Conformal holonomy, symmetric spaces, and skew symmetric torsion

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[arXiv:1208.2191]

Conformal geometry

Conformal manifold: $(M, [g])$, $[g]$ = class of conformally equivalent semi-Riemannian metrics, $dim(M) = p + q$.

- Flat model: $\mathbb{S}^{p,q} := \mathcal{N}/\mathbb{R}^+ = G/P$ with
  - $\mathcal{N}$ null cone in $\mathbb{R}^{p+1,q+1}$ with $\mathbb{R}^+$-action,
  - $G := \mathrm{SO}^0(p+1, q+1)$, $P := \mathrm{Stab}_G(\textit{null line in } \mathbb{R}^{p+1,q+1})$.

The curved version is described by

- A P-bundle $\mathcal{G}$ (conformal Cartan bundle)

$$\begin{array}{ccccc} \mathcal{G} & \overset{G_+}{\rightarrow} & \mathcal{G}^0 = \{\textit{conformal frames}\} & \overset{G_0=\mathrm{CO}(p,q)}{\rightarrow} & M \\ \uparrow & & & & \end{array}$$

horizontal subspaces in $T\mathcal{G}^0$, kernel of some ω^g, $g \in [g]$

- Normal conformal *Cartan connection* $\omega \in T^*\mathcal{G} \otimes \mathfrak{g}$,
  - $\omega : T\mathcal{G} \rightarrow \mathfrak{g}$ parallelism, $R_p^*\omega = \mathrm{Ad}(p^{-1})\omega$, $\omega(\widetilde{X}) = X \in \mathfrak{p}$,
  - $\Omega(X, Y) \in \mathfrak{p}$ ("torsion-free") and curvature condition.
- E.g., flat model: $\mathcal{G} := G \rightarrow G/P$ and ω = Maurer-Cartan form of G.

What is conformal holonomy?

ω does not give horizontal subspaces and no parallel transport.

- ω defines connection $\hat{\omega}$ on G-bundle $\hat{\mathcal{G}} = \mathcal{G} \times_P G$ by $\hat{\omega}|_{\mathcal{G}} = \omega$.
- *tractor connection* $\hat{\nabla}$ on *(standard) tractor bundle*

$$\mathcal{T} = \hat{\mathcal{G}} \times_G \mathbb{R}^{p+1,q+1} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1}.$$

Conformal holonomy: $\mathrm{Hol}_x(M,[g]) := \mathrm{Hol}_x(\mathcal{T},\hat{\nabla}) \simeq \mathrm{Hol}_p(\hat{\mathcal{G}},\hat{\omega}) \subset G$.

1. Which groups can occur?
2. Are they holonomy groups of semi-Riemannian metrics?
3. Which structures correspond to holonomy reductions?

Obstacles:

- No obvious algebraic criterion for holonomy algebra.
- Hol is defined up to conjugation in G, not only in P.
- Reduction to subgroup H might not define a Cartan connection on M, as we could have $dim(H/H \cap P) \neq dim(M)$.

Classification of semi-Riemannian holonomy

Let $\mathfrak{h}$ be the holonomy algebra of a semi-Riemannian manifold.
Ambrose-Singer holonomy theorem,

$$\mathfrak{h} = \mathrm{span}\left\{ \mathcal{P}_\gamma^{-1} \circ \mathcal{R}(X,Y) \circ \mathcal{P}_\gamma \in \mathrm{SO}(T_pM) \mid \gamma(0) = p, X, Y \in T_{\gamma(1)}M \right\}$$

and 1st Bianchi-identity for $\mathcal{R}$ imply

$$(B) \qquad \mathfrak{h} = \mathrm{span}\left\{R(x,y) \mid R \in \mathcal{K}(\mathfrak{h}), x, y \in \mathbb{R}^n\right\},$$

with $\mathcal{K}(\mathfrak{h}) := \left\{ R \in \Lambda^2 \mathbb{R}^{n^*} \otimes \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0 \right\}$. For $\mathfrak{h} \subset \mathfrak{so}(p,q)$ irreducible, (B) yields a classification (Berger '55).

No such algebraic criterion known for conformal holonomy.

Conformally Einstein metrics and parallel tractors

Q: When can we rescale a given metric g to an Einstein metric?

$\rightsquigarrow$ Overdetermined PDE system: $\tilde{g} = \sigma^{-2} \cdot g$ is Einstein $\Longleftrightarrow$

$$Hess^g(\sigma) + P^g \cdot \sigma \;=\; \lambda \cdot g$$

for some function λ. $P^g = \frac{1}{n-2}(Ric^g - \frac{S}{2n-2)} g)$ is the Schouten tensor of g.

Let $g_\Lambda \in [g]$ be an Einstein metric, i.e. $Ric = (n-1)\Lambda \cdot g_\Lambda$. Then

1. $\mathcal{T}$ admits a parallel section η with $\hat{g}(\eta, \eta) = -\Lambda$ and hence, $\mathrm{Hol}(M, [g])$ admits an invariant vector.
2. the *Fefferman-Graham ambient* metric $\widetilde{g}$ is given as
   - $\Lambda \neq 0$: $\widetilde{g} = -\frac{1}{\Lambda} \mathrm{d}s^2 + \underbrace{\frac{1}{\Lambda} \mathrm{d}r^2 + r^2 g_\Lambda}_{cone\ metric}$
   - $\Lambda = 0$: $\widetilde{g} = -\mathrm{d}u\mathrm{d}t + t^2 g_\Lambda$

   and $\mathrm{Hol}(\widetilde{M}, \widetilde{g}) = \mathrm{Hol}(M, [g])$, i.e., the conformal holonomy is a semi-Riemannian holonomy.

Conversely, if $\mathrm{Hol}(M, [g])$ admits an invariant line, then on an open dense subset M_0 of M, locally, there exist an Einstein metric $g_\Lambda \in [g|_{M_0}]$, and all of the above holds for $\mathrm{Hol}(M_0, [g|_{M_0}])$.

Tractor bundle and its constant sections [Bailey/Eastwood/Gover]

Let $P = \mathrm{Stab}_G(L)$, $L =$ null line.

- Filtration $L \subset L^\perp \subset \mathbb{R}^{p+1,q+1}$ gives $\mathcal{L} \subset \mathcal{L}^\perp \subset \mathcal{T}$.
- Projection $\mathcal{L}^\perp = \mathcal{G} \times_P L^\perp \to \mathcal{L}^\perp/\mathcal{L} \simeq TM \simeq \mathcal{G}^0 \times_{\mathrm{CO}_0(p,q)} (L^\perp/L)$.

Every $g \in [g]$ splits $\mathcal{T} = \mathcal{L}^\perp \oplus \underline{\mathbb{R}} = \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$ with

$$\hat{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\nabla}_X \begin{pmatrix} \tau \\ Y \\ \sigma \end{pmatrix} = \begin{pmatrix} \mathrm{d}\tau(X) - P(X,Y) \\ \nabla_X Y + \tau X + \sigma (X \lrcorner P)^\sharp \\ \mathrm{d}\sigma(X) - g(X,Y) \end{pmatrix},$$

$\hat{X} = \begin{pmatrix} \rho \\ X \\ \sigma \end{pmatrix}$ with $\hat{\nabla}\hat{X} = 0 \iff \sigma^{-2} g$ is Einstein metric on the open and dense complement of $zero(\sigma)$.

Irreducible case: Riemannian conf. structures

Theorem (Berger '55, Di Scala/Olmos '00)

If $H \subset \mathrm{SO}^0(1, n+1)$ acts irreducibly, then $H = \mathrm{SO}^0(1, n+1)$.

⇒ A Riemannian conformal manifold has generic conformal holonomy unless: On an open dense subset of M

- $[g]$ contains an Einstein metric or
- a certain product of Einstein metrics:
  Decomposition thm by S. Armstrong '04: $\mathrm{Hol}(M, [g])$ has invariant subspace of dim $k > 1 \iff$ locally, $[g]$ contains product of Einstein metrics g_1 and g_2 of dim $(k-1)$ and $(n-k+1)$ with

$$\tfrac{n-k+1}{k-1}\Lambda_1 = -\tfrac{k-1}{n-k+1}\Lambda_2$$

  and the conformal holonomy is given by the holonomy of the products of cones. (cf. Leitner '04, Leitner/Gover '09).

Irreducible case: Lorentzian conf. structures

Theorem (Di Scala/L '11)

Let $H \subset \mathrm{SO}^0(2,n)$ act irreducibly. Then H is conjugated to

1. $\mathrm{SO}^0(2,n)$,
2. $\mathrm{SU}(1,p)$, $\mathrm{U}(1,p)$, $\mathrm{U}(1)\cdot\mathrm{SO}^0(1,p)$ *if* $p > 1$, n *even*
3. $\mathrm{SO}^0(1,2) \overset{irr.}{\subset} \mathrm{SO}(2,3)$, *for* $n = 3$.

- $\mathrm{U}(1,p)$ and $\mathrm{U}(1)\cdot\mathrm{SO}^0(1,p)$ cannot be conformal holonomy groups: $\mathrm{Hol}([g]) \subset \mathrm{U}(r,s) \Rightarrow \mathrm{Hol}([g]) \subset \mathrm{SU}(r,s)$ [Leitner'06, Cap/Gover'06]
- $\mathrm{Hol}([g]) = \mathrm{SU}(1,p)$: Fefferman space in conformal class
- What about (3)?
  (3) corresponds to the symmetric space $M^5 := \mathrm{SL}_3\mathbb{R}/\mathrm{SO}^0(1,2)$ with signature $(2,3)$ metric given by the Killing form of $\mathfrak{sl}_3\mathbb{R}$.

Isotropy representation of $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(p,q)$

Semi-Riemannian irreducible symmetric space $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(p,q)$, $p+q=n$.

- symmetric decomposition $\mathfrak{sl}_n\mathbb{R} = \mathfrak{so}(p,q) \oplus \mathfrak{m}$, where $\mathfrak{m} = \{X \in \mathfrak{gl}_n\mathbb{R} \mid X^\top \mathbf{1}_{p,q} = \mathbf{1}_{p,q} X\}$, inv. under $Ad(\mathrm{SO}(p,q))$.
- irred. rep'n

$$Ad : \mathrm{SO}(p,q) \to \mathrm{SO}(\mathfrak{m}, K_{\mathfrak{sl}_n\mathbb{R}}) = \mathrm{SO}\left(pq, \tfrac{p(p+1)+q(q+1)-2}{2}\right)$$

Theorem (Alt/DiScala/L '12)

If the conformal holonomy of a conformal manifold $(M,[g])$ of signature $(2,1)$ is contained in $\mathrm{Ad}(\mathrm{SO}(2,1))$, then g is locally conformally flat.

Corollary

If the conformal holonomy group of a Lorentzian conformal manifold acts irreducibly, then it is equal to $\mathrm{SO}(2,n)$ or $\mathrm{SU}(1,p)$.

Holonomy reductions via parallel sections

Principal G-bundle $\hat{\mathcal{G}} \to M$ with connection $\hat{\omega}$ on $\hat{\mathcal{G}}$. Assume:

- $H \subset G$ closed such that $\mathrm{Hol}_p(\hat{\mathcal{G}}, \hat{\omega}) \subset H$,
- Reduction to the H-bundle (depending on $p \in \hat{\mathcal{G}}$),

$$\mathcal{H}_p = \underbrace{\{\gamma(1) \mid \gamma(0) = p, \gamma \text{ horizontal}\}}_{\text{holonomy bundle}} \cdot H \subset \hat{\mathcal{G}}$$

$\mathbb{W}$ a G-module, $\mathcal{W} = \hat{\mathcal{G}} \times_G \mathbb{W}$ associated vector bundle.

$$C^\infty(\hat{\mathcal{G}}, \mathbb{W})^G \simeq \Gamma(\mathcal{W}), \; s \mapsto \sigma(x) = [p, s(p)] \text{ for } p \in \hat{\mathcal{G}}_x.$$

- $\sigma \in \Gamma(\mathcal{W})$ defines map $M \ni x \mapsto s(\hat{\mathcal{G}}_x) =: \mathcal{O}_x = G$-orbit in $\mathbb{W}$.
- $\hat{\omega}$ on $\hat{\mathcal{G}}$ induces a covariant derivative $\hat{\nabla}$ on $\mathcal{W}$.
  $\hat{\nabla}\sigma = 0 \iff s \circ \gamma$ const. for all horizontal curves γ in $\hat{\mathcal{G}}$.
  Hence, $\hat{\nabla}\sigma = 0$ implies $\mathrm{Hol}_p(\hat{\omega}) \subset \mathrm{Stab}_G(s(p))$ and defines $s(\hat{\mathcal{G}}_x) \equiv G \cdot s(p) =: \mathcal{O}$.

$$\sigma \in \Gamma(\mathcal{W}), \hat{\nabla}\sigma = 0 \;\mapsto\; \mathcal{O} = G/\mathrm{Stab}_G(w) \text{ for } w \in \mathcal{O}$$

Curved orbits [Čap/Gover/Hammerl '11]

$\hat{\mathcal{G}}$ and $\hat{\omega}$ defined by (normal conformal) Cartan connection ω on $\mathcal{G} \xrightarrow{P} M$ via $\hat{\mathcal{G}} = \mathcal{G} \times_P G$ and $\hat{\nabla}$ on $\mathcal{W} = \mathcal{G} \times_P \mathbb{W}$.
ω has a holonomy reduction of type $\mathcal{O}$, i.e., $\exists\, \sigma \in \Gamma(\mathcal{W})$ with $\hat{\nabla}\sigma = 0$ defining $s \in C^\infty(\hat{\mathcal{G}}, \mathbb{W})^G$ with G-orbit $\mathcal{O}$. Note:

- $\hat{\omega}$-horizontal curves leave $\mathcal{G}$ if $\mathrm{Hol}_p(\hat{\omega}) \not\subset P$.
- Although G-orbits are the same, P-orbits $s(\mathcal{G}_x) \subset \mathcal{O}$ might change with $x \in M$.
- $s(\mathcal{G}_x) = P{\cdot}w =: [w] \in P\backslash\mathcal{O}$ is the P-orbit type of σ at $x \in M$,

$$M = \bigcup_{[w] \in P\backslash\mathcal{O}} M_{[w]}, \text{ with } M_{[w]} := \{x \in M \mid s(\mathcal{G}_x) = [w]\}.$$

- For $w \in \mathcal{O}$ set $G_w = \mathrm{Stab}_G(w)$. Then $P\backslash\mathcal{O} = P\backslash G/G_w \simeq G_w\backslash G/P = G_w\backslash\mathbb{S}^{p,q}$. I.e.,

$$\begin{array}{rcl} P\text{-orbits in } \mathcal{O} = G/G_w & \leftrightarrow & G_w\text{-orbits in } G/P \\ P \cdot g \cdot G_w & \mapsto & G_w \cdot g^{-1} \cdot P \end{array}$$

Curved orbits and holonomy reduction

Theorem (Čap/Gover/Hammerl '11)

Let ω be a Cartan connection of type $P \subset G$ with curvature Ω and with a holonomy reduction of type $\mathcal{O} = G/G_w$.

- *Let $w \in \mathcal{O}$ with P-orbit $[w] := P \cdot w = PeG_w$ in $\mathcal{O}$,*
- *$G_w/P_w = G_w eP$ the corresponding G_w-orbit in G/P, $P_w := G_w \cap P$.*

Then, $\forall\ x \in M_{[w]}\ \exists$ nbhd. U of x in M and a diffeom. $\phi : U \to V \subset G/P$:

- *$\phi(x) = eP$, $\phi(U \cap M_{[w]}) = V \cap G_w eP$,* $\begin{array}{ccc} U & \xrightarrow{\phi} & V \subset G/P \\ \downarrow & & \downarrow \\ P\backslash\mathcal{O} & \to & G_w\backslash G/P \end{array}$ *comm.*
- *ω induces a Cartan connection of type $P_w \subset G_w$ on* $\begin{array}{ccc} \mathcal{G}_w & \subset & \mathcal{G} \\ \downarrow & & \downarrow \\ M_{[w]} & \subset & M \end{array}$
  whose curvature is the restriction of Ω to $\mathcal{G}_w$ with values in $\mathfrak{p}_w$.

Proof of Thm for $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(p,q)$

$\mathrm{Ad}(\mathrm{SO}(p,q))$-invariant decomposition $\mathfrak{sl}_n\mathbb{R} = \mathfrak{so}(p,q) \oplus \mathfrak{m}$. Then

- $H := \mathrm{Ad}(\mathrm{SO}(p,q)) \subset \mathrm{SO}(\mathfrak{m}, K_{\mathfrak{sl}_n\mathbb{R}})$ is the stabilisier of a curvature tensor $R \in \mathbb{W} := \Lambda^2\mathfrak{m} \otimes \mathfrak{so}(\mathfrak{m}, K_{\mathfrak{sl}_n\mathbb{R}})$.
- The null cone $\mathcal{N}$ in $\mathfrak{m}$ consists of matrices S with $\mathrm{tr}(S^2) = 0$ and defines the Möbius sphere $\mathcal{N} \to \mathbb{S}^{\hat{p},\hat{q}} = \mathcal{N}/\mathbb{R}^*$.

Proposition

Let $\mathcal{N}_0 := \{S \in \mathcal{N} \mid S$ has n distinct eigenvalues, possibly in $\mathbb{C}\}$.
Then $\mathcal{N}_0$ is dense in $\mathcal{N}$ and, for all $S \in \mathcal{N}_0$, $\mathfrak{stab}_{\mathrm{ad}(\mathfrak{h})}(\mathbb{R}\cdot S) = \{0\}$.
I.e., the union of H-orbits of codimension $n-3$ is dense in $\mathbb{S}^{\hat{p},\hat{q}}$.

CGH-Thm $\Rightarrow$
$M_0 := \{x \in M \mid s(\mathcal{G}_x)$ corresponds to orbit of max dim in $\mathbb{S}^{\hat{p},\hat{q}}\}$ is dense.
$\mathfrak{p}_w = \{0\}$ and invariance of $\Omega \Rightarrow \Omega \equiv 0$ along maximal orbits.
Hence, for $n = 3$ we have $\Omega \equiv 0$, i.e., locally conformally flat.

$\mathrm{SL}_n\mathbb{C}/\mathrm{SU}(p,q)$ and $\mathrm{SL}_n\mathbb{H}/\mathrm{Sp}(p,q)$

Let $H = \mathrm{SU}(p,q)$ or $H = \mathrm{Sp}(p,q) = \mathrm{SU}(2p,2q) \cap \mathrm{Sp}_n\mathbb{C}$.
$\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{sl}_n\mathbb{K} = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$, respectively.
Let $\mathcal{N}$ be the null-cone w.r.t. the Killing form of $\mathfrak{sl}_n\mathbb{K}$.

Proposition

$\mathcal{N}_0 := \{S \in \mathcal{N} \mid S$ has n distinct eigenvalues$\}$ is dense in $\mathcal{N}$ and, for $S \in \mathcal{N}_0$ there is an $1 \leq r \leq \frac{n}{2}$ such that $\mathfrak{stab}_{\mathrm{ad}(\mathfrak{h})}(\mathbb{R}\cdot S)$ is given as

- $\Big(r \cdot \mathfrak{so}(1,1) \oplus (n-r)\cdot \mathfrak{u}(1)\Big) \cap \mathfrak{sl}_n\mathbb{C} =$

  $\Big\{\mathrm{diag}(z_1, \dots, z_r, ix_1, \dots, ix_{n-2r}, -\overline{z_r}, \dots, -\overline{z_1}) \mid z_i \in \mathbb{C}, x_j \in \mathbb{R}\Big\} \cap \mathfrak{sl}_n\mathbb{C},$

  if $\mathbb{K} = \mathbb{C}$,
- $r \cdot \mathfrak{sl}_2\mathbb{C} \oplus (n-2r)\cdot \mathfrak{sp}(1)$, *if* $\mathbb{K} = \mathbb{H}$.

Again, the union of H-orbits of codimension $n-3$ is dense in $\mathbb{S}^{\hat{p},\hat{q}}$.

Consequences for the holonomy reduction?

Reductive Cartan connections

A Cartan connection η of type $B \subset H$ is *reductive* if $\mathfrak{b}$ has an $\mathrm{Ad}(B)$-inv complement $\mathfrak{n}$ in $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{n}$. Then η decomposes into

$$\eta = \eta^{\mathfrak{b}} \oplus \eta^{\mathfrak{n}}$$

- $\eta^{\mathfrak{b}}$ a connection on B-bundle $\mathcal{H}$,
- $\eta^{\mathfrak{n}} \in T^*\mathcal{H} \otimes \mathfrak{b}$ is $\mathrm{Ad}(B)$-inv.
- $\eta^{\mathfrak{n}}$ defines an isom $\psi_u : T_x M \to \mathfrak{h}/\mathfrak{b} \to \mathfrak{n}$, yielding a reduction of the frame bdle of M to $\mathcal{H}$. I.e., $\eta^{\mathfrak{b}}$ induces a linear connection ∇^η on TM.

Proposition

Let η be a reductive, torsion-free Cartan connection of type $B \subset H$. Assume that $\mathfrak{h}$ admits an Ad_H-invariant metric $K : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ such that $\mathfrak{h} = \mathfrak{b} \oplus^\perp \mathfrak{n}$. Then there is a canonical metric g^η on M and an affine connection ∇^η with torsion T^η such that:

- $\nabla^\eta T^\eta = 0$ *and* $\nabla^\eta g^\eta = 0$,
- $g^\eta(T^\eta(.,.),.)$ *is totally skew-symmetric,*
- $\mathrm{Hol}(\nabla^\eta) \subset \mathrm{Ad}_H(B) \subset \mathrm{O}(\mathfrak{n}, K)$.

Holonomy reduction to isotropy groups

Theorem

Let G/H be a symmetric space with $\mathfrak{g}$ and $\mathfrak{h}$ simple of non-compact type, and invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.
Let $(M, [g])$ be a conformal manifold of signature (p, q) with holonomy reduction to $\mathrm{Ad}_G(H) \subset \mathrm{SO}(\mathfrak{m}) \simeq \mathrm{SO}(p+1, q+1)$.
Assume there is a null vector $S \in \mathfrak{m}$ with stabilizer $B = \mathrm{Stab}_H(\mathbb{R}S)$ with

1. *$\mathfrak{b} = LA(B)$ is invariant under a Cartan involution of $\mathfrak{h}$,*
2. *the H-orbit of $[S]$ is open in the Möbius sphere $\mathbb{S}^{p,q}$ of $\mathfrak{m}$.*

Then $M_0 \subset M$ corresponding to the H-orbit of $[S]$ in $\mathbb{S}^{p,q}$ has

- *a canonical metric $g_0 \in [g|_{M_0}]$,*
- *a connection ∇^0 with $\nabla^0 g^0 = 0$ and with skew-symmetric, ∇^0-parallel torsion T^0, and*
- *$\mathrm{Hol}(\nabla^0) \subset \mathrm{Ad}_H(B) \subset \mathrm{SO}(\mathfrak{h}/\mathfrak{b})$.*

$\mathrm{SL}_3\mathbb{C}/\mathrm{SU}(2,1)$ and nearly para-Kähler structures

$\mathrm{SL}_n\mathbb{C}/\mathrm{SU}(p,q)$ satisfies assumption (1) of the Thm and, for $n = 3$ also assumption (2). We find: $\nabla^0 =$ canonical connection for a para-nearly Kähler structure (g, J) of constant type $\frac{1}{2}$, i.e.,

- $J \in End(TM^0)$ with $J^2 = \mathbf{1}$ and $J^*g = -g$,
- $\nabla_X J(X) = 0$ for all $X \in TM^0$, where $\nabla = \nabla^{LC}$,
- $g(\nabla_X J(Y), \nabla_X J(Y)) = \frac{1}{2}\left(g(X,X)g(Y,Y) - g^2(X,Y) + g^2(JX,Y)\right)$

Fact [Ivanov/Zamkovoy '05]:

Six-dim'l nearly para-Kähler manifolds are of constant type Λ and Einstein with Einstein constant 5Λ.

Theorem

If $(M, [g])$ has conformal holonomy in $\mathrm{Ad}(\mathrm{SU}(2,1)) \subset \mathrm{SO}(4,4)$, then, on an open dense subset, there exists a nearly para-Kähler metric in $[g]$. In particular, the conformal holonomy preserves a time-like vector in $\mathbb{R}^{4,4}$, and is properly contained in $\mathrm{PSU}(2,1)$.

$\mathrm{SL}_2\mathbb{H}/\mathrm{Sp}(2,1)$ and $\mathrm{Sp}(2,1)/\mathrm{SL}_2\mathbb{C}\times\mathrm{Sp}(1)$

$\mathrm{SL}_2\mathbb{H}/\mathrm{Sp}(2,1)$ satisfies the assumptions of the Thm.

- The open orbits in the Möbius sphere are given by $\mathrm{PSp}(2,1)/B$ with $B = \mathrm{SL}_2\mathbb{C}\times\mathrm{Sp}(1)$.
- This is a naturally reductive homogeneous space with metric Einstein K of signature $(5,7)$.
- The Ricci tensor of g^0 in $[g]$ is related to the one of K via

$$\mathrm{Ric}^{g_0}(X,Y) = \mathrm{Ric}^K(\psi_u(X),\psi_u(Y)),$$

  and is thus also Einstein.

Theorem

If $(M,[g])$ is a conformal manifold of signature $(5,7)$ with conformal holonomy in $\mathrm{PSp}(2,1)\subset\mathrm{SO}(6,8)$, then on an open dense subset there is an Einstein metric $[g]$. In particular, the conformal holonomy is a proper subgroup of $\mathrm{PSp}(2,1)$.

Ambient construction 1: the ambient space

- A conformal class $[g]$ on M corresponds to an $\mathbb{R}^+$-principle fibre bundle, the cone

$$\bigcup_{x\in M}\left\{g_x\in\odot^2 T_xM \mid g\in[g]\right\} =: \mathcal{C} \quad \circlearrowright \mathbb{R}^+, \quad \delta_t(g_x):=(t^2g_x)$$
$$\pi\downarrow$$
$$M$$

- Tautological tensor on $\mathcal{C}$: $\mathbf{g}(U,V)|_{g_x} := g_x(\mathrm{d}\pi(U),\mathrm{d}\pi(V))$,
  - $\mathbf{g}(T,.)=0$ for the fundamental vector field T of δ.
  - of degree 2 w.r.t. the $\mathbb{R}^+$-action, i.e. $\delta_t^*\mathbf{g}=t^2\mathbf{g}$.
- Each $g\in[g]$ trivialises $\mathcal{C}$,

$$\begin{array}{ccc}\mathbb{R}^+\times M & \simeq & \mathcal{C}\\ (t,x) & \mapsto & (t^2g_x,x)\end{array}$$

- Get the ambient space from the cone:

$$\widetilde{M}:=\mathcal{C}\times(-\varepsilon,\varepsilon)$$

The $\mathbb{R}^+$ action on $\mathcal{C}$ extends trivially to $\widetilde{M}$: $\delta_t(g_x,\rho):=(\delta_tg_x,\rho)$.

Ambient construction 2: ambient metric

Definition

Let $(M, [g])$ be smooth manifold with conformal class of signature (t, s). An *ambient metric* for $(M, [g])$ is a smooth metric $\widetilde{g}$ on $\widetilde{M}$ such that

1. $\widetilde{g}$ is homogeneous of degree 2 w.r.t. the $\mathbb{R}^+$-action δ.
2. $\iota^*\widetilde{g} = \mathbf{g}$, for the inclusion $\iota : \mathcal{C} = \mathcal{C} \times \{0\} \subset \widetilde{M}$.
3. $\mathrm{Ric}(\widetilde{g}) = 0$.

By fixing $g \in [g]$, and thus trivialising $\mathcal{C}$, and a coordinate ρ such that $\widetilde{M} \simeq \mathbb{R}^+ \times M \times (-\varepsilon, \varepsilon) \ni (t, x, \rho)$, we have

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2 g_\rho,$$

for a ρ-dependent family of metrics g_ρ with $g_0 = g$ and subject to the condition $\mathrm{Ric}(\widetilde{g}) = 0$.

Ambient construction 3: The Fefferman-Graham result

Theorem (C. Fefferman & C.R. Graham '85, '07)

- *If* $n := \dim M$ *is odd, then*
  1. *there exists a formal power series solution* g_ρ *to* $\mathrm{Ric}(\widetilde{g}) = 0$,
  2. *this solution is unique up to* $\mathbb{R}^+$ *invariant diffeomorphisms fixing* $\mathcal{C} \subset \widetilde{M}$,
  3. *if* $[g]$ *is analytic, then there exists an ambient metric.*
- *If n is even, then there exists a formal power series solution to* $\mathrm{Ric}(\widetilde{g}) = O(\rho^{\frac{n-2}{2}})$ *which is uniquely determined up to terms of order* $\frac{n}{2}$ *in* ρ. *Furthermore, there exists a conformally invariant tensor* $\mathcal{O} \in \Gamma(\odot^2 TM)$, *such that* $\mathcal{O} \equiv 0 \iff \mathrm{Ric}(\widetilde{g}) = 0$ *to infinite order.*

$\mathcal{O} = \Delta_g^{n/2-2}\left(\Delta_g P - \nabla^2 \mathrm{tr}(P)\right) +$ lower order terms, no trace and no div.

Expanding g_ρ as a power series $g_\rho = g + \sum_{k=1}^{\infty} \rho^k \mu_k$, then

$$
\begin{aligned}
(\mu_1)_{ab} &= 2P_{ab} \\
(n-4)(\mu_2)_{ab} &= -B_{ab} + (n-4)P_a{}^c P_{bc}, \ \textit{etc...}
\end{aligned}
$$

where $B_{ab} = \nabla_c C_{ab}{}^c - P_{cd} W^c{}_{ab}{}^d$ is the Bach tensor.

Fefferman-Graham ambient metric and conf. holonomy

Relation to tractors:

Theorem (Graham/Willse '11)

Let $(M,[g])$ be a real analytic conformal structure on an odd-dim'l mf M. Then parallel tractors in $\otimes^k\mathcal{T}$ can be uniquely extended to parallel ambient tensors for Ricci flat ambient space $(\widetilde{M},\widetilde{g})$.

$\mathrm{Hol}(\widetilde{M},\widetilde{g}) = \mathrm{Stab}(\widetilde{R}) \neq \mathrm{SO}(p+1,q+1)$ irreducible, with $\tilde{R}$ an algebraic curvature tensor, then $Ric = 0$ implies $(\widetilde{M},\widetilde{g})$ flat.
Using this and the inclusion $\mathrm{Hol}(M,[g]) \subset \mathrm{Hol}(\widetilde{M},\widetilde{g})$ we obtain:

Theorem

Let $(M,[g])$ be a real analytic conformal structure on an odd-dim'l mf M with irreducible conformal holonomy $H = \mathrm{Stab}(w)$. Then H is equal to $\mathrm{SO}(p+1,q+1)$ or $G_{2(2)}$.

Speculations

1. Isotropy groups of irreducible symmetric spaces cannot be conformal holonomy groups.
2. Conformal holonomy groups are always pseudo-Riemannian holonomy groups of Ricci flat manifolds.
3. Lie algebras $\mathfrak{h} \subset \mathfrak{so}(\mathbb{T})$ for which $Ric : \mathcal{K}(\mathfrak{h}) \to \odot^2\mathbb{T}^*$ is injective cannot be conformal holonomy algebras.

Thank you!