Ambient metrics for n-dimensional pp-waves

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Conformal structures

- Let $(M^n, [g])$ be a smooth manifold equipped with a conformal structure of semi-Riemannian metrics, $[g] := \left\{ e^{2\varphi} g \mid \varphi \in C^\infty(M) \right\}$.
- Problem: Invariant description of conformal structures (e.g. Weyl tensor$= 0 \iff$ conformally flat).
- One solution: Fefferman-Graham ambient metric: Describe the conformal geometry of $(M^n, [g])$ in terms of the metric geometry of an *ambient space* $(\widetilde{M}^{n+2}, \widetilde{g})$ as in the following

Example

Let S^n be the n sphere. Consider the $n+2$ dimensional Minkowski space $\mathbb{M}^{n+2}$ with flat metric $\widetilde{g}_0$, and $\mathcal{C}^+$ be the light cone. Then $S^n \simeq \mathcal{C}^+/\mathbb{R}^+$ and every section $\sigma \in \Gamma(\mathcal{C}^+ \to S^n)$ defines a metric $g := \sigma^* \widetilde{g}_0$ on S^n. Two metrics obtained in this way are conformally equivalent.

The ambient metric construction generalises this picture to arbitrary conformal classes.

Construction of the ambient space

- A conformal class $[g]$ on M corresponds to an $\mathbb{R}^+$-principle fibre bundle, the cone

$$\mathcal{C} := \left\{(g,x) \in \odot^2 TM \mid (g,x) = g_x \text{ for a } g \in [g]\right\} \quad \xrightarrow{\pi} \quad M$$

$$\mathbb{R}^+\text{-action } \delta_t(g,x) := (t^2 g, x) \text{ on the fibres}$$

- Tautological tensor on $\mathcal{C}$: $\quad \mathbf{g}(U,V)|_{(g,x)} := g_x(\mathrm{d}\pi(U), \mathrm{d}\pi(V))$,
  - $\mathbf{g}(T,T) = 0$ for the fundamental vector field T of δ.
  - of degree 2 w.r.t. the $\mathbb{R}^+$-action, i.e. $\delta_t^* \mathbf{g} = t^2 \mathbf{g}$.
- Every $g \in [g]$ defines a trivialisation of $\mathcal{C}$,

$$\begin{array}{ccc} \mathbb{R}^+ \times M & \simeq & \mathcal{C} \\ (t,x) & \mapsto & (t^2 g_x, x) \end{array}$$

- Now, *thicken* the cone to obtain the ambient space

$$\widetilde{M} := \mathcal{C} \times (-\varepsilon, \varepsilon)$$

The $\mathbb{R}^+$ action on $\mathcal{C}$ extends trivially to $\widetilde{M}$: $\delta_t(g_x, \rho) := (\delta_t g_x, \rho)$.

Definition

Let $(M, [g])$ be smooth manifold with conformal class of signature (t, s). An *ambient metric* for $(M, [g])$ is a smooth metric $\widetilde{g}$ on $\widetilde{M}$ such that

1. $\widetilde{g}$ is homogeneous of degree 2 w.r.t. the $\mathbb{R}^+$-action δ.
2. $\iota^*\widetilde{g} = \mathbf{g}$, for the inclusion $\iota : \mathcal{C} = \mathcal{C} \times \{0\} \subset \widetilde{M}$.
3. $\mathrm{Ric}(\widetilde{g}) = 0$.

(1), (2) and $\mathrm{Ric}(\widetilde{g}) = 0$ imply that by fixing $g \in [g]$ that trivialises $\mathcal{C}$ and a coordinate ρ such that $\widetilde{M} \simeq \mathbb{R}^+ \times M \times (-\varepsilon, \varepsilon) \ni (t, x, \rho)$, we have

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2 g_\rho,$$

for a ρ-dependent family of metrics g_ρ with $g_0 = g$ and subject to the condition $\mathrm{Ric}(\widetilde{g}) = 0$.

Theorem (C. Fefferman & C.R. Graham '85,'07)

- *If* $n := \dim M$ *is odd, then*
  1. *there exists a formal power series solution* g_ρ *to* $\mathrm{Ric}(\widetilde{g}) = 0$,
  2. *this solution is unique up to* $\mathbb{R}^+$ *invariant diffeomorphisms fixing* $\mathcal{C} \subset \widetilde{M}$,
  3. *if* $[g]$ *is analytic, then there exists an ambient metric.*
- *If n is even, then there exists a conformally invariant tensor* $\mathcal{O} \in \Gamma(\odot^2 TM)$, *the* obstruction tensor, *such that* $\mathcal{O} \equiv 0 \iff$ *(1),(2),(3) as in the odd case.*
  Furthermore, there exists a formal power series solution to the problem $\mathrm{Ric}(\widetilde{g}) = \mathcal{O}(\frac{n-2}{2})$ *in* ρ *which is uniquely determined up to terms of order* $\frac{n}{2}$.

$\mathcal{O}$ is trace and divergence free, conf. invariant of weight $(2-n)$, given by

$$\mathcal{O} = \Delta_g^{n/2-2}\left(\Delta_g P - \nabla^2 \mathrm{tr}(P)\right) + \text{lower order terms},$$

where $P = \frac{1}{n-2}\left(Ric - \frac{scal}{2(n-1)}g\right)$ is the Schouten tensor and Δ_g is the Laplacian of $g \in [g]$.

Expanding g_ρ as a power series we get

$$\widetilde{g} \;=\; 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2\left(g + \sum_{k=1}^{\infty} \rho^k \mu_k\right).$$

with

$$\begin{aligned} (\mu_1)_{ab} &= 2P_{ab} \\ (n-4)(\mu_2)_{ab} &= -B_{ab} + (n-4)P_a{}^c P_{bc} \\ 3(n-4)(n-6)(\mu_3)_{ab} &= \Delta_g B_{ab} + \ldots \end{aligned}$$

where $B_{ab} = \nabla_c C_{ab}{}^c - P_{cd} W^c{}_{ab}{}^d$ is the Bach tensor, W_{abcd} is the Weyl tensor, $C_{abc} := \nabla_c P_{ab} - \nabla_b P_{ac}$ is the Cotton tensor.

Examples of ambient metrics

$[g]$ contains a (local) Einstein metric $\iff$ $\widetilde{g}$ admits a parallel vector field:
If $g_\Lambda \in [g]$ is Einstein with $P = \Lambda g$. Then

$$\begin{aligned}\widetilde{g} &= 2t\mathrm{d}(\rho t)\mathrm{d}t + \left(t^2 + 2t^2\rho\Lambda + t^2\rho^2\lambda^2\right) g_\Lambda \\ &\overset{-u=t\rho}{=} -2\mathrm{d}u\mathrm{d}t + \left(t^2 - 2\Lambda ut + \Lambda^2 u^2\right) g_\Lambda\end{aligned}$$

- $\Lambda \neq 0$: $\widetilde{g} \overset{\substack{r=t-\Lambda u \\ s=t+\Lambda u}}{=} -\frac{1}{2\Lambda}\mathrm{d}s^2 + \underbrace{\frac{1}{2\Lambda}\mathrm{d}r^2 + r^2 g_\Lambda}_{cone\ metric}$
- $\Lambda = 0$: $\widetilde{g} = -\mathrm{d}u\mathrm{d}t + t^2 g_\Lambda$.

Examples of explicit ambient metrics with truncated ambient metric:

- Products of Einstein metrics with related Einstein constants [R. Gover & F. Leitner '06]
- G_2-conformal structures [P. Nurowski '07, Nurowski & L. 09]

An n-dimensional Lorentzian manifold (M, g) is a *pp-wave* $\iff$

- $\exists$ parallel null vector field K,
- $R(U, V)$ maps $K^\perp$ to $\mathbb{R} \cdot K$ for all $U, V \in TM$ ($\iff$ $R_{abij}R^{ij}{}_{cd} = 0$).

$\iff$ (M, g) has abelian holonomy $\mathbb{R}^{n-2}$
$\iff$ (M, g) admits coordinates $(x^1, \ldots, x^{n-2}, u, r)$ such that

$$g = \sum_{i=1}^{n-2} (\mathrm{d}x^i)^2 \; + \; 2\mathrm{d}u \left(\mathrm{d}r + h\mathrm{d}u\right), \quad \text{with } h = h(x^1, \ldots, x^{n-2}, u)$$

pp-waves satisfy $scal = 0$ and

$$P = \tfrac{1}{n-2}\mathrm{Ric} = -\tfrac{1}{n-2}\Delta h \mathrm{d}u^2 \quad \text{and} \quad B = -\tfrac{1}{n-2}\Delta^2 h \mathrm{d}u^2$$

where $\Delta h = \sum_{i=1}^{n-2} \partial_i^2 h$.

Let $(M, g = \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2 \ + \ 2\mathrm{d}u\,(\mathrm{d}r + h\mathrm{d}u))$ be a pp-wave. *Ansatz* for ambient metric

$$\widetilde{g} \;\; = \;\; 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(g + H\mathrm{d}u^2)$$

where $H = H(\rho, x^i, u)$, and $H(\rho, x^i, u)_{|\rho=0} = 0$. Then we have

$$\mathrm{Ric}(\widetilde{g}) = \Big((2-n)H_\rho + 2\rho H_{\rho\rho} - \Delta H - \Delta h\Big)\mathrm{d}u^2,$$

with $H_\rho := \frac{\partial H}{\partial \rho}$, etc. I.e., $\widetilde{g}$ is ambient metric for $[g]$ if

$$(2-n)H_\rho + 2\rho H_{\rho\rho} - \Delta H = \Delta h. \qquad (*)$$

Solution to $(*)$ is given by

$$H := \sum_{k=1}^{\infty} \frac{\rho^k \Delta^k h}{k! \prod_{i=1}^{k}(2i-n)}$$

Theorem (Nurowski & L. '10)

Let $g = \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2 \; + \; 2\mathrm{d}u\,(\mathrm{d}r + h\mathrm{d}u)$ be a pp-wave metric. If n is odd, then the ambient metric for $\widetilde{g}$ for the conformal class $[g]$ is given by

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g + t^2\left(\sum_{k=1}^{\infty}\frac{\rho^k\Delta^k h}{k!\prod_{i=1}^{k}(2i-n)}\right)\mathrm{d}u^2.$$

If $n = 2s$ is even, the obstruction tensor is given as

$$\mathcal{O} = \Delta^s h\mathrm{d}u^2.$$

If it vanishes, the ambient metric truncates at order $(s-1)$ in ρ.

Bach flat pp-waves in dimension 4

In dimension $n = 4$ the obstruction to the existence of the ambient metric for a pp-wave

$$g = \mathrm{d}z\mathrm{d}\overline{z} \; + \; 2\mathrm{d}u\mathrm{d}r + 2h\mathrm{d}u^2$$

is given by the Bach tensor $B = -\frac{1}{n-2}\Delta^2 h\mathrm{d}u^2$.

Proposition (Nurowski & L. '10)

A 4-dimensional pp-wave is Bach flat $\Longleftrightarrow$

$$h = z\overline{\alpha} + \overline{z}\alpha + \beta + \overline{\beta} \qquad \text{for } \alpha = \alpha(z,u) \text{ and } \beta = \beta(z,u) \text{ holom. in } z$$

Furthermore, a 4-dimensional Bach flat pp-wave satisfies

$$g \text{ is conformally flat} \Longleftrightarrow g \text{ is conformally Einstein} \Longleftrightarrow \partial_z^2\alpha = 0.$$

This gives an abundance of examples for which the ambient metric truncates but which are not conformally Einstein.