Cauchy problems for Lorentzian manifold with special holonomy

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(With H. Baum & A. Lischewski, Humboldt-Universität Berlin, arXiv:1411.3059)

Constraint and evolution equations: the general idea

$(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ Lorentzian manifold, $\mathcal{M} \subset \overline{\mathcal{M}}$ spacelike hypersurface, of the form

$$\boxed{\overline{\mathrm{g}} = -\lambda^2\, dt^2 + \mathrm{g}_t} \qquad t \in \mathcal{I} \subset \mathbb{R}$$

where g_t = family of Riemannian metrics on $\mathcal{M}$, $\lambda = \lambda(t, x)$ *"lapse function"*.
Assume we have some geometric (PDE) on $\overline{\mathrm{g}}$, e.g., $\overline{\mathrm{Ric}} = 0$.

- Constraint conditions: Conditions (C_t) on the geometry of g_t, i.e., PDEs without ∂_t derivatives,
- Evolution equations: PDEs (E) involving ∂_t-derivatives that preserve the conditions (C_t).
- $(C_t)_{t\in\mathcal{I}}$ & (E) are equivalent to (PDE).

Cauchy problem: Given $(\mathcal{M}, \mathrm{g}_0)$ satisfying (C_0), show that for given initial conditions the system (E) has a (unique) solution.
Then: Obtain a Lorentz metric $\overline{\mathrm{g}}$ satisfying (PDE).

Example: Cauchy problem for $\overline{\mathrm{Ric}} = 0$

Let $\overline{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}_t$ on $\mathcal{I} \times \mathcal{M}$ and $T = \frac{1}{\lambda}\partial_t$ be the unit normal.

- $\mathrm{W} := -\overline{\nabla} T|_{T\mathcal{M}}$ the Weingarten operator of $(\mathcal{M}, \mathrm{g}_0 =: \mathrm{g})$, $\mathrm{II} = \mathrm{g}(\mathrm{W}\cdot, \cdot) = -\frac{1}{2\lambda}\dot{\mathrm{g}}$.
- Fundamental curvature equations:

$$
\begin{aligned}
\overline{\mathrm{R}}|_{T\mathcal{M}} &= \mathrm{R} + \mathrm{II} \wedge \mathrm{II} && \text{Gauß} \\
\overline{\mathrm{R}}(\cdot,\cdot,\cdot,T)|_{T\mathcal{M}} &= d^{\nabla}\mathrm{II} && \text{Codazzi} \\
\overline{\mathrm{R}}(\cdot,T,T,\cdot)|_{T\mathcal{M}} &= \mathrm{g}(\mathrm{W}^2\cdot,\cdot) + \frac{1}{\lambda}\left(\dot{\mathrm{II}} + \mathrm{Hess}(\lambda)\right) && \text{Mainardi}
\end{aligned}
$$

- $\overline{\mathrm{Ric}} = 0 \iff \left\{ \begin{array}{rcl} \mathrm{scal} & = & \mathrm{tr}(\mathrm{II}^2) - \mathrm{tr}(\mathrm{II})^2 \\ d\,\mathrm{tr}(\mathrm{II}) & = & -\mathrm{div}(\mathrm{II}) \end{array} \right\}$ (constraints) and

$$\dot{\mathrm{II}} = \lambda\Big(\mathrm{Ric} + \mathrm{tr}(\mathrm{II})\mathrm{II} - 2\mathrm{g}(\mathrm{W}^2\cdot,\cdot)\Big) - \mathrm{Hess}(\lambda) \quad \text{(evolution)}$$

Form of (E): $\ddot{\mathrm{g}} = F(\mathrm{g}, \dot{\mathrm{g}}, \partial_i \mathrm{g}, \partial_i \dot{\mathrm{g}}, \partial_i\partial_j \mathrm{g})$, with initial data $\mathrm{g}|_{t=0} = \mathrm{g}$, $\dot{\mathrm{g}}|_{t=0} = -2\lambda\mathrm{II}$.

- λ and initial data real analytic: apply Cauchy-Kowalevski to get unique solution.
- Solution in the smooth setting: Choquet-Bruhat.
- Riemannian: solution for the analytic data, but in general no solution for smooth, non-analytic.

Parallel spinors on Lorentzian manifolds

- $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ a Lorentzian spin manifold with spinor bundle $\overline{\mathbb{S}} \to \overline{\mathcal{M}}$.
- $\psi \in \Gamma(\overline{\mathbb{S}})$ a parallel spinor,

$$\nabla\psi := d\psi + \frac{1}{2}\sum_{i,j=1}^{m} \overline{\mathrm{g}}(\nabla s_i, s_j)\, s_i \cdot s_j \cdot \psi = 0.$$

- ψ induces Dirac current V_ψ by

$$\overline{\mathrm{g}}(X, V_\psi) = -\langle X \cdot \psi, \psi\rangle \quad \forall X \in T\overline{\mathcal{M}}$$

  V_ψ is causal ($\overline{\mathrm{g}}(V_\psi, V_\psi) \leq 0$) and parallel ($\nabla V_\psi = 0$) vf V_ψ
  - $\overline{\mathrm{g}}(V_\psi, V_\psi) = -1$:
    $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ locally is a product $(\mathcal{N}, \mathrm{h}) \times (\mathbb{R}, -dt^2)$
    with $(\mathcal{N}, \mathrm{h})$ Riemannian, $\mathrm{Ric} = 0$ & carrying a parallel spinor
    $\rightsquigarrow$ Solve for $\mathrm{Ric} = 0$ and then check that the result has parellel spinor [Ammann, Moroianu & Moroianu, CMP '14]
  - $\overline{\mathrm{g}}(V_\psi, V_\psi) = 0$:
    No induced product structure and *not Ricci-flat* $\rightsquigarrow$ Lorentzian manifold with special holonomy, $\mathfrak{hol}(\overline{\mathcal{M}}, \overline{\mathrm{g}}) \subset \mathfrak{so}(m-2) \ltimes \mathbb{R}^{m-2} = \mathfrak{stab}_{\mathfrak{so}(1,m-1)}(\textit{null vector})$.
- Cauchy problem for Lorentzian manifolds with parallel null vector field.

Constraint equations for a parellel null vector field

Let $(\overline{\mathcal{M}}, \overline{\mathrm{g}}) = (\mathcal{I} \times \mathcal{M}, -\lambda^2 dt^2 + \mathrm{g}_t)$.
A null vector field V induces a space-like vector field

$$U = pr_{T\mathcal{M}}(V) = uT - V$$

depending on t with $u := \mathrm{g}(U, U)$.

Proposition

If $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ admits a parallel null vector field V, then $U = pr_{T\mathcal{M}}(V)$ satisifies

$$\nabla U + u\,\mathrm{W} = 0,$$

with W the Weingarten operator and $u^2 = \mathrm{g}(U, U)$.
Moreover we have

$$\dot{U} := [\partial_t, U] = u\,\mathrm{grad}(\lambda) + \lambda\,\mathrm{W}(U).$$

Towards evolution equations for a parallel null vector field

Lemma

V is parallel for $\overline{\mathrm{g}}$ if and only if for all $X, Y \in T\mathcal{M}$ we have

$$\begin{aligned} \overline{\mathrm{R}}(\partial_t, X, Y, V) \;=\; \overline{\mathrm{R}}(\partial_t, X, V, \partial_t) = 0 & \quad \rightsquigarrow \textit{evolution for } \mathrm{g} \quad (*) \\ \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t} V = 0 & \quad \rightsquigarrow \textit{evolution for for } U = pr_{T\mathcal{M}}(V) \\ \overline{\nabla}_X V|_{\{0\}\times\mathcal{M}} = 0 & \quad \rightsquigarrow \textit{constraint for } U \\ \overline{\nabla}_{\partial_t} V|_{\{0\}\times\mathcal{M}} = 0 & \quad \rightsquigarrow \textit{initial condition for } \dot{U} \end{aligned}$$

This leads to evolution equations in Cauchy-Kowalevski form:

$$\begin{pmatrix} \ddot{\mathrm{g}} \\ \ddot{U} \\ \ddot{u} \end{pmatrix} = \mathcal{F}(\mathrm{g}, \dot{\mathrm{g}}, \partial_i \mathrm{g}, \partial_i \dot{\mathrm{g}}, \partial_i\partial_j \mathrm{g}, U, \dot{U}, \ldots, \partial_i\partial_j u),$$

however the first component of $\mathcal{F}$ is not necessarily symmetric!

Observation:

In the analytic case, $(*)$ can be replaced by $\overline{\mathrm{R}}(X, V, V, Y) = 0$ for all $X, Y \in T\mathcal{M}$.

Evolution equations for a parallel null vector field

Theorem (Baum, Lischewski & L '14)

Let $(\mathcal{M}, \mathrm{g})$ *be a Riemannian mfd,* II *a symmetric bilinear tensor field,* U *a vector field,* u *function on* $\mathcal{M}$*, all real analytic, with constraints*

$$\nabla_i U_j = -u\mathrm{II}_{ij}, \quad \mathrm{g}(U, U) = u^2 > 0.$$

Then for any analytic fct. $\lambda = \lambda(t, x)$ *the Lorentzian metric* $\overline{g} = -\lambda^2 dt^2 + \mathrm{g}_t$ *has parallel null vector field* $V = \frac{u_t}{\lambda}\partial_t - U_t \iff$

$$\begin{aligned}
\ddot{\mathrm{g}}_{ij} &= \tfrac{1}{u} U^k \left(\lambda \nabla_{[k} \dot{\mathrm{g}}_{(j]i)} - \dot{\mathrm{g}}_{(i[j)} \nabla_{k]} \lambda\right) + \frac{1}{2} \dot{\mathrm{g}}_{ik} \dot{\mathrm{g}}^k{}_j + \tfrac{\dot{\lambda}}{\lambda} \dot{\mathrm{g}}_{ij} + 2\lambda \nabla_i \nabla_j \lambda + \tfrac{2\lambda^2}{u^2} U^k U^\ell R_{ik\ell j} \\
\ddot{U}_i &= \tfrac{1}{2u} U^k U^l \left(\dot{\mathrm{g}}_{l[k} \nabla_{i]} \lambda - \lambda \nabla_{[i} \dot{\mathrm{g}}_{k]l}\right) - U^k \left(\dot{\mathrm{g}}_{ki} - \tfrac{\dot{\lambda}}{\lambda 2} \dot{\mathrm{g}}_{ki} - \lambda \nabla_k \nabla_i \lambda - \nabla_k \lambda \nabla_i \lambda\right) \\
&\quad + u \mathrm{g}_{ki} \nabla^k \dot{\lambda} + \tfrac{u}{2} \dot{\mathrm{g}}_{ki} \nabla^k \lambda + 2\dot{u} \nabla_i \lambda \\
\ddot{u} &= U^k \left(g_{kl} \nabla^l \dot{\lambda} + \tfrac{3}{2} \dot{\mathrm{g}}_{kl} \nabla^l \lambda\right) + 2\dot{U}^k \nabla_k \lambda - u \nabla_k \lambda \nabla^k \lambda
\end{aligned}$$

with initial conditions
$$\begin{cases}
\mathrm{g}_{ij}(0) = \mathrm{g}_{ij}, & \dot{\mathrm{g}}_{ij}(0) = -2\lambda \mathrm{II}_{ij}, \\
U(0) = U, & \dot{U}_i(0) = u\, \nabla_i \lambda + \lambda U^k \mathrm{II}_{ki} \\
u(0) = u & \dot{u}(0) = U^k \nabla_k \lambda
\end{cases}.$$

Consequences

Cauchy-Kowalevski $\Longrightarrow$

Corollary

If $(M, \mathrm{g}, \mathrm{II}, U, u)$ are real analytic satisfying the constraint equations, λ real analytic, then (M, g) can be extended to a Lorentzian manifold with parallel null vector field. This extension is unique when specifying the above initial conditions.

Example

$\lambda \equiv 1$, II Codazzi tensor, i.e. $\nabla_{[i}\mathrm{II}_{j]k} = 0 \Longrightarrow$

$$\begin{aligned} \mathrm{g}_{ij}(t) &= \mathrm{g}_{ij} - 2t\mathrm{II}_{ij} + t^2\mathrm{II}_{ik}\mathrm{II}^k{}_j \\ U^i(t) &= \mathrm{A}^i{}_k(t)U^k, \text{ mit } \mathrm{A}^i{}_k \text{ inverse of } (\delta_i{}^j - t\mathrm{II}_i{}^j) \\ u(t) &= u \end{aligned}$$

solves the above system.

Constraints for a parallel spinor

Identify the spinor bundle $\mathbb{S} \to \mathcal{M}$ of $\mathcal{M}$ with $\overline{S}_{|\mathcal{M}}$ if $n := \dim(\mathcal{M})$ even and with $\overline{\mathbb{S}}^{+}_{|\mathcal{M}}$ if n is odd. Clifford multiplication:

$$X \cdot \varphi = i\, T \cdot X \cdot \psi\,|_{\mathcal{M}},$$

Proposition

If $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ admits a parallel null spinor field ψ, then $(\mathcal{M}, \mathrm{g})$ has a spinor field φ with

$$\begin{aligned} \nabla_X \varphi &= \tfrac{i}{2}\,\mathrm{W}(X) \cdot \varphi, \qquad \forall X \in T\mathcal{M}, \\ U_\varphi \cdot \varphi &= i\, u_\varphi\, \varphi, \end{aligned} \tag{1}$$

in which U_φ is defined by $\mathrm{g}(U_\varphi, X) = -i(X \cdot \varphi, \varphi)$, $u_\varphi = \sqrt{\mathrm{g}(U_\varphi, U_\varphi)} = ||\varphi||^2$.

- A spinor with (1) is called *imaginary* W*-Killing spinor*.
- $U_\varphi = pr_{T\mathcal{M}} V_\psi$.
- Observe that (1) implies the constraint for a parallel null vector field

$$\nabla_X U_\varphi + u_\varphi \mathrm{W}(X) = 0$$

Evolution for parallel spinor

Theorem (Baum, Lischewski & L '14)

Let $(\mathcal{M}, \mathrm{g})$ be an analytic Riemannian spin manifold with an analytic g-symmetric endomorphism field W and φ an imaginary W-Killing spinor on $(\mathcal{M}, \mathrm{g})$. Then on the Lorentzian manifold obtained in the first Theorem, and with parallel null vector field V, there exists a parallel null spinor field ϕ with Dirac current V. The parallel spinor ϕ is obtained by parallel transport of φ along the lines $t \mapsto (t, x)$.

Proof: Take the Lorentzian manifold obtained as solution with parallel null vector field. Translate the initial spinor φ parallel along t-lines $\rightsquigarrow$ spinor ψ with $\overline{\nabla}_{\partial_t}\psi = 0$.

- $\mathcal{E} := \left(T^*\mathcal{M} \otimes \overline{\mathbb{S}}\right) \oplus \left(\Lambda^2 T^*\mathcal{M} \otimes \overline{\mathbb{S}}\right) \longrightarrow \overline{\mathcal{M}}$,
- $\begin{pmatrix} A := \overline{\nabla}\psi \\ B := \overline{\mathrm{R}}(\cdot, \cdot)\psi \end{pmatrix} \in \Gamma(\mathcal{E})$
- Check that $\begin{pmatrix} A \\ B \end{pmatrix}$ satisfies a PDE $\overline{\nabla}_{\partial_t}\begin{pmatrix} A \\ B \end{pmatrix} = Q\begin{pmatrix} A \\ B \end{pmatrix}$, with Q linear on $\mathcal{E}$.
- $A = B = 0$ along initial hypersurface and hence for all t.

An example satisfying the constraints

Let $(\mathcal{F}, \mathrm{g}_{\mathcal{F}})$ be a complete Riemannian spin manifold with

- a parallel spinor field and a Codazzi tensor T
- $b \in C^\infty(\mathbb{R}, \mathbb{R})$ a smooth function.
- Let A be the $(1,1)$-tensor field on $\mathcal{M} := \mathbb{R} \times \mathcal{F}$ given by

$$\mathrm{A} = \begin{pmatrix} b(s) & 0 \\ 0 & e^s\Big(\mathrm{T} - \int_0^s b(r)e^{-r}dr \cdot \mathrm{Id}_{\mathrm{T}\mathcal{F}}\Big) \end{pmatrix}.$$

Then

- $\mathrm{g} := \mathrm{A}^*(ds^2 + e^{-2s}\mathrm{g}_{\mathcal{F}})$ is a complete Riemannian metric on $\mathcal{M} := \mathbb{R} \times \mathcal{F}$,
- $\mathrm{W} := \mathrm{A}^{-1}$ is an invertible Codazzi tensor on $(\mathcal{M}, \mathrm{g})$
- $(\mathcal{M}, \mathrm{g})$ admits an imaginary W-Killing spinor.

All complete Riemannian spin manifolds $(\mathcal{M}, \mathrm{g})$ with imaginary W-Killing spinor for an invertible Codazzi tensor W arise in this way [Baum & Müller ’08].

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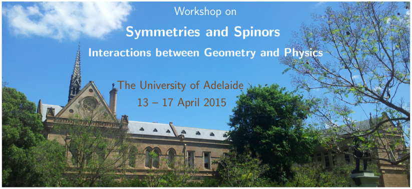

Welcome | Venue | Poster | Program | Registration | Information for participants

The Workshop

The interplay between physics and geometry has lead to stunning advances and enriched the internal structure of each field. This is vividly exemplified in the theory of supergravity, which is a supersymmetric extension of Einstein's relativity theory to the small scales governed by the laws of quantum physics. Sophisticated mathematics is being employed for finding solutions to the generalised Einstein equations and in return, they provide a rich source for new exotic geometries. This workshop brings together world-leading scientists from both, geometry and mathematical physics, as well as young researchers and students, to meet and learn about each others work.

The workshop is funded by The University of Adelaide's Institute for Geometry and its Applications (IGA) and the Australian Mathematical Sciences Institute (AMSI), which also offer support to cover travelling costs of students and early career researchers from AMSI member institutions. Further support is given by the Australian National University's Special Year in Geometry and Physics.

Accompanying the workshop will be a Women in Mathematic meeting on one of the afternoons.

Plenary Speakers

- José Figueroa-O'Farrill (University of Edinburgh)
- Maxim Zabzine (Uppsala University)

Invited Speakers

- David Baraglia (University of Adelaide)
- Evgeny Buchbinder (University of Western Australia)
- Vladimir Chernov (Dartmouth College)
- Michael Eastwood (Australian National University) (to be confirmed)
- Sergei Kuzenko (University of Western Australia)
- Paul Norbury (University of Melbourne)
- David Ridout (Australian National University)
- Dennis The (Australian National University)
- Simon Wood (Australian National University)