gover '09).
Theorem (Di Scala/L ’11)

Let $H \subset SO^0(2, n)$ act irreducibly. Then $H$ is conjugated to

1. $SO^0(2, n)$,
2. $SU(1, p), U(1, p), U(1) \cdot SO^0(1, p)$ if $p > 1, n$ even
3. $SO^0(1, 2)^{\text{irr.}} \subset SO(2, 3)$, for $n = 3$.

$U(1, p)$ and $U(1) \cdot SO^0(1, p)$ cannot be conformal holonomy groups:

$\text{Hol}([g]) \subset U(r, s) \Rightarrow \text{Hol}([g]) \subset SU(r, s)$

[Leitner’06, Cap/Gover’06]

$\text{Hol}([g]) = SU(1, p)$: Fefferman space in conformal class

What about (3)?

(3) corresponds to the symmetric space $M^5 := SL_3 \mathbb{R}/SO^0(1, 2)$ with signature $(2, 3)$ metric given by the Killing form of $\mathfrak{s}l_3 \mathbb{R}$. 
Semi-Riemannian irreducible symmetric space $\text{SL}_n \mathbb{R} / \text{SO}(p, q)$, $p + q = n$.

- symmetric decomposition $\mathfrak{s}l_n \mathbb{R} = \mathfrak{so}(p, q) \oplus m$, where $m = \{ X \in \mathfrak{gl}_n \mathbb{R} \mid X^\top 1_{p,q} = 1_{p,q} X \}$, inv. under $\text{Ad}(\text{SO}(p, q))$.

- irred. rep’n

$$\text{Ad} : \text{SO}(p, q) \to \text{SO}(m, K_{\mathfrak{s}l_n \mathbb{R}}) = \text{SO}\left(pq, \frac{p(p+1)+q(q+1)-2}{2}\right)$$

**Theorem (Alt/DiScala/L ’12)**

*If the conformal holonomy of a conformal manifold $(M, [g])$ of signature $(2, 1)$ is contained in $\text{Ad}(\text{SO}(2, 1))$, then $g$ is locally conformally flat.*

**Corollary**

*If the conformal holonomy group of a Lorentzian conformal manifold acts irreducibly, then it is equal to $\text{SO}(2, n)$ or $\text{SU}(1, p)$.***
Holonomy reductions via parallel sections

Principal $G$-bundle $\hat{\mathcal{G}} \to M$ with connection $\hat{\omega}$ on $\hat{\mathcal{G}}$. Assume:

- $H \subset G$ closed such that $\text{Hol}_p(\hat{\mathcal{G}}, \hat{\omega}) \subset H$,
- Reduction to the $H$-bundle (depending on $p \in \hat{\mathcal{G}}$),

\[ \mathcal{H}_p = \{ \gamma(1) | \gamma(0) = p, \gamma \text{ horizontal} \} \cdot H \subset \hat{\mathcal{G}} \]

holonomy bundle

$\hat{\omega}$ on $\hat{\mathcal{G}}$ induces a covariant derivative $\hat{\nabla}$ on $\mathcal{W}$.

\[ \hat{\nabla}\sigma = 0 \iff s \circ \gamma \text{ const. for all horizontal curves } \gamma \text{ in } \hat{\mathcal{G}}. \]

Hence, $\hat{\nabla}\sigma = 0$ implies $\text{Hol}_p(\hat{\omega}) \subset \text{Stab}_G(s(p))$ and defines

\[ s(\hat{\mathcal{G}}_x) \equiv G \cdot s(p) =: O. \]

$\sigma \in \Gamma(\mathcal{W})$, $\hat{\nabla}\sigma = 0 \iff O = G/\text{Stab}_G(w)$ for $w \in O$
Curved orbits [Čap/Gover/Hammerl '11]

\( \hat{G} \) and \( \hat{\omega} \) defined by (normal conformal) Cartan connection \( \omega \) on \( G^P \to M \) via \( \hat{G} = G \times_P G \) and \( \hat{\nabla} \) on \( W = G \times_P W \).

\( \omega \) has a holonomy reduction of type \( O \), i.e., \( \exists \sigma \in \Gamma(W) \) with \( \hat{\nabla}_\sigma = 0 \) defining \( s \in C^\infty(\hat{G}, W)^G \) with \( G \)-orbit \( O \). Note:

- \( \hat{\omega} \)-horizontal curves leave \( G \) if \( \text{Hol}_p(\hat{\omega}) \notin P \).
- Although \( G \)-orbits are the same, \( P \)-orbits \( s(G_x) \subset O \) might change with \( x \in M \).
- \( s(G_x) = P \cdot w =: [w] \in P \setminus O \) is the \( P \)-orbit type of \( \sigma \) at \( x \in M \),

\[
M = \bigcup_{[w] \in P \setminus O} M_{[w]}, \text{ with } M_{[w]} := \{x \in M \mid s(G_x) = [w]\}.
\]

- For \( w \in O \) set \( G_w = \text{Stab}_G(w) \). Then \( P \setminus O = P \setminus G/G_w \approx G_w \setminus G/P = G_w \setminus S^{p,q} \). i.e.,

\[
P\text{-orbits in } O = G/G_w \iff G_w\text{-orbits in } G/P
\]

\[
P \cdot g \cdot G_w \iff G_w \cdot g^{-1} \cdot P
\]
Theorem (Čap/Gover/Hammerl ’11)

Let $\omega$ be a Cartan connection of type $P \subset G$ with curvature $\Omega$ and with a holonomy reduction of type $O = G/G_w$.

- Let $w \in O$ with $P$-orbit $[w] := P \cdot w = PeG_w$ in $O$,
- $G_w/P_w = G_w eP$ the corresponding $G_w$-orbit in $G/P$, $P_w := G_w \cap P$.

Then, $\forall \ x \in M_{[w]} \ \exists$ nbhd. $U$ of $x$ in $M$ and a diffeom. $\phi : U \to V \subset G/P$:

- $\phi(x) = eP$, $\phi(U \cap M_{[w]}) = V \cap G_w eP$,
- $U \downarrow \phi \to V \subset G/P$ comm.
- $P \backslash O \to G_w \backslash G/P$
- $G_w \subset G$

$\omega$ induces a Cartan connection of type $P_w \subset G_w$ on $M_{[w]} \subset M$ whose curvature is the restriction of $\Omega$ to $G_w$ with values in $\mathfrak{p}_w$. 
Proof of Thm for $\text{SL}_n\mathbb{R}/\text{SO}(p, q)$

$\text{Ad}(\text{SO}(p, q))$-invariant decomposition $\mathfrak{sI}_n\mathbb{R} = \mathfrak{so}(p, q) \oplus m$. Then

- $H := \text{Ad}(\text{SO}(p, q)) \subset \text{SO}(m, K_{\mathfrak{sI}_n\mathbb{R}})$ is the stabiliser of a curvature tensor $R \in \mathcal{W} := \Lambda^2 m \otimes \mathfrak{so}(m, K_{\mathfrak{sI}_n\mathbb{R}})$.

- The null cone $\mathcal{N}$ in $m$ consists of matrices $S$ with $\text{tr}(S^2) = 0$ and defines the Möbius sphere $\mathcal{N} \to \mathbb{S}^{\hat{p}, \hat{q}} = \mathcal{N}/\mathbb{R}^*$.

**Proposition**

Let $\mathcal{N}_0 := \{S \in \mathcal{N} \mid S \text{ has } n \text{ distinct eigenvalues, possibly in } \mathbb{C}\}$. Then $\mathcal{N}_0$ is dense in $\mathcal{N}$ and, for all $S \in \mathcal{N}_0$, $\text{stab}_{\text{ad}(\mathfrak{i})}(R \cdot S) = \{0\}$. I.e., the union of $H$-orbits of codimension $n - 3$ is dense in $\mathbb{S}^{\hat{p}, \hat{q}}$.

**CGH-Thm $\Rightarrow$**

$\mathcal{M}_0 := \{x \in \mathcal{M} \mid s(\mathcal{G}_x) \text{ corresponds to orbit of max dim in } \mathbb{S}^{\hat{p}, \hat{q}} \}$ is dense.

$p_w = \{0\}$ and invariance of $\Omega \Rightarrow \Omega \equiv 0$ along maximal orbits.

Hence, for $n = 3$ we have $\Omega \equiv 0$, i.e., locally conformally flat.
Let $H = \text{SU}(p, q)$ or $H = \text{Sp}(p, q) = \text{SU}(2p, 2q) \cap \text{Sp}_n \mathbb{C}$.

Ad($H$)-invariant decomposition $\mathfrak{sl}_n K = \mathfrak{h} \oplus \mathfrak{m}$ for $K = \mathbb{C}, H$, respectively. Let $N$ be the null-cone w.r.t. the Killing form of $\mathfrak{sl}_n K$.

**Proposition**

$N_0 := \{ S \in N \mid S \text{ has } n \text{ distinct eigenvalues} \}$ is dense in $N$ and, for $S \in N_0$ there is an $1 \leq r \leq \frac{n}{2}$ such that $\text{stab}_{\text{ad}(\mathfrak{h})}(\mathbb{R} \cdot S)$ is given as

$$
\left( r \cdot \mathfrak{so}(1,1) \oplus (n-r) \cdot \mathfrak{u}(1) \right) \cap \mathfrak{sl}_n \mathbb{C} =
\left\{ \text{diag}(z_1, \ldots, z_r, ix_1, \ldots, ix_{n-2r}, -\overline{z}_r, \ldots, -\overline{z}_1) \mid z_i \in \mathbb{C}, x_j \in \mathbb{R} \right\} \cap \mathfrak{sl}_n \mathbb{C},
$$

if $K = \mathbb{C}$,

$$
\left( r \cdot \mathfrak{sl}_2 \mathbb{C} \oplus (n-2r) \cdot \mathfrak{sp}(1) \right), \text{ if } K = H.
$$

Again, the union of $H$-orbits of codimension $n - 3$ is dense in $\mathfrak{sp}_{\hat{p}, \hat{q}}$.

**Consequences for the holonomy reduction?**
Reductive Cartan connections

A Cartan connection $\eta$ of type $B \subset H$ is **reductive** if $\mathfrak{h}$ has an $\text{Ad}(B)$-inv complement $\mathfrak{n}$ in $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{n}$. Then $\eta$ decomposes into

$$\eta = \eta^\mathfrak{b} \oplus \eta^\mathfrak{n}$$

- $\eta^\mathfrak{b}$ a connection on $B$-bundle $\mathcal{H}$,
- $\eta^\mathfrak{n} \in T^*\mathcal{H} \otimes \mathfrak{b}$ is $\text{Ad}(B)$-inv.
- $\eta^\mathfrak{n}$ defines an isom $\psi_u : T_xM \to \mathfrak{h}/\mathfrak{b} \to \mathfrak{n}$, yielding a reduction of the frame bundle of $M$ to $\mathcal{H}$. I.e., $\eta^\mathfrak{b}$ induces a linear connection $\nabla^\eta$ on $TM$.

**Proposition**

Let $\eta$ be a reductive, torsion-free Cartan connection of type $B \subset H$. Assume that $\mathfrak{h}$ admits an $\text{Ad}_H$-invariant metric $K : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ such that $\mathfrak{h} = \mathfrak{b} \oplus \perp \mathfrak{n}$. Then there is a canonical metric $g^\eta$ on $M$ and an affine connection $\nabla^\eta$ with torsion $T^\eta$ such that:

- $\nabla^\eta T^\eta = 0$ and $\nabla^\eta g^\eta = 0$,
- $g^\eta(T^\eta(.,.),.)$ is totally skew-symmetric,
- $\text{Hol}(\nabla^\eta) \subset \text{Ad}_H(B) \subset O(\mathfrak{n}, K)$. 
Holonomy reduction to isotropy groups

Theorem

Let $G/H$ be a symmetric space with $\mathfrak{g}$ and $\mathfrak{h}$ simple of non-compact type, and invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Let $(M, [g])$ be a conformal manifold of signature $(p, q)$ with holonomy reduction to $\text{Ad}_G(H) \subset \text{SO}(\mathfrak{m}) \cong \text{SO}(p + 1, q + 1)$.

Assume there is a null vector $S \in \mathfrak{m}$ with stabilizer $B = \text{Stab}_H(\mathbb{R}S)$ with

1. $\mathfrak{b} = \text{LA}(B)$ is invariant under a Cartan involution of $\mathfrak{h}$,
2. the $H$-orbit of $[S]$ is open in the Möbius sphere $\mathbb{S}^{p,q}$ of $\mathfrak{m}$.

Then $M_0 \subset M$ corresponding to the $H$-orbit of $[S]$ in $\mathbb{S}^{p,q}$ has

- a canonical metric $g_0 \in [g|_{M_0}]$,
- a connection $\nabla^0$ with $\nabla^0 g^0 = 0$ and with skew-symmetric, $\nabla^0$-parallel torsion $T^0$, and
- $\text{Hol}(\nabla^0) \subset \text{Ad}_H(B) \subset \text{SO}(\mathfrak{h}/\mathfrak{b})$. 
\( \text{SL}_3 \mathbb{C}/\text{SU}(2, 1) \) and nearly para-Kähler structures

\( \text{SL}_n \mathbb{C}/\text{SU}(p, q) \) satisfies assumption (1) of the Thm and, for \( n = 3 \) also assumption (2). We find: \( \nabla^0 = \text{canonical connection} \) for a para-nearly Kähler structure \((g, J)\) of constant type \( \frac{1}{2} \), i.e.,

- \( J \in \text{End}(TM^0) \) with \( J^2 = 1 \) and \( J^* g = -g \),
- \( \nabla_X J(Y) = 0 \) for all \( X \in TM^0 \), where \( \nabla = \nabla^{LC} \),
- \( g(\nabla_X J(Y), \nabla_X J(Y)) = \frac{1}{2} \left(g(X, X)g(Y, Y) - g^2(X, Y) + g^2(JX, Y)\right) \)

Fact [Ivanov/Zamkovoy ’05]:

Six-dim’l nearly para-Kähler manifolds are of constant type \( \Lambda \) and Einstein with Einstein constant \( 5\Lambda \).

Theorem

If \((M, [g])\) has conformal holonomy in \( \text{Ad(\text{SU}(2, 1))} \subset \text{SO}(4, 4) \), then, on an open dense subset, there exists a nearly para-Kähler metric in \([g]\). In particular, the conformal holonomy preserves a time-like vector in \( \mathbb{R}^{4, 4} \), and is properly contained in \( \text{PSU}(2, 1) \).
SL₂\H/Sp(2, 1) and Sp(2, 1)/SL₂\C × Sp(1)

SL₂\H/Sp(2, 1) satisfies the assumptions of the Thm.

- The open orbits in the Möbius sphere are given by PSp(2, 1)/B with B = SL₂\C × Sp(1).
- This is a naturally reductive homogeneous space with metric Einstein \( K \) of signature (5, 7).
- The Ricci tensor of \( g^0 \) in \([g]\) is related to the one of \( K \) via

\[
\text{Ric}^{g_0}(X, Y) = \text{Ric}^K(\psi_u(X), \psi_u(Y)),
\]

and is thus also Einstein.

**Theorem**

*If \((M, [g])\) is a conformal manifold of signature (5, 7) with conformal holonomy in PSp(2, 1) ⊂ SO(6, 8), then on an open dense subset there is an Einstein metric \([g]\). In particular, the conformal holonomy is a proper subgroup of PSp(2, 1).*
Ambient construction 1: the ambient space

- A conformal class \([g]\) on \(M\) corresponds to an \(\mathbb{R}^+\)-principle fibre bundle, the cone
  \[
  \bigcup_{x \in M} \left\{ g_x \in \odot^2 T_xM \mid g \in [g] \right\} =: C \cup \mathbb{R}^+, \quad \delta_t(g_x) := (t^2 g_x)
  \]

- Tautological tensor on \(C\):
  \[g(U, V)|_{g_x} := g_x(d\pi(U), d\pi(V)),\]
  \[g(T, .) = 0\] for the fundamental vector field \(T\) of \(\delta\).

- Each \(g \in [g]\) trivialises \(C\),
  \[
  \mathbb{R}^+ \times M \cong C
  \]
  \[\quad (t, x) \mapsto (t^2 g_x, x)\]

- Get the ambient space from the cone:
  \[
  \tilde{M} := C \times (-\varepsilon, \varepsilon)
  \]

The \(\mathbb{R}^+\) action on \(C\) extends trivially to \(\tilde{M}\):
  \[\delta_t(g_x, \rho) := (\delta_t g_x, \rho).\]
Ambient construction 2: ambient metric

Definition

Let \((M, [g])\) be smooth manifold with conformal class of signature \((t, s)\). An ambient metric for \((M, [g])\) is a smooth metric \(\widetilde{g}\) on \(\widetilde{M}\) such that

1. \(\widetilde{g}\) is homogeneous of degree 2 w.r.t. the \(\mathbb{R}^+\)-action \(\delta\).
2. \(\iota^*\widetilde{g} = g\), for the inclusion \(\iota: C = C \times \{0\} \subset \widetilde{M}\).
3. \(\text{Ric}(\widetilde{g}) = 0\).

By fixing \(g \in [g]\), and thus trivialising \(C\), and a coordinate \(\rho\) such that \(\widetilde{M} \cong \mathbb{R}^+ \times M \times (-\varepsilon, \varepsilon) \ni (t, x, \rho)\), we have

\[
\widetilde{g} = 2t d\rho dt + 2\rho dt^2 + t^2 g_\rho,
\]

for a \(\rho\)-dependent family of metrics \(g_\rho\) with \(g_0 = g\) and subject to the condition \(\text{Ric}(\widetilde{g}) = 0\).
Theorem (C. Fefferman & C.R. Graham ’85, ’07)

- If $n := \dim M$ is odd, then
  1. there exists a formal power series solution $g_\rho$ to $\text{Ric}(\tilde{g}) = 0$,
  2. this solution is unique up to $\mathbb{R}^+$ invariant diffeomorphisms fixing $C \subset \tilde{M}$,
  3. if $[g]$ is analytic, then there exists an ambient metric.

- If $n$ is even, then there exists a formal power series solution to $\text{Ric}(\tilde{g}) = O(\rho^{n-2})$ which is uniquely determined up to terms of order $\frac{n}{2}$ in $\rho$. Furthermore, there exists a conformally invariant tensor $O \in \Gamma(\otimes^2 TM)$, such that $O \equiv 0 \iff \text{Ric}(\tilde{g}) = 0$ to infinite order.

$$O = \Delta_g^{n/2-2} \left( \Delta_g P - \nabla^2 \text{tr}(P) \right) + \text{lower order terms, no trace and no div.}$$

Expanding $g_\rho$ as a power series $g_\rho = g + \sum_{k=1}^\infty \rho^k \mu_k$, then

$$
\begin{align*}
(\mu_1)_{ab} &= 2P_{ab} \\
(n-4)(\mu_2)_{ab} &= -B_{ab} + (n-4)P_a^c P_{bc}, \text{ etc...}
\end{align*}
$$

where $B_{ab} = \nabla_c C_{ab}^c - P_{cd} W_{ab}^c d$ is the Bach tensor.
Relation to tractors:

**Theorem (Graham/Willse ’11)**

Let \((M, [g])\) be a real analytic conformal structure on an odd-dim’l mf \(M\). Then parallel tractors in \(\otimes^k T\) can be uniquely extended to parallel ambient tensors for Ricci flat ambient space \((\tilde{M}, \tilde{g})\).

\[
\text{Hol}(\tilde{M}, \tilde{g}) = \text{Stab}(\tilde{R}) \neq \text{SO}(p + 1, q + 1) \text{ irreducible, with } \tilde{R} \text{ an algebraic curvature tensor, then } \text{Ric} = 0 \implies (\tilde{M}, \tilde{g}) \text{ flat.}
\]

Using this and the inclusion \(\text{Hol}(M, [g]) \subset \text{Hol}(\tilde{M}, \tilde{g})\) we obtain:

**Theorem**

Let \((M, [g])\) be a real analytic conformal structure on an odd-dim’l mf \(M\) with irreducible conformal holonomy \(H = \text{Stab}(w)\). Then \(H\) is equal to \(\text{SO}(p + 1, q + 1)\) or \(G_2(2)\).
Speculations

1. Isotropy groups of irreducible symmetric spaces cannot be conformal holonomy groups.

2. Conformal holonomy groups are always pseudo-Riemannian holonomy groups of Ricci flat manifolds.

3. Lie algebras $\mathfrak{h} \subset \mathfrak{so}(\mathbb{T})$ for which $Ric : \mathcal{K}(\mathfrak{h}) \to \bigodot^2 \mathbb{T}^* \otimes \mathbb{T}^*$ is injective cannot be conformal holonomy algebras.

Thank you!