Irreducible subgroups of $\mathrm{SO}(2, n)$

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A naive question

For a given n, what are all possible connected subgroups of $\mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$?

- No general answer because of Weyl's trick.
- All the more remarkable is

Classification of holonomy groups of Riemannian mf's [Berger '55]

The connected component of an irreducible holonomy group of a Riemannian manifold of dimension n is conjugated to

- $\mathrm{SO}(n)$, for arbitrary n,
- $\mathrm{U}(n/2)$ or $\mathrm{SU}(n/2)$, for n even,
- $\mathrm{Sp}(n/4)$ or $\mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$, for n divisible by 4,
- G_2, for $n = 7$, $\mathrm{Spin}(7)$, for $n = 8$, or

the isotropy group of an irreducible Riemannian symmetric space.

A better question

For given n and $p + q = n$, $0 < p < n$, what are possible connected subgroups of $\mathrm{SO}(p, q)$ that act irreducibly on $\mathbb{R}^{p,q}$?

Theorem (Berger '55, Di Scala/Olmos '00, Benoist/de la Harpe '04)

The only connected subgroup of the Lorentz group $\mathrm{SO}(1, n-1)$ that acts irreducible on the n-dimensional Minkowski space is its connected component $\mathrm{SO}^0(1, n-1)$.

- A Lorentzian manifolds admits no parallel tensors/spinors unless it is a product or admits a parallel null line bundle.
- A Riemannian conformal manifold has generic conformal holonomy unless it is locally conformally equivalent to a product of Einstein metrics or locally conformally Einstein.

Irreducible subgroups of $\mathrm{SO}(2,n)$

Theorem (Di Scala/L '11)

Every connected Lie group that acts irreducibly on $\mathbb{R}^{2,n}$ is conjugated to one of the following:

1. *for arbitrary $n \geq 1$: $\mathrm{SO}^0(2,n)$,*
2. *for $n = 2p$ even: $\mathrm{U}(1,p)$, $\mathrm{SU}(1,p)$, or $\mathrm{U}(1)\cdot\mathrm{SO}^0(1,p)$ if $p > 1$,*
3. *for $n = 3$: $\mathrm{SO}^0(1,2) \overset{\text{irr.}}{\subset} \mathrm{SO}(2,3)$.*

- The last group in (2) uses the inclusion $\mathrm{SO}(1,p) \subset \mathrm{SU}(1,p)$, the $\mathrm{U}(1)$ factor makes it irreducible (no Berger algebra).
- The group in (3) corresponds to the symmetric space $M^5 := \mathrm{SL}_3\mathbb{R}/\mathrm{SO}^0(1,2)$ which is of signature $(2,3)$ w.r.t. the Killing form of $\mathrm{SL}_3\mathbb{R}$. Hence, $\mathrm{SO}^0(1,2) = \mathrm{Hol}(M^5) \overset{\text{irr.}}{\subset} \mathrm{SO}(2,3)$.

Symmetric spaces (of non-compact type)

(M, g) Riemannian symmetric space (simply conected)

- $\Leftrightarrow$ $\forall\ p \in M\ \exists \phi \in G := \mathrm{Iso}(M) :\ d\phi_p = -Id$
- $\Leftrightarrow$ $K \subset G$ closed: $\exists$ involution σ: $Fix^0(\sigma) \subset K \subset Fix(\sigma)$.
- $\Leftrightarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, $ad(\mathfrak{k})|_{\mathfrak{m}} \subset \mathfrak{gl}(\mathfrak{m})$ comp.
- $\Leftrightarrow$ Lie triple system $T = (\mathfrak{m}, R, \langle .,. \rangle)$, R curvature, $R(x, y) \in \mathfrak{aut}(T)$

of non-compact type $\Leftrightarrow$ not flat and $sec \leq 0$

- $\Leftrightarrow$ $G := \mathrm{Iso}(M)$ is non compact and semisimple, K max. compact
- $\Leftrightarrow$ no flat and no compact factor in De Rham decomposition
- $\Leftrightarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is Cartan decomposition.

N totally geodesic submanifold in $M = G/K$

- $\Leftrightarrow$ geodesics starting tangent to N remain in N
- $\Leftrightarrow$ $\forall p \in N :\ \phi(N) \subset N$.
- $\Leftrightarrow$ sub-Lie triple system $\mathfrak{n} \subset \mathfrak{m}$: $R|_{\mathfrak{n}\times\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$

The Karpelevich - Mostow - Theorem

Geometric version [Karpelevich '53]

Let M be a Riemannian symmetric space of noncompact type and $G \subset \mathrm{Iso}(M)$ connected and semisimple. Then G has a totally geodesic orbit in M.

Algebraic version [Mostow '55]

Let $\hat{\mathfrak{g}}$ be a real semisimple, non-compact Lie algebra and $\mathfrak{g} \subset \hat{\mathfrak{g}}$ a semisimple subalgebra. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a Cartan decomposition for $\mathfrak{g}$, then there exists a Cartan decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{m}}$ for $\hat{\mathfrak{g}}$ such that $\mathfrak{k} \subset \hat{\mathfrak{k}}$ and $\mathfrak{m} \subset \hat{\mathfrak{m}}$.

- Not true for compact symmetric spaces (e.g., take S^n and $G \subset \mathrm{SO}(n+1)$ irreducible).
- Implies uniqueness of symmetric pairs: If $M = G/K$ is non-compact type with $G \subset \mathrm{Iso}(M)$, then G and K are unique.

Idea for proving the theorems geometrically

Consider symmetric spaces associated to

$$\hat{G} = \mathrm{SO}^0(1,n): \quad H^n = \frac{\mathrm{SO}^0(1,n)}{SO(n)}, \qquad \text{hyperbolic space,}$$

$$\hat{G} = \mathrm{SU}(1,n): \quad \mathbb{C}H^n = \frac{\mathrm{SU}(1,n)}{\mathrm{U}(n)}: \qquad \text{complex hyperbolic space}$$

$$\hat{G} = \mathrm{SO}^0(2,n): \quad \mathcal{L}^n := \frac{\mathrm{SO}^0(2,n)}{\mathrm{SO}(2)\cdot SO(n)}, \qquad \text{Lie ball,}$$

and find their totally geodesic submanifolds!

Problems:

1. $G \subset \hat{G}$ is not assumed to be semisimple, only irreducible.
2. $G \subset \hat{G}$ might not act effectively on totally geodesic submanifolds M in H^n or $\mathcal{L}^n$.

Idea: When G is simple, then it must act effectively, as $\{A \in G \mid A|_M = Id_M\}$ is normal.

Proof for $\mathrm{SO}(1,n)$

Lemma

Let $G \subset \mathrm{SO}^0(1,n)$ act irreducibly. Then G is simple unless $n = 1$.

Hence, we can apply Karpelevich-Mostow:

- Totally geodesic submanifold in H^n are given by

$$H^n \cap V, \quad \text{with } V \subset \mathbb{R}^{1,n} \text{ subspace.}$$

- Since G is irreducible, $V = \mathbb{R}^{1,n}$ and $H^n = G/K$.
- Uniqueness of symmetric pairs implies $G = \mathrm{SO}^0(1,n)$. □

Lemma

Let $G \subset \mathrm{SO}^0(2,n)$ act irreducibly on $\mathbb{R}^{2,n}$. Then G is simple unless $n = 2$ or $G \subset \mathrm{U}(1,\frac{n}{2})$.

Case 1: $G \subset \mathrm{U}(1,n)$

Proposition

If $G \subset \mathrm{U}(1,\frac{n}{2})$ act irreducibly on $\mathbb{R}^{2,2n}$, then G is equal to

$$\mathrm{SU}(1,n),\ \mathrm{U}(1,n),\ \text{or } \mathrm{U}(1)\cdot\mathrm{SO}^0(1,n).$$

Proof: Complex hyperbolic space

$$\mathbb{C}H^n = \{z \in \mathbb{C}^n \mid ||z||^2 < 1\} = \frac{\mathrm{SU}(1,n)}{\mathrm{U}(n)} = \frac{\mathrm{U}(1,n)}{\mathrm{U}(1)\cdot\mathrm{U}(n)}.$$

has the following totally geodesic submanifolds M [e.g. Goldman]:

1. M totally real, $M = H^n$ real hyperbolic,
2. M totally complex, $M = \mathbb{C}H^k = \mathbb{C}H^n \cap V^k$, $V^k \subset \mathbb{C}^n$ subspace.

Then split $G = Z \cdot S$ into centre and semisimple part and apply Karpelevich-Mostow to S:

1. $M = H^n \Rightarrow S = \mathrm{SO}^0(1,n)$
2. $\mathbb{C}H^n \cap V^k \Rightarrow S$ irreducible, hence $k = n$. □

Case 2: G simple: Totally geodesic submanifolds of the Lie ball

Set $q(x) = -x_0^2 - x_1^2 + x_2^2 + \ldots + x_{n+1}^2$,

$$\mathcal{L}^n = \frac{\mathrm{SO}^0(2,n)}{\mathrm{SO}(2)\cdot SO(n)} = \{[z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid q^h(z) < 0, q^{\mathbb{C}}(z) = 0\}$$

Based on the classification of tot. geod.submanifolds of the complex quadric $Q^n = \frac{\mathrm{SO}^0(n+2)}{\mathrm{SO}(2)\cdot SO(n)}$ [Chen/Nagano '77, Klein '08] and duality we obtain tot. geod. submf's in $\mathcal{L}^n$ and their isometry groups:

$$\mathcal{L}^m, \quad \mathbb{C}H^k = \frac{\mathrm{SU}(1,k)}{\mathrm{U}(k)}, \quad H^k = \frac{\mathrm{SO}(1,k)}{\mathrm{SO}(k)}, \quad H^p \times H^q, \quad \mathbb{C}H^1 \times H^1,$$

for $m \leq n$, $k \leq 2n$, $p + q \leq n$, and one exceptional $H^2 \subset \mathcal{L}^3$, which corresponds to $\mathrm{SO}^0(1,2) \subset \mathrm{SO}(2,3)$.

Applications to conformal holonomy

Conformal structure $(M,[g]) \rightsquigarrow$ unique *normal conformal Cartan connection* ω with values in $\mathfrak{so}(p+1,q+1)$.

- If the corresponding vector bundle connection admits a parallel line bundle, then $[g]$ contains a local Einstein metric.
- $\mathrm{Hol}(\omega) \subset \mathrm{U}(p,q) \Rightarrow \mathrm{Hol}(\omega) \subset \mathrm{SU}(p,q)$ [Leitner'06, Cap/Gover'06]

Proposition (Alt/Di Scala/L, in progress)

If the holonomy of a 3-dim Lorentzian conformal manifold $(M,[g])$ is contained in $\mathrm{SO}^0(1,2) \overset{irr.}{\subset} \mathrm{SO}(2,3)$, then $(M,[g])$ is conformally flat.

Corollary

If the conformal holonomy of a Lorentzian manifold is irreducible, then it is equal to $\mathrm{SU}(1,p)$ or $\mathrm{SO}^0(2,n)$.

Hol not irreducible $\Rightarrow$ $[g] \ni$ Einstein, product of Einstein, or aligned pure radiation metric [L' 06].