Conformal classes with explicit ambient metrics

Thomas Leistner

SCHOOL OF MATHEMATICAL SCIENCES
THE UNIVERSITY of ADELAIDE

The Geometry of Differential Equations
Mathematical Sciences Institute, ANU
September 23, 2011

[Joint work with Paweł Nurowski (Warsaw)]

Outline

Let $(M^n,[g])$ be a smooth manifold equipped with a conformal structure of semi-Riemannian metrics, $[g]:=\left\{\mathrm{e}^{2\varphi}g \mid \varphi\in C^\infty(M)\right\}$. Invariant descriptions:

1. Fefferman-Graham ambient metric: Describe the conformal geometry of $(M^n,[g])$ in terms of the metric geometry of an *ambient space* $(\widetilde{M}^{n+2},\widetilde{g})$ as in the the flat case $S^n\simeq\mathcal{C}^+/\mathbb{R}^+$ with $\mathcal{C}^+\subset\mathbb{R}^{1,n+1}=:\widetilde{M}$.
2. Tractor bundle associated to the Cartan bundle with normal conformal Cartan connection.

Construct explicit ambient metrics for

1. n-dimensional pp-waves
2. Nurowski's conformal structures associated to $(2,3,5)$-distributions with split G_2 Cartan connection $\rightsquigarrow$ Holonomy split G_2 ambient metric

Construction of the ambient space

- A conformal class $[g]$ on M corresponds to an $\mathbb{R}^+$-principle fibre bundle, the cone

$$\bigcup_{x\in M}\left\{g_x \in \odot^2 T_xM \mid g\in[g]\right\} =: \mathcal{C} \quad \circlearrowleft \mathbb{R}^+, \quad \delta_t(g_x) := (t^2 g_x)$$
$$\pi \downarrow$$
$$M$$

- Tautological tensor on $\mathcal{C}$: $\quad \mathbf{g}(U,V)|_{g_x} := g_x(\mathrm{d}\pi(U), \mathrm{d}\pi(V))$,
    - $\mathbf{g}(T,.) = 0$ for the fundamental vector field T of δ.
    - of degree 2 w.r.t. the $\mathbb{R}^+$-action, i.e. $\delta_t^*\mathbf{g} = t^2\mathbf{g}$.
- Each $g\in[g]$ trivialises $\mathcal{C}$,

$$\begin{array}{ccc} \mathbb{R}^+\times M & \simeq & \mathcal{C} \\ (t,x) & \mapsto & (t^2 g_x, x) \end{array}$$

- Get the ambient space from the cone:

$$\widetilde{M} := \mathcal{C}\times(-\varepsilon,\varepsilon)$$

The $\mathbb{R}^+$ action on $\mathcal{C}$ extends trivially to $\widetilde{M}$: $\delta_t(g_x,\rho) := (\delta_t g_x, \rho)$.

Ambient metric

Definition

Let $(M,[g])$ be smooth manifold with conformal class of signature (t,s). An *ambient metric* for $(M,[g])$ is a smooth metric $\widetilde{g}$ on $\widetilde{M}$ such that

1. $\widetilde{g}$ is homogeneous of degree 2 w.r.t. the $\mathbb{R}^+$-action δ.
2. $\iota^*\widetilde{g} = \mathbf{g}$, for the inclusion $\iota : \mathcal{C} = \mathcal{C} \times \{0\} \subset \widetilde{M}$.
3. $\mathrm{Ric}(\widetilde{g}) = 0$.

This implies that by fixing $g \in [g]$ trivialising $\mathcal{C}$ and a coordinate ρ such that $\widetilde{M} \simeq \mathbb{R}^+ \times M \times (-\varepsilon, \varepsilon) \ni (t, x, \rho)$, we have

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2 g_\rho,$$

for a ρ-dependent family of metrics g_ρ with $g_0 = g$ and subject to the condition $\mathrm{Ric}(\widetilde{g}) = 0$.

The Fefferman-Graham ambient metric

Theorem (C. Fefferman & C.R. Graham '85, '07)

- *If* $n := \dim M$ *is odd, then*
  1. *there exists a formal power series solution* g_ρ *to* $\mathrm{Ric}(\widetilde{g}) = 0$,
  2. *this solution is unique up to* $\mathbb{R}^+$ *invariant diffeomorphisms fixing* $\mathcal{C} \subset \widetilde{M}$,
  3. *if* $[g]$ *is analytic, then there exists an ambient metric.*
- *If n is even, then there exists a formal power series solution to* $\mathrm{Ric}(\widetilde{g}) = O(\rho^{\frac{n-2}{2}})$ *which is uniquely determined up to terms of order* $\frac{n}{2}$ *in* ρ. *Furthermore, there exists a conformally invariant tensor* $\mathcal{O} \in \Gamma(\odot^2 TM)$, *the* obstruction tensor, *such that* $\mathcal{O} \equiv 0 \iff \mathrm{Ric}(\widetilde{g}) = 0$ *to infinite order.*

$\mathcal{O}$ is trace and divergence free, conf. invariant of weight $(2-n)$, given by

$$\mathcal{O} = \Delta_g^{n/2-2}\left(\Delta_g \mathrm{P} - \nabla^2 \mathrm{tr}(P)\right) + \text{lower order terms},$$

where $\mathrm{P} = \frac{1}{n-2}\left(Ric - \frac{scal}{2(n-1)}g\right)$ is the Schouten tensor and Δ_g is the Laplacian of $g \in [g]$.

First terms for g_ρ

Expanding g_ρ as a power series we get

$$\widetilde{g} \quad = \quad 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2\left(g + \sum_{k=1}^{\infty}\rho^k\mu_k\right).$$

with

$$\begin{aligned} (\mu_1)_{ab} &= 2\mathrm{P}_{ab} \\ (n-4)(\mu_2)_{ab} &= -B_{ab} + (n-4)\mathrm{P}_a{}^c\mathrm{P}_{bc} \\ 3(n-4)(n-6)(\mu_3)_{ab} &= \Delta_g B_{ab} + \dots \end{aligned}$$

where $B_{ab} = \nabla_c C_{ab}{}^c - \mathrm{P}_{cd}W^c{}_{ab}{}^d$ is the Bach tensor, W_{abcd} is the Weyl tensor, $C_{abc} := \nabla_c\mathrm{P}_{ab} - \nabla_b\mathrm{P}_{ac}$ is the Cotton tensor.

Ambient metrics for Einstein classes

$[g]$ contains a Einstein metric $\Longrightarrow \widetilde{g}$ admits a parallel vector field:
If $g_\Lambda \in [g]$ is Einstein with $\mathrm{P} = \Lambda g$. Then

$$\begin{aligned} \widetilde{g} &= 2\mathrm{d}(\rho t)\mathrm{d}t + \left(t^2 + 2t^2\rho\Lambda + t^2\rho^2\Lambda^2\right) g_\Lambda \\ &\overset{-u=t\rho}{=} -2\mathrm{d}u\mathrm{d}t + \left(t^2 - 2\Lambda u t + \Lambda^2 u^2\right) g_\Lambda \end{aligned}$$

- $\Lambda \neq 0$: $\widetilde{g} \overset{\substack{r=t-\Lambda u \\ s=t+\Lambda u}}{=} -\frac{1}{2\Lambda}\mathrm{d}s^2 + \underbrace{\frac{1}{2\Lambda}\mathrm{d}r^2 + r^2 g_\Lambda}_{cone\ metric}$
- $\Lambda = 0$: $\widetilde{g} = -\mathrm{d}u\mathrm{d}t + t^2 g_\Lambda.$

Further examples of explicit ambient metrics with truncated ambient metric:

- Products of Einstein metrics with related Einstein constants [R. Gover & F. Leitner '06]

pp-waves

An n-dimensional Lorentzian (pseudo-Riemannian) manifold (M, g) is a *pp-wave* (*"plane fronted with parallel rays"*) $\iff$

- $\exists$ parallel null vector field K,
- $R(U, V) : K^\perp \to \mathbb{R}K$ for all $U, V \in TM$

$\iff$ (M, g) has abelian holonomy contained in $\mathbb{R}^{n-2} \subset \mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$\iff$ (M, g) admits coordinates $(x^1, \dots, x^{n-2}, u, r)$ such that

$$g = \sum_{i=1}^{n-2} (\mathrm{d}x^i)^2 + 2\mathrm{d}u \left(\mathrm{d}r + h\mathrm{d}u\right), \quad \text{with } h = h(x^1, \dots, x^{n-2}, u)$$

All scalar invariants vanish, in particular $scal = 0$ and

$$P = \tfrac{1}{n-2}\mathrm{Ric} = -\tfrac{1}{n-2}\Delta h\, \mathrm{d}u^2 \quad \text{and} \quad B = -\tfrac{1}{n-2}\Delta^2 h\, \mathrm{d}u^2$$

where $\Delta h = \sum_{i=1}^{n-2} \partial_i^2 h$.
Note: Einstein $\iff$ $Ric = 0$ $\iff$ $\Delta h = 0$.

The ambient metric of a pp-wave

Let $(M, g = \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2 + 2\mathrm{d}u\mathrm{d}r + 2h\mathrm{d}u^2)$ be a pp-wave.
Ansatz for ambient metric

$$\begin{aligned}\widetilde{g} &= 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(g + H\mathrm{d}u^2) \\ &= 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\Big(2\mathrm{d}u\mathrm{d}r + 2(h+H)\mathrm{d}u^2 + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2\Big),\end{aligned}$$

where $H = H(\rho, x^i, u)$, and $H(\rho, x^i, u)_{|\rho=0} = 0$. Then

$$\mathrm{Ric}(\widetilde{g}) = \Big((2-n)\partial_\rho H + 2\rho\,\partial_\rho^2 H - \Delta H - \Delta h\Big)\mathrm{d}u^2,$$

I.e., $\widetilde{g}$ is ambient metric for $[g]$ if

$$\Big((2-n)\partial_\rho + 2\rho\,\partial_\rho^2 - \Delta\Big)(H) = \Delta h. \tag{1}$$

Solution to (1) is given by

$$H := \sum_{k=1}^{\infty} \frac{\rho^k \Delta^k h}{k!\prod_{i=1}^{k}(2i-n)}$$

Theorem (Nurowski & L. '10)

Let $g = \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2 + 2\mathrm{d}u\,(\mathrm{d}r + h\mathrm{d}u)$ be a pp-wave metric. If n is odd, then the ambient metric for $\widetilde{g}$ for the conformal class $[g]$ is given by

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g + t^2\left(\sum_{k=1}^{\infty}\frac{\rho^k \Delta^k h}{k!\prod_{i=1}^{k}(2i-n)}\right)\mathrm{d}u^2.$$

If n is even, the obstruction tensor is given as $\mathcal{O} = \Delta^{\frac{n}{2}} h\mathrm{d}u^2$. If $\mathcal{O} = 0$, then

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g + t^2\left(\sum_{k=1}^{\frac{n-2}{2}}\frac{\rho^k \Delta^k h}{k!\prod_{i=1}^{k}(2i-n)}\right)\mathrm{d}u^2. \tag{2}$$

is an ambient metric.

Note that even if $\mathcal{O} = \Delta^{\frac{n}{2}} h\,\mathrm{d}u^2 = 0$ the ambient metric is not unique since we can add terms of the form

$$\rho^{\frac{n}{2}} \Delta^{\frac{n-2}{2}} h\,\mathrm{d}u^2$$

in (2) and still get a Ricci flat ambient metric.

Bach flat pp-waves in dimension 4

In dimension $n = 4$ the obstruction to the existence of the ambient metric for a pp-wave

$$g = \mathrm{d}z\mathrm{d}\overline{z} \; + \; 2\mathrm{d}u\mathrm{d}r + 2h\mathrm{d}u^2$$

is given by the Bach tensor $B = -\frac{1}{n-2}\Delta^2 h\mathrm{d}u^2$.

Proposition

A 4-dimensional pp-wave is Bach flat $\Longleftrightarrow$

$$h = z\overline{\alpha} + \overline{z}\alpha + \beta + \overline{\beta} \qquad \text{for } \alpha = \alpha(z,u) \text{ and } \beta = \beta(z,u) \text{ holom. in } z$$

Furthermore, a 4-dimensional Bach flat pp-wave satisfies

$$g \text{ is conformally flat} \Longleftrightarrow g \text{ is conformally Einstein} \Longleftrightarrow \partial_z^2\alpha = 0.$$

$\rightsquigarrow$ Examples for which the ambient metric truncates but which are not conformally Einstein.

Normal conformal tractor bundle

Let $(M,[g])$ be a conformal manifold of signature (p,q) and set $G := \mathrm{SO}^0(p+1,q+1)$, $P = \mathrm{Stab}_G(L)$ and $\mathfrak{g} := \mathfrak{so}(p+1,q+1)$:

- $\omega \in T\mathcal{P} \to \mathfrak{g}$ be the normal conformal Cartan connection on the Cartan bundle $\mathcal{P}$ with structure group P.
- $\overline{\omega} : T\overline{\mathcal{P}} := T(\mathcal{P} \times_P G) \to \mathfrak{g}$ extension to a G-bundle connection, and
- $\overline{\nabla}$ the corresponding covariant derivative on the tractor bundle $\mathbb{T} = \overline{\mathcal{P}} \times_G \mathbb{R}^{p+1,q+1}$.

Every $g \in [g]$ splits $\mathbb{T} = \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$ with

$$\overline{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{\nabla}_X \begin{pmatrix} \tau \\ Y \\ \sigma \end{pmatrix} = \begin{pmatrix} \mathrm{d}\tau(X) - \mathrm{P}(X,Y) \\ \nabla_X Y + \tau X + \sigma(X \lrcorner \mathrm{P})^\sharp \\ \mathrm{d}\sigma(X) - g(X,Y) \end{pmatrix}.$$

Relation to the ambient metric

Computing the LC connection $\widetilde{\nabla}$ of $\widetilde{g}$ along $M \subset \widetilde{M}$ yields that the vector bundle isomoprhism

$$\Phi : T\widetilde{M}|_M \ni \sigma\tfrac{\partial}{\partial\rho} + \tau\tfrac{\partial}{\partial t} + Y \quad \mapsto \quad \begin{pmatrix} \tau \\ Y \\ \sigma \end{pmatrix} \in \mathbb{T}$$

is affine for $\widetilde{\nabla}$ and $\overline{\nabla}$, i.e. $\Phi(\widetilde{\nabla}_X U) = \overline{\nabla}_X(\Phi(U))$. In particular, $\mathrm{Hol}_x(\mathbb{T}, \overline{\nabla}) = \mathrm{Hol}_x(T\widetilde{M}|_M, \widetilde{\nabla})$ and hence

$$\mathrm{Hol}_x(\mathbb{T}, \overline{\nabla}) \subset \mathrm{Hol}_x(T\widetilde{M}, \widetilde{\nabla}). \tag{3}$$

- Extension of parallel tractors to $\widetilde{M}$ yields $=$ in (3) at least when n is odd: Graham & Willse [last week].

Parallel objects for the tractor connection

Let $(M, [g])$ be a conformal manifold.
"[g] is conformal ..." $:=$
"on an open dense subset of M there is a metric g in $[g]$ such that g is ..."

1. $\overline{\nabla}$ admits a parallel line bundle $\Longleftrightarrow$ $[g]$ is conformal Einstein.
2. $\overline{\nabla}$ admits a parallel plane bundle of totally null planes $\Longleftrightarrow$ $[g]$ is conformal *aligned pure radiation*. g is aligned pure radiation metric if
   - $\exists$ parallel null line bundle $\mathcal{K}$,
   - $\mathcal{K}^\perp \lrcorner Ric = 0$.
3. $[g]$ is a conformal pp-wave $\Longleftrightarrow$ $\overline{\nabla}$ admits a parallel totally null plane subbundle $\mathcal{H}$ such that $\overline{R}$ satisfies

$$\overline{R}_{abCD} X^C Y^D = 0,$$

   for all X^C and Y^D orthogonal to $\mathcal{H}$. [2,3: L '06, Nurowski & L '11].

Replace tractor connection $\overline{\nabla}$ by ambient connection $\widetilde{\nabla}$ we get $\Rightarrow$ in the above statements.

Nurowski's conformal structures

For a function F on $\mathbb{R}^5 = \{(x, y, p, q, z)\}$ with $F_{qq} \neq 0$ consider the $(2, 3, 5)$-distribution $\mathcal{D}_F$ spanned by

$$\partial_q \ \text{ and } \ \partial_x + p\partial_y + q\partial_p + F\partial_z.$$

$\mathcal{D}_F$ defines a conformal class $[g_F]$ on $\mathbb{R}^{3,2}$ such that $\mathcal{D}_F^\perp = \mathrm{span}(\mathcal{D}_F, [\mathcal{D}_F, \mathcal{D}_F])$ and whose normal conformal Cartan connection reduces to split G_2.

Nurowski observed: For F of the form

$$F = q^2 + \sum_{i=1}^{6} a_i p^i + bz \tag{4}$$

the ambient metric for $[g_F]$ is of the form

$$\widetilde{g}_F = -2\mathrm{d}t\mathrm{d}u + t^2 g_F - 2ut\mathrm{P} - u^2 B, \tag{5}$$

where P and B are the Schouten and Bach tensors of g_F and $u = -t\rho$.

Properties of $[g_F]$

1. $F = F(q) \implies [g_F]$ contains Ricci-flat metric $\implies$ $\mathrm{Hol}(\widetilde{g}) \subset \mathrm{Stab}_{\mathrm{G}_2}$(null vector).
2. If at least one of a_4, a_5, or a_6 is not zero, then the conformal class $[g_F]$ is not conformal Cotton flat and thus, not conformal Einstein.
3. For $a_4 = a_5 = a_6 = 0$ and $a_3 \neq 0$ there is a unique Cotton flat metric in $[g_F]$, but $[g_F]$ is not conformal Einstein.
4. If at least one of a_3, a_4, a_5, a_6 not equal to zero. Then $[g_F]$ is not conformal aligned pure radiation.

Proposition

For F as in (4) the ambient metric $\widetilde{g}_F$ of $[g_F]$ admits a parallel spinor of non-zero length. Hence, $\mathrm{Hol}(\widetilde{g}_F) \subset \mathrm{G}_2$.

Ambient holonomy G_2

Theorem (Nurowski & L '11)

If $\mathrm{Hol}(\widetilde{g}_F) \neq \mathrm{G}_2$, then $[g_F]$ is conformal Einstein or conformal aligned pure radiation. In particular, if at least one of a_3, a_4, a_5 or a_6 in F is not zero then $\mathrm{Hol}(\widetilde{g}_F) = \mathrm{G}_2$.

Proof: Berger's list in signature $(4,3)$: G_2 and $\mathrm{SO}^0(4,3)$. Have to check that $\mathrm{Hol}(\widetilde{g}_F)$ has no invariant subspace. Let $V \subset \mathbb{R}^{4,3}$ be invariant.

- V non-degenerate. De Rham $\Rightarrow \widetilde{g}_F$ is locally a product with both factors Ricci-flat. One factor has dim $\leq 3 \Rightarrow$ flat $\Rightarrow$ conformal Ricci flat
- $V \cap V =: W \neq 0$ totally null.
  - $\dim(W) = 1 \Rightarrow$ conformal Ricci flat.
  - $\dim(W) = 2 \Rightarrow$ conformal aligned pure radiation.
  - $\dim(W) = 3$ maximal null space $\Rightarrow \exists$ invariant null line of spinors. Then

    Non-null spinor ψ + null line of spinors $\mathbb{R}\varphi \Rightarrow$ null line of vectors $\mathbb{R}V_{\psi,\varphi}$

    via $\widetilde{g}_F(V_{\psi,\varphi}, X) = \langle X \cdot \psi, \varphi \rangle$. Hence, $[g_F]$ is conformal Ricci flat. □