Half flat structures and special holonomy

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Outline

1. Linear algebra of stable forms
2. Half flat structures and their evolution under the Hitchin flow
3. Examples

Joint work with V. Cortés, F. Schulte-Hengesbach (Hamburg), and L. Schäfer (Hannover) [Proc. London Math. Soc., 2010]

Stable forms

Definition

Let V be a real vector space of dimension n. A k-form $\omega \in \Lambda^k V^* =: \Lambda^k$, with $1 < k < n-1$, is called *stable* if and ω has an open orbit under the action of $\mathrm{GL}_n\mathbb{R}$.

- Stable forms exists if $k = 2, n-2$ or $k = 3, n-3$ and $n = 6, 7, 8$ [Kimura & Sato '77]
- Stable forms come in pairs: There is a $\mathrm{GL}_n^+\mathbb{R}$-invariant map $\phi : \Lambda^k \to \Lambda^n$ which sends stable forms to volume forms which and defines a *dual stable form* $\hat{\omega}$ by

$$\hat{\omega} \wedge \omega = \tfrac{n}{k}\phi(\omega)$$

  for every stable form ω. The connected components of their stabilisers are the same.
- Example: $k = 2$, $n = 2m$: $\omega \in \Lambda^2$ stable iff it is non-degenerate.

$$\phi(\omega) = \tfrac{1}{m!}\omega^m\,, \quad \hat{\omega} = \tfrac{1}{(m-1)!}\omega^{m-1}\,, \quad \text{and} \quad \mathrm{Stab}_{\mathrm{GL}_n}(\omega) = \mathrm{Sp}_m\mathbb{R}.$$

Examples of stable forms

- $k = 3$, $n = 6$: A stable form $\rho \in \Lambda^3 V^*$ on an oriented V defines a linear map

$$V \ni v \;\mapsto\; v \lrcorner \rho \wedge \rho \in \Lambda^5 V^* \simeq V$$

which can be rescaled to a (para-)complex structure $J_\rho : V \to V$ with $J_\rho^2 = \epsilon \, \mathbb{I}$ with $\epsilon = \pm 1$. We have

$$\phi(\rho) = J_\rho^* \rho \wedge \rho, \;\; \hat{\rho} = J_\rho^* \rho, \;\; \mathrm{Stab}_{\mathrm{GL}_6^+}(\rho) = \begin{cases} \mathrm{SL}_3\mathbb{C}\,, & \epsilon = -1 \\ \mathrm{SL}_3\mathbb{R} \times \mathrm{SL}_3\mathbb{R}\,, & \epsilon = 1 \end{cases}$$

- $k = 3$, $n = 7$: $\varphi \in \Lambda^3 V^*$ is stable if the bilinear form (values in $\Lambda^7 V^*$)

$$b_\varphi : (v, w) \;\mapsto\; \tfrac{1}{6} v \lrcorner \varphi \wedge w \lrcorner \varphi \wedge \varphi \;\in \Lambda^7 V^*$$

is non-degenerate. Then $\phi(\varphi) = \det(b_\varphi)^{1/9}$ defines a volume form and a scalar product $g_\varphi := \frac{1}{\phi(\varphi)} b_\varphi$. We have

$$3\hat{\varphi} = *\varphi\,, \;\; 7\phi(\varphi) = \varphi \wedge *\varphi\,, \;\; \mathrm{Stab}_{\mathrm{GL}_7}(\varphi) = \begin{cases} \mathrm{G}_2 \subset \mathrm{SO}(7)\,, & g_\varphi > 0 \\ \mathrm{G}_2^* \subset \mathrm{SO}(3,4)\,, & \text{else.} \end{cases}$$

Compatible stable forms

- $n = 6$: A pair of stable forms $\omega \in \Lambda^2 V^*$ and $\rho \in \Lambda^3 V^*$ is *compatible* if

$$\omega \wedge \rho = 0 \text{ and } \phi(\rho) = 2\phi(\omega)$$

In this case $h := \epsilon\omega(\cdot, J_\rho \cdot)$ defines a scalar product and

$$\mathrm{Stab}_{\mathrm{GL}_6}(\rho, \omega) = \begin{cases} \mathrm{SU}(3) \text{ or } SU(1,2)\,, & \epsilon = -1 \\ \mathrm{SL}_3\mathbb{R} \subset \mathrm{SO}(3,3)\,, & \epsilon = 1 \end{cases}$$

- Let $\rho \in \Lambda^3 V^*$ and $\omega \in \Lambda^2 V^*$ be a pair of compatible stable forms defining the scalar product h. Then, on $W := \mathbb{R} \cdot e^0 \oplus V$ the form

$$\varphi := \omega \wedge e^0 + \rho \;\in\; \Lambda^3 W^*$$

is stable, defines scalar product $g_\varphi = h - \epsilon(e^0)^2$, and

$$\mathrm{Stab}_{\mathrm{GL}_7}(\varphi) = \begin{cases} \mathrm{G}_2\,, & \epsilon = -1 \text{ and } h > 0 \\ \mathrm{G}_2^*\,, & \text{else} \end{cases}$$

- Conversely, fix $v \in V^7$ with $g_\varphi(v, v) = -\epsilon$, then $\omega := (v \lrcorner \varphi)|_{v^\perp}$ and $\rho := \varphi|_{v^\perp}$ are compatible stable forms on $v^\perp$.

Half flat structures

- Let $G \subset GL_n\mathbb{R}$. A G-structure on a smooth manifold M is a reduction of the frame bundle of M to G.
  For $G \subset \mathrm{O}(s,t)$, a G-structure is parallel if the bundle of G-frames is invariant under parallel transport ($\iff$ Holonomy of $\nabla^{LC} \subset G$.)
- For a real form H of $\mathrm{SL}_3\mathbb{C}$ a H-structure is equivalent to the existence of $\rho \in \Omega^3 M$ and $\omega \in \Omega^2 M$ which define a pair of compatible stable forms at each point in M.
- A H-structure (ρ, ω) on a 6-manifold is half flat if

  $$\mathrm{d}\rho = 0 \quad \text{and} \quad \mathrm{d}\hat{\omega} = 0.$$

  This generalises CY 3-manifolds for which we have $\mathrm{d}\rho = \mathrm{d}\hat{\rho} = 0$ and $\mathrm{d}\omega = 0$.
- A $G_2^{(*)}$-structure on a 7-manifold is given by $\varphi \in \Omega^3 M$ that defines a stable form at each point of M.
  It is parallel if $\nabla^{LC}\varphi = 0$, or equivalently, if $\mathrm{d}\varphi = 0$ and $\mathrm{d}{*}\varphi = 0$.

Let H be a real form H of $\mathrm{SL}_3\mathbb{C}$ and (ρ_t, ω_t) be a one-parameter family of H structures on a 6-manifold M^6 with $t \in I$. Then

$$\varphi = \omega \wedge \mathrm{d}t + \rho$$

defines a parallel $G_2^{(*)}$-structure on $I \times M^6 \iff (\rho_t, \omega_t)$ is half flat $\forall t$ and

$$\partial_t \rho = \mathrm{d}\omega \quad \text{and} \quad \partial_t \hat{\omega} = \mathrm{d}\hat{\rho} \qquad \text{Hitchin flow eq's} \tag{1}$$

Theorem (Hitchin '01 for M compact & $H = \mathrm{SU}(3)$, CLSS in gen.)

Let (ρ, ω) be a 1-parameter family of stable forms on M^6 satisfying the Hitchin flow eq's. If $(\rho_{t_0}, \omega_{t_0})$ is half flat for a $t_0 \in I$, then (ρ, ω) is a family of half flat H-structures.
In particular, the three-form $\varphi = \omega \wedge dt + \rho$ defines a parallel $G_2^{(*)}$-structure on $M \times I$ with induced metric $g_\varphi = g_t - \epsilon dt^2$.

Using Cauchy-Kovalevskaya Theorem we can now construct $G_2^{(*)}$ structures from real analytic half flat structures.

Corollary

Let M be a real analytic 6*-mf with real analytic half flat structure* (ω_0, ρ_0)*.*

- $\exists!$ *maximal solution* (ω, ρ) *of* (1) *with initial value* (ω_0, ρ_0)*, which is defined on an open neighbourhood* $\Omega \subset \mathbb{R} \times M$ *of* $\{0\} \times M$*. In particular, there is a parallel* $G_2^{(*)}$*-structure on* Ω*.*
- *The evolution is natural, i.e. automorphisms of the initial structures extend to automorphisms of the evolved structures.*
- *Furthermore, if M is compact or a homogeneous space* $M = G/K$ *such that the* (ω_0, ρ_0) *is G-invariant, then there is unique maximal open interval I and a unique solution* (ω, ρ) *of* (1) *with initial value* (ω_0, ρ_0) *on* $I \times M$*. In particular, there is a parallel* $G_2^{(*)}$*-structure on* $I \times M$*.*

In general, the $G_2^{(*)}$-metrics obtained in this way will only be geodesically complete if $I = \mathbb{R}$. But they can be conformally changed to a complete metric.

Evolution of nearly-Kähler 6-manifolds

- CY 3-manifolds are half flat, $\mathrm{d}\omega = 0$ and $\mathrm{d}\rho = \mathrm{d}\hat{\rho} = 0$. The resulting $\mathrm{G}_2^{(*)}$-metric is a direct product: $g_\varphi = -\epsilon \mathrm{d}t^2 + g_0$.
- Let (M, g, J) be an *almost ϵ-Hermitian manifold*, i.e. $J^2 = \epsilon \mathbb{I}$ and $J^* g = -\epsilon g$. If ∇J is skew, (M^{2m}, g, J) is called nearly-ε-Kähler. On a 6-manifold M, a nearly ε-Kähler structure with $|\nabla J|^2 \equiv 4$ is equivalent to a half flat structure (ω, ρ) with $\rho := \nabla \omega$ which satisfies

$$\mathrm{d}\omega = 3\rho \quad \text{and} \quad \mathrm{d}\hat{\rho} = 4\hat{\omega} \tag{2}$$

The solutions to the Hitchin flow are given as

$$\omega_t = t^2 \omega_0 \ , \quad \rho_t = t^3 \rho_0 \ \text{ defining the metric } g_t = t^2 g_0$$

for an initial half flat structure (ω_0, ρ_0). Indeed, because of (2): $\partial_t \rho_t = 3t^2 \rho_0 = t^2 d\omega_0 = d\omega_t$ and $\partial_t \hat{\omega}_t = 4t^3 \hat{\omega}_0 = t^3 d\hat{\rho}_0 = d\hat{\rho}_t$. The resulting $\mathrm{G}_2^{(*)}$ metric is a cone metric $g_\varphi = -\epsilon \mathrm{d}t^2 + t^2 g_0$ on $\mathbb{R}^+ \times M^6$.
- Conversely, $G_2^{(*)}$-cone-metrics define nearly-ϵ-Kähler metrics on the base.

Let G be a 6-dimensional Lie group. Then:

$$\{\text{left-inv half flat structures on } G\} \leftrightarrow \left\{ \begin{array}{l} \text{compatible forms } (\omega, \rho) \text{ on } \mathfrak{g}^* \\ \text{with } d\omega^2 = d\rho = 0 \end{array} \right\}$$

$\rightsquigarrow$ algebraic problem as $d\alpha(X, Y) = \alpha([X, Y])$ for $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$.

$\rightsquigarrow$ Classification of left inv half flat structures on the product of two 3-dim Lie groups [Schulte-Hengesbach, J. Geom. Phys. '10]

Let H_3 be the 3-dim Heisenberg group and $G = H_3 \times H_3$.

- Every stable $\omega \in \Lambda^2\mathfrak{g}^*$ with $d\omega^2 = 0$ has one of the normal forms:

$$\begin{aligned} \omega_1 &= e^1f^1 + e^2f^2 + e^3f^3, & \omega_4 &= e^1f^3 + e^2f^2 + e^3f^1 + e^{13} + \beta f^{13} \\ \omega_2 &= e^2f^2 + e^{13} + f^{13}, & \omega_5 &= e^1f^3 + e^2f^2 + e^{13} + f^{13} \\ \omega_3 &= e^1f^3 + e^2f^2 + e^3f^1, \end{aligned}$$

  for a basis $(e^1, e^2, e^3, f^1, f^2, f^3)$ be a basis of $H_3 \times H_3$ with commutator $de^3 = e^{12}$ and $df^3 = f^{12}$ and $\beta \neq -1$ a parameter.
- Compatible stable closed 3-forms ρ are given by a linear 8-parameter family $\rho_i = \rho_i(a^1, \ldots, a^8)$ subject to a quartic non-degeneracy condition.

Half flat structures on $H_3 \times H_3$ with $\omega = \omega_1$

- $\omega = \omega_1$ is a necessary condition for the existence of a half flat $\mathrm{SU}(3)$ structure on $G := H_3 \times H_3$
- There are half flat $\mathrm{SU}(1,2)$ and $\mathrm{SL}_3\mathbb{R}$ structures on G with $\omega \neq \omega_1$.
- Examples of half flat structures on G with $\omega = \omega_1 = e^1 f^1 + e^2 f^2 + e^3 f^3$:
  - $\rho = \frac{1}{\sqrt{2}}(e^{123} - f^{123} - e^1 f^{23} + e^{23} f^1 - e^2 f^{31} + e^{31} f^2 - e^3 f^{12} + e^{12} f^3)$ $\rightsquigarrow$ half flat $\mathrm{SU}(3)$-structure.
  - $\rho = \frac{1}{\sqrt{2}}(e^{123} - f^{123} - e^1 f^{23} + e^{23} f^1 + e^2 f^{31} - e^{31} f^2 + e^3 f^{12} - e^{12} f^3)$ $\rightsquigarrow$ half flat $\mathrm{SU}(1,2)$-structure, e_1 and e_4 being spacelike.
  - $\rho = \sqrt{2}\,(e^{123} + f^{123}) \rightsquigarrow$ half flat $\mathrm{SL}_3\mathbb{R}$-structure such that the $\mathfrak{h}_3$'s are the J_ρ-eigenspaces, i.e. the metric is $g = 2(e^1 \cdot f^1 + e^2 \cdot f^2 + e^3 \cdot f^3)$.
- In order to evolve these structures we define $\kappa : I \to \mathbb{R}$:

$$
\begin{array}{rcll}
\mathrm{SU}(3) & : & \kappa(x) = (x - \sqrt{2})^3 (x + \sqrt{2}) & , \quad I = (-\sqrt{2}, \sqrt{2}) \\
\mathrm{SU}(1,2) & : & \kappa(x) = (x - \sqrt{2})(x + \sqrt{2})^3 & , \quad I = (-\sqrt{2}, \sqrt{2}) \\
\mathrm{SL}_3\mathbb{R} & : & \kappa(x) = (2 + x)^2 & , \quad I = \mathbb{R}
\end{array}
$$

Evolving half flat structures on $H_3 \times H_3$ with $\omega = \omega_1$

Theorem

Let ρ_0 be one of the stable forms compatible to $\omega = \omega_1$ defining a half flat structure on $H_3 \times H_3$ and $\kappa : I \to \mathbb{R}$ as on the previous slide. Let x be a solution to the ODE $\dot{x} = \frac{2}{\sqrt{\varepsilon\kappa(x(t))}}$. Then

$$\begin{aligned} \rho_x &= \rho_0 + x(e^{12}f^3 - e^3f^{12}), \\ \omega_x &= \tfrac{1}{2}(\varepsilon\kappa(x))^{-\frac{1}{2}} \left(\varepsilon\kappa(x)\, e^1 f^1 + \varepsilon\kappa(x)\, e^2 f^2 + 4e^3 f^3 \right), \end{aligned}$$

give a solution to the Hitchin flow on the intervall I. The parallel stable three-form and the metric on $M \times I$ are

$$\varphi = \tfrac{1}{2}\sqrt{\varepsilon\kappa(x)}\,\omega_x \wedge dx + \rho_x, \qquad g_\varphi = g_x - \tfrac{1}{4}\kappa(x)dx^2,$$

and $(M \times I, g_\varphi)$ has holonomy equal *to $G_2^{(*)}$.*
Moreover, by varying ρ we obtain an 8*-parameter family of metrics with holonomy* equal *to $G_2^{(*)}$.*

Half flat structures on $H_3 \times H_3$ with $\omega \neq \omega_1$

What happens in the case when ω is not of the form ω_1?

Let (ω, ρ) be a left-invariant half flat structure with $\omega \neq \omega_1$ on $G = H_3 \times H_3$ and let g be the pseudo-Riemannian metric induced by (ω, ρ). Then

- The pseudo-Riemannian manifold $(H_3 \times H_3, g)$ is either flat or isometric to the product of
  - a two-dimensional flat factor and
  - the unique 4-dimensional simply connected para-hyper Kähler symmetric space (N, g_N) with 1-dimensional holonomy group [Alekseevsky et al '05]

  In particular, the metric g is Ricci-flat.
- The Hitchin flow is defined for all times and defines a G_2^*-metric on $G \times \mathbb{R}$.
- However, this metric is either flat or isometric to a product of (N, g_N) and a 3-dim flat factor, so its holonomy is at most one-dimensional.