Global aspects of Lorentzian manifolds with special holonomy

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[Joint work with D. Schliebner, arXiv:1306.0120, and with H. Baum & K. Lärz, arXiv:1204.5657, all Humboldt University Berlin]

Holonomy groups in a nutshell

- Let $(\mathcal{M}, \mathbf{g})$ be a semi-Riemannian manifold $\rightsquigarrow$ Parallel transport

$$\mathcal{P}_\gamma \quad : \quad T_{\gamma(0)}\mathcal{M} \ni X_0 \quad \stackrel{\sim}{\longmapsto} \quad X(1) \in T_{\gamma(1)}M$$

where $X(t)$ is the solution to the ODE $\nabla_{\dot{\gamma}(t)}X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (connected) holonomy group

$$\mathrm{Hol}_p^0(\mathcal{M}, \mathbf{g}) := \left\{\mathcal{P}_\gamma \,\middle|\, \gamma(0) = \gamma(1) = p, \gamma \sim \{p\}\right\} \subset \mathrm{O}(T_p\mathcal{M}, \mathbf{g}_p) \simeq \mathrm{O}(r, s)$$

- For $p, q \in M$: $\mathrm{Hol}_p(\mathcal{M}, \mathbf{g}) \sim \mathrm{Hol}_q(\mathcal{M}, \mathbf{g})$ conjugated in $\mathrm{O}(r, s)$.
- $\mathrm{Hol}_p^0(\mathcal{M}, \mathbf{g}) \subset \mathrm{Hol}_p(\mathcal{M}, \mathbf{g})$ normal and the fundamental group

$$\Pi_1(\mathcal{M}, p) \ni [\gamma] \quad \stackrel{surjects}{\twoheadrightarrow} \quad \left[P_\gamma\right] \in \mathrm{Hol}_p(\mathcal{M}, \mathbf{g})/\mathrm{Hol}_p^0(\mathcal{M}, \mathbf{g})$$

- Ambrose-Singer: The Lie algebra $\mathfrak{hol}_p(\mathcal{M}, \mathbf{g})$ is spanned by

$$P_\gamma^{-1} \circ R_{\gamma(1)}(X, Y) \circ P_\gamma \in \mathfrak{so}(T_p\mathcal{M}, \mathbf{g}_p),$$

where $\gamma(0) = p$, $R_{\gamma(1)}$ the curvature at $\gamma(1)$, $X, Y \in T_{\gamma(1)}M$.

Special Lorentzian holonomy

A Lorentzian manifold $(\mathcal{M}^{n+2}, \mathbf{g})$ has special holonomy if

1. $\mathrm{Hol}^0 \neq \mathrm{SO}^0(1, n)$ and
2. $T_p\mathcal{M}$ has no non-degenerate Hol^0-invariant subspaces.

This is in accordance with the terminology for Riemannian manifolds (indecomposable = irreducible, Berger's list),

- De Rham/ Wu decomposition
  $\Longrightarrow (\mathcal{M}, \mathbf{g})$ is not a product, not even locally.

but with a fundamental difference:

- $\mathrm{Hol}^0 \subset \mathrm{SO}^0(1, n+1)$ irreducible $\Longrightarrow \mathrm{Hol}^0 = \mathrm{SO}^0(1, n+1)$
  [Berger, DiScala/Olmos]
  I.e., special holonomy $\Longrightarrow \mathrm{Hol}^0$-invariant null line $\mathbb{L} \subset T_p\mathcal{M}$.
- $\mathrm{Nor}_{\mathrm{O}(1,n+1)}(\mathrm{Hol}^0) \subset \mathrm{Stab}_{\mathrm{O}(1,n+1)}(\mathbb{L}) \simeq (\mathbb{R}^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n \Longrightarrow$
  $\mathbb{L}$ is Hol-invariant.

Geometrically: $\mathcal{M}$ admits a parallel null line bundle, i.e., fibres are invariant under parallel transport.

Classification [Bérard-Bergery & Ikemakhen '93, TL '03]

If $(\mathcal{M}^{n+2}, \mathbf{g})$ is Lorentzian with special holonomy, then $\mathrm{Hol}^0(\mathcal{M}, \mathbf{g}) \simeq$

1. $G \ltimes \mathbb{R}^n$ or $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$, where G is a Riemannian holonomy group,
2. $(A \times G^s) \ltimes \mathbb{R}^{n-k}$, where G^s is the semisimple part of a Riemannian holonomy group G and $A \subset \mathbb{R}^+ \times Z(G)$ if $k = 0$, or $A \subset Z(G) \times \mathbb{R}^k$.
   In fact, $A = \mathrm{graph}(\Psi)$ for $\Psi \in \mathrm{Hom}(Z(G), \mathbb{R}^+$ or $\mathbb{R}^k)$

- For all possible groups there exist metrics [... Galaev '06].
- E.g.: $(\mathcal{N}^n, \mathbf{h})$ Riemannian, $H \in C^\infty(\mathbb{R}^2 \times \mathcal{N})$, $\exists p$: $\det(\nabla^{\mathbf{h}} dH)_p \neq 0 \Rightarrow$

$$\left(\mathcal{M} = \mathbb{R}^2 \times \mathcal{N}\ ,\ \mathbf{g} = \mathbf{g}^{\mathbf{h},H} := 2du(dv + Hdu) + \mathbf{h}\right)$$

  has holonomy $(\mathbb{R}^+ \times \mathrm{Hol}(\mathcal{N}, \mathbf{h})) \ltimes \mathbb{R}^{n-2}$ or $\mathrm{Hol}(\mathcal{N}, \mathbf{h}) \ltimes \mathbb{R}^{n-2}$, if $\frac{\partial H}{\partial v} = 0$.

Are there compact or geodesically complete examples for all groups?

Screen bundle and screen distribution

Let $(\mathcal{M},\mathbf{g})$ be Lorentzian of dim. $n+2$, Hol^0 indecomp. & not irred.,

- $\exists$ bundle of parallel null lines $\mathbb{L} \subset \mathbb{L}^\perp \subset T\mathcal{M}$
- Screen bundle $\Sigma := \mathbb{L}^\perp/\mathbb{L} \longrightarrow \mathcal{M}$ with pos. def. metric induced by $\mathbf{g}$ and connection ∇^Σ: $\nabla^\Sigma_X[Y] := [\nabla_X Y]$. Then

$$G = \mathrm{pr}_{\mathrm{SO}(n)}\mathrm{Hol}_p(\mathcal{M},\mathbf{g}) = \mathrm{Hol}_p(\Sigma,\nabla^\Sigma)$$

- Screen distribution $\mathbb{S} \subset \mathbb{L}^\perp$, rank n and $\mathbf{g}|_{\mathbb{S}\times\mathbb{S}}$ positive definite, i.e., a splitting of $0 \to \mathbb{L} \to \mathbb{L}^\perp \to \Sigma \to 0$.

Assume $\mathbb{L}$ is spanned by a *parallel* null vector field $V \in \Gamma(\mathbb{L})$

- screen vector field Z: null, $\mathbf{g}(Z,V) = 1 \overset{1-1}{\longleftrightarrow} \mathbb{S} := V^\perp \cap Z^\perp$.
- As $(\mathcal{M},\mathbf{g})$ is time-oriented, there exists a screen distribution/vf

Involutive and horizontal screen distributions

$$\left.\begin{array}{lcl} \mathbb{S} \text{ horizontal} & \Longleftrightarrow & [V,S] \in \Gamma(\mathbb{S})\ \forall S \in \Gamma(\mathbb{S}) \\ \mathbb{S} \text{ involutive} & \Longleftrightarrow & [S_1,S_2] \in \Gamma(\mathbb{S})\ \forall S_1,S_2 \in \Gamma(\mathbb{S}) \end{array}\right\} \Longleftrightarrow V^\flat \wedge dZ^\flat = 0$$

In the above example: $\mathbb{S} = \mathrm{span}(\partial_1,\ldots,\partial_n)$, $Z = \partial_u - H\partial_v$, $Z^\flat = dv + Hdu$.

pp-waves ("plane fronted with parallel rays" [Ehlers-Kundt])

Definition: A Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ is a

- *pp-wave* if it admits a parallel null vf V and $\mathrm{R}(U, W) = 0\ \forall U, W \in V^\perp$.
- *standard pp-wave* if $\mathcal{M} = \mathbb{R}^{n+2} \ni (u, v, x^1, \ldots, x^n)$ and

$$\mathbf{g} = \mathbf{g}^H := 2du(dv + Hdu) + \delta_{ij}dx^i dx^j \qquad (1)$$

for a smooth function H with $\partial_v H = 0$.

Equivalences: $(\mathcal{M}, \mathbf{g})$ is a pp-wave

$\Leftrightarrow$ it is locally of the form (1),

$\Leftrightarrow$ $\nabla V = 0$ & $\mathrm{R}(X, Y) : V^\perp \to \mathbb{R}V$, $\forall\ X, Y \in T\mathcal{M}$,

$\Leftrightarrow$ $\nabla V = 0$ & the screen bundle is flat,

$\Leftrightarrow$ $\mathrm{Hol}^0(\mathcal{M}, \mathbf{g}) \subset \mathbb{R}^n$,

$\Leftrightarrow$ $\mathrm{Hol}(\mathcal{M}, \mathbf{g}) \subset \Gamma \ltimes \mathbb{R}^n$ for $\Gamma \subset \mathrm{O}(n)$ discrete,

$\Leftrightarrow$ $\nabla V = 0$ & locally, $\exists\ S_1, \ldots S_n \in \Gamma(V^\perp)$ with $g(S_i, S_j) = \delta_{ij}$ and $\nabla S_i = \alpha^i \otimes V$, where α^i local one-forms with $d\alpha^i|_{V^\perp \wedge V^\perp} = 0$.

Geodesic completeness for compact pp-waves

- Compact Lorentzian manifolds are not always geodesically complete.
- They are if: homogeneous (Marsden '72), of constant curvature (Carriére '89, Klingler '96), or have a time-like conformal vf (Romero/Sánchez '95) Are compact pp-waves complete?
- Ehlers-Kundt '62: *"Prove that complete, Ricci-flat pp-waves are plane waves, no matter which topology one chooses!"* *(EK)*
- *plane wave* = *pp-wave with* $\nabla_X \mathrm{R} = 0 \ \forall \ X \in V^{\perp}$.

Theorem (Schliebner/TL '13)

Let $(\mathcal{M}, \mathbf{g})$ *be a compact pp-wave. Then:*

1. *Its universal cover is globally isometric to a standard pp-wave.*
2. $(\mathcal{M}, \mathbf{g})$ *is geodescically complete.*

Corollary

Every compact Ricci-flat pp-wave is a plane wave.

Thm and Corollary give a proof of (EK) in the compact case (and any dim).

Examples

1. η flat metric on the torus $\mathbb{T}^n$, $H \in C^\infty(\mathbb{T}^n)$ smooth. $\mathcal{M} := \mathbb{T}^2 \times \mathbb{T}^n$ with

$$\mathbf{g}^H = 2d\theta d\varphi + 2Hd\theta^2 + \eta,$$

$\Rightarrow$ complete pp-wave metric on the torus $\mathbb{T}^{n+2}$, in gen. no plane wave.

2.
   - Torus $\mathbb{T}^{n+1}$ with canonical 1-forms $\xi^0, \ldots, \xi^n$. Set

$$\omega = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \xi^i \wedge \xi^j \in \Omega^2(\mathbb{T}^{n+1})$$

   with const's $a_{ij} = -a_{ji}$ such that $0 \neq c := [\omega] \in H^2(\mathbb{T}^{n+1}, \mathbb{Z})$.
   - Let $\pi : \mathcal{M} \to \mathbb{T}^{n+1}$ be the S^1-bundle with 1st Chern class $= c$, $A \in T^*\mathcal{M} \otimes \mathrm{i}\mathbb{R}$ the S^1-connection with curvature $F := dA = -2\pi \mathrm{i} \pi^* \omega$.
   - Pull backs $\eta := \pi^*\xi^0$, $\sigma^i := \pi^*\xi^i$ $i = 1, \ldots n$, function $H \in C^\infty(\mathbb{T}^{n+1})$,

$$\mathbf{g} = 2\left(H\eta - \mathrm{i}A\right) \cdot \eta + \sum_{i=1}^{n} (\sigma^i)^2 \quad = \quad \text{pp-wave metric on } \mathcal{M}.$$

   - Note: $(\mathcal{M}, \mathbf{g})$ does *not* admit an involute screen distribution

Step 1: A compact pp-wave is univ. covered by $\mathbb{R}^{n+2}$.

1. Screen distribution $\mathbb{S} := V^\perp \cap Z^\perp \rightsquigarrow$ complete Riemannian metric:

$$\mathbf{h}(V,.) := \mathbf{g}(Z,.),\ \mathbf{h}(Z,.) := \mathbf{g}(V,.),\ \mathbf{h}|_{\mathbb{S}\times\mathbb{S}} := \mathbf{g}|_{\mathbb{S}\times\mathbb{S}},$$

   with complete lift $\widetilde{\mathbf{h}}$ to the universal cover $\widetilde{\mathcal{M}}$.

2. Z complete $\Rightarrow \widetilde{\mathcal{M}} \overset{diff.}{\simeq} \mathbb{R}\times\widetilde{\mathcal{N}}$, with $\widetilde{\mathcal{N}}$ is a leaf of $\widetilde{V}^\perp$, and $\widetilde{\mathbf{h}}|_{\widetilde{\mathcal{N}}}$ complete.

3. Σ flat $\Rightarrow \exists\ S_1,\dots S_n \in \Gamma(\widetilde{V}^\perp \to \widetilde{\mathcal{M}})$ with
   - $g(S_i,S_j) = \delta_{ij}$, $\widetilde{\nabla}S_i = \alpha^i \otimes \widetilde{V}$, with $d\alpha^i|_{\widetilde{V}^\perp\wedge\widetilde{V}^\perp} = 0$
   - geodetic for $\widetilde{\mathbf{h}}|_{\widetilde{\mathcal{N}}}$ and hence complete.

4. $\exists\ b_i \in C^\infty(\widetilde{\mathcal{M}})$: $db_i|_{\widetilde{V}^\perp} = \alpha^i|_{\widetilde{V}^\perp}$. $\hat{S}_i := S_i - b_i V$ spans an involutive and horizontal screen and satisfies: $\widetilde{\nabla}_U\hat{S}_i = 0, \quad$ whenever $U \in \widetilde{V}^\perp$.

5. S_i complete $\Rightarrow \hat{S}_i$ complete.
   $\widetilde{V}$ and $\hat{S}_i$ complete & parallel on $(\widetilde{\mathcal{N}},\widetilde{\nabla})$, i.e., $\widetilde{\mathcal{N}} \simeq \mathbb{R}^{n+1}$.

Step 2: The universal cover is a standard pp-wave

Let Z be a screen vf and γ be the integral curve of $\widetilde{Z} \in \Gamma(T\widetilde{\mathcal{M}})$ through o, S_i be a global frame for the screen $\widetilde{\mathbb{S}}$.

$$\Phi : \mathbb{R}^{n+2} \ni (u, v, x^1, \dots, x^n) \mapsto \exp^{\widetilde{\mathbf{g}}}_{\gamma(u)} \Big(v\,\widetilde{V}(\gamma(u)) + x^k S_k(\gamma(u)) \Big) \in \widetilde{\mathcal{M}}.$$

- $(\widetilde{\mathcal{N}}, \widetilde{\nabla}|_{\widetilde{\mathcal{N}}})$ is complete, flat and simply connected, for each leaf $\widetilde{\mathcal{N}}$
  $\Rightarrow \exp^{\widetilde{\mathbf{g}}}_p|_{\widetilde{V}^\perp} : \widetilde{V}^\perp_p \longrightarrow \widetilde{\mathcal{N}}$ is a diffeomorphism for each $p \in \widetilde{\mathcal{M}}$.
  $\Rightarrow \Phi$ is a diffeomorphism.
- $\Phi^* \widetilde{\mathbf{g}}$ is a standard pp-wave metric on $\mathbb{R}^{n+2}$ with $2H := (\Phi^* \widetilde{g})(\partial_u, \partial_u)$.

Step 3: Completeness

Lemma (C.f. results by Candela et al)

A standard pp-wave metric $\mathbf{g}^H$ *is complete if all* $\left|\frac{\partial^2 H}{\partial x^i \partial x^j}\right|$ *are bounded.*

Let $(\mathcal{M}, \mathbf{g})$ be a compact pp-wave with screen vf Z and let $\Phi^* \mathbf{g} = g^H$.

- Define a bilinear form $Q := \mathrm{R}(., Z, Z, .)$ on $\mathcal{M}$.
- With $\mathcal{M}$ compact, $\mathbf{g}(Q, Q) = \sum_{i,j=1}^{n} \mathrm{R}(S_i, Z, Z, S_j)^2$ is bounded.
- We have $\Phi^* Q(\partial_i, \partial_j) = -\partial_i \partial_j (H)$, and thus

$$C^2 > \mathbf{g}(Q, Q) = \mathbf{g}^H(\Phi^* Q, \Phi^* Q)^2 = \sum_{i,j=1}^{n} \Phi^* Q(\partial_i, \partial_i) = \sum_{i,j=1}^{n} (\partial_i \partial_j H)^2 \geq 0,$$

  i.e., all $\partial_i \partial_j H$ bounded.

By the Lemma, a compact pp-wave is complete. □

Proof of corollary: $\mathrm{Ric} = 0 \Rightarrow H$ and thus $\partial_i \partial_j H$ harmonic for $\Delta^0 = \sum_{i=1}^{n} \partial_i^2$. $\partial_i \partial_j H$ bounded $\Rightarrow$ independent of x^i.

$(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ semi-Riemannian, $\Gamma \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ properly discontinuous on $\widetilde{\mathcal{M}}$
$\Longrightarrow$ covering $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}}) \xrightarrow{\pi} (\mathcal{M} := \widetilde{\mathcal{M}}/\Gamma, \mathbf{g})$.
For $p \in M$ and $\widetilde{p} \in \pi^{-1}(p)$:

1. injective group homomorphism

$$\iota : \mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}}) \hookrightarrow \mathrm{Hol}_p(M, \mathbf{g}), \quad \widetilde{P}_{\widetilde{\gamma}} \longmapsto P_{\pi\circ\widetilde{\gamma}},$$

   for $\widetilde{\gamma}$ a loop at $\widetilde{p}$, and the image is a normal subgroup.

2. surjective group homomorphism

$$\Phi : \Gamma \twoheadrightarrow \mathrm{Hol}_p(M)/\mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}), \quad \sigma \longmapsto \left[P_\gamma\right],$$

   γ loop at p that, when lifted to a curve $\widetilde{\gamma}$ starting at $\widetilde{p}$, ends at $\sigma^{-1}(\widetilde{p})$.

For a loop γ at $p \in M$, we have:

$$P_\gamma = d\sigma_{\sigma^{-1}(\widetilde{p})} \circ \tilde{P}_{\tilde{\gamma}} \quad (\text{ using } T_{\widetilde{p}}\widetilde{\mathcal{M}} \overset{d\pi_{\widetilde{p}}}{\simeq} T_pM),$$

$\widetilde{\gamma}$ is the lift of γ starting at $\widetilde{p}$ and ending at $\sigma^{-1}(\widetilde{p})$ with $\sigma \in \Gamma$. I.e.,

$$\left[d\sigma_{\sigma^{-1}(\widetilde{p})} \circ \tilde{P}_{\tilde{\gamma}} = (d\sigma^{-1}|_{\widetilde{p}})^{-1} \circ \tilde{P}_{\tilde{\gamma}}\right] \in \Phi(\sigma) \in \mathrm{Hol}_p(M)/\mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}).$$

Isometries of special Lorentzian manifolds

Let $(\mathcal{N}^n, \mathbf{h})$ be Riemannian, $H \in C^\infty(\mathbb{R}^2 \times \mathcal{N})$, $\exists p$: $\det(\nabla^{\mathbf{h}} dH)_p \neq 0$.

$$\left(\widetilde{\mathcal{M}} = \Omega \times \mathcal{N}\ ,\ \widetilde{\mathbf{g}} = \mathbf{g}^{\mathbf{h},H} := 2du(dv + Hdu) + \mathbf{h}\right)$$

$\Omega \subset \mathbb{R}^2$ open domain. Isometries of $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ are of the form:

$$\sigma \begin{pmatrix} u \\ v \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \frac{1}{a}u + b \\ v + \tau(u, \mathbf{x}) \\ \rho(u, \mathbf{x}) \end{pmatrix}, \text{ with } \rho(u, .) \in \mathrm{Iso}(\mathcal{N}, \mathbf{h})\ \forall u.$$

Theorem (Baum, Lärz,TL '12)

Let $\pi : (\widetilde{\mathcal{M}}, \mathbf{g}^{H,\mathbf{h}}) \to (\mathcal{M}, \mathbf{g}) := \widetilde{M}/\Gamma$ *be a covering. Then, for* $\sigma \in \Gamma$ *a representative of* $\Phi(\sigma) \in \mathrm{Hol}_p(M)/\mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}})$ *is given by*

$$\hat{\phi}(\sigma) = \begin{pmatrix} a & 0 & 0 \\ 0 & (\mathrm{d}\rho^{-1}(u, v, .)|_{\mathbf{x}})^{-1} \circ \mathrm{P}^{\mathbf{h}}_{\sigma} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in \Phi(\sigma),$$

In particular, $\mathrm{Hol}_{\pi(\widetilde{q})}(M) = \left\{ \hat{\phi}(\sigma) \middle| \sigma \in \Gamma \right\} \cdot \mathrm{Hol}_p(N, h) \ltimes \mathbb{R}^n.$

Examples with disconnected holonomy groups [BLL '12]

Using certain $\Gamma \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ we obtain examples with $\mathrm{Hol} =$

$\mathbb{Z}^p \ltimes \mathbb{R}^n$, $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \ltimes \mathrm{R}^n$, $(\mathbb{Z} \oplus \mathbb{Z}) \ltimes \mathrm{R}^n$, $(\mathbb{Z} \ltimes \mathrm{SU}(n)) \ltimes \mathbb{R}^{2n}$, $(\mathbb{Z}_2 \ltimes \mathrm{SU}(n)) \ltimes \mathbb{R}^{2n}$

Example with infinitely generated holonomy group

- $\mathcal{N} := \mathbb{R}^2 \setminus \mathbb{Z}^2$ with flat metric $\mathbf{h} = dx^2 + dy^2$, $\Gamma := \Pi_1(\mathcal{N}) = \mathbb{Z} * \mathbb{Z} * \dots$ infinitely generated free group, $\mathrm{Hol}(\mathcal{N}, \mathbf{h})$ trivial, $H \in C^\infty(\mathcal{N})$.
- $\pi : \mathbb{R}^2 \to \mathcal{N} = \mathbb{R}^2/\Gamma$ univ. cover, $\widetilde{\mathbf{h}} = \pi^*\mathbf{h}$, $\widetilde{H} := H \circ \pi$ are Γ-invariant.
- $\Omega := \left\{(v, u) \in \mathbb{R}^2 \mid u > 0\right\}$, $\widetilde{\mathcal{M}} := \Omega \times \mathbb{R}^2$, $\widetilde{\mathbf{g}} = 2du(dv + \frac{\widetilde{H}}{u^2}du) + \mathbf{h}$.
- Fix generators $(\gamma_1, \gamma_2, \dots)$ of Γ, $\underline{\lambda} := (\lambda_1, \lambda_2, \dots)$ lin. indep. over $\mathbb{Q}$, $\sigma_i(v, u, x) := (\mathrm{e}^{\lambda_i} v, \mathrm{e}^{-\lambda_i} u, \gamma_i(x))$, $\Gamma_{\underline{\lambda}} := \left\langle \sigma_i \mid i = 1, 2, \dots \right\rangle \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$.
- $\Gamma_{\underline{\lambda}}$ acts properly discontinuous on $\widetilde{\mathcal{M}}$ and $\mathcal{M} = \widetilde{\mathcal{M}}/\Gamma_{\underline{\lambda}}$ is LMf with metric $\mathbf{g}$, $\mathrm{Hol}(\mathcal{M}, \mathbf{g})$ is infinitely generated by

$$\begin{pmatrix} \mathrm{e}^{\lambda_i} & w & * \\ 0 & 1_2 & * \\ 0 & 0 & \mathrm{e}^{-\lambda_i} \end{pmatrix} \in \mathrm{O}(1,3), \quad w \in \mathbb{R}^2$$