Locally homogeneous pp-waves

Thomas Leistner

(joint work with W. Globke)

THE UNIVERSITY OF ADELAIDE AUSTRALIA

AMSI workshop "Differential Geometry, Complex Analysis and Lie Theory"
La Trobe University, Melbourne, December 5 – 7, 2014

- Which implication has the homogeneity for the geometry/curvature of a semi-Riemannian manifold?
  E.g., homogeneous and $\mathrm{Ric} = 0$ implies flat [Alekseevskiĭ & Kimel′fel′d '75]
- Study such question for Lorentzian manifolds with special holonomy:
  - Holonomy group **Hol** := group of parallel transports along loops
  - *special holonomy* $\iff$ $\mathfrak{hol} \subsetneq \mathfrak{so}(p,q)$ but the manifold is *indecomposable*, i.e. does *not* (locally) decompose as a product.
- Riemannian special Holonomy
  $\rightsquigarrow$ Berger's list: $\mathbf{U}(p)$ $\mathbf{SU}(p)$, $\mathbf{Sp}(q)$ $\mathbf{Sp}(q) \cdot \mathbf{Sp}(1)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$.
- Lorentzian special holonomy: no irreducible subalgebras of $\mathfrak{so}(1, n+1)$!
  $\Longrightarrow$

$$\mathfrak{hol} \subset \mathfrak{stab}(\textit{null line}) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \mathfrak{sim}(n).$$

  There is a classification of Lorentzian special holonomy algebras [Berard-Bergery & Ikemakhen '93, L '03, Galaev '05]
- Study the geometry of Lorentzian manifold with special holonomy, e.g. in relation to homogeneity.

pp-waves (plane fronted waves with parallel rays)

Lorentzian metric on $\mathbb{R}^{n+2} \ni (u, v, x^1, \ldots, x^n)$, $H = H(u, x^i) \in C^\infty(\mathbb{R}^{n+1})$,

$$\mathrm{g} = \mathrm{g}^H := 2\, du\, dv + 2H(u, x^i)\, du^2 + \delta_{ij}\, dx^i\, dx^j \tag{1}$$

Definition

A Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ is a pp-wave, if g is locally of the form (1).

Question: When are pp-waves locally homogeneous?

Answer (in the best of all possible worlds): Only when H is a quadratic polynomial in the x^i's with u-dependent coefficients. (Not quite true.)

Definition

A pp-wave is a plane wave, if $\exists$ coord as in (1) such that $H(u, x^i) = A_{ij}(u) x^i x^j$.

- Equivalent definition for pp-waves: $(\mathcal{M}, \mathrm{g})$ Lorentzian manifold with
  - parallel, light-like vector field V
  - $\mathrm{R}(X, Y) : V^\perp \to \mathbb{R} \cdot V, \quad \forall X, Y \in T\mathcal{M}$.
    (equivalent: $\mathrm{R} : \Lambda^2 T\mathcal{M} \to \Lambda^2 T\mathcal{M}$ has kernel $\Lambda^2 V^\perp$, or $\mathrm{R}(X, Y) = 0\ \forall\ X, Y \in V^\perp$)
- Plane waves: $\nabla_X \mathrm{R} = 0\ \forall X \in V^\perp$.
- Cahen-Wallach spaces: $\nabla \mathrm{R} = 0$ (equivalent to A constant).

Some history

- Brinkmann ['25]: conformally equivalent Einstein metrics
- GR: wavelike solutions to the Vacuum Einstein equations

$$\mathrm{Ric}^g = \Delta(H)\, du^2 = 0,$$

Δ flat Laplacian w.r.t x^i-coordinates

  - Einstein '16: linearised Einstein equations
  - Einstein & Rosen '35: gravitational waves
  - Penrose '76: Every spacetime has a plane wave as "Penrose limit".

- Parallel light-like vector field $\frac{\partial}{\partial v}$ = direction of propagation, wave fronts $\{(v, u) = const\}$ with induced *flat* metric.

$\rightsquigarrow$ "pp-wave=plane-fronted wave with parallel rays" [Jordan, Ehlers & Kundt '60].

- pp-waves have no scalar invariants, i.e, all functions made from covariant derivatives and traces of the curvature vanish.
- pp-waves have Abelian holonomy

$$\mathbb{R}^n \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n = \mathfrak{stab}_{\mathfrak{so}(1,n+1)}(\textit{null vector}).$$

Locally homogeneous spaces

$(\mathcal{M}, \mathrm{g})$ is locally homogeneous $\iff \forall\, p, q \in \mathcal{M}$

$$\exists \text{ nbhds } \mathcal{U}_p, \mathcal{U}_q \text{ and } \Phi : (\mathcal{U}_p, \mathrm{g}|_{\mathcal{U}_p}) \stackrel{isometry}{\longrightarrow} (\mathcal{U}_q, \mathrm{g}|_{\mathcal{U}_q}) : \ \phi(p) = q. \qquad (2)$$

Equivalent: $\forall\, p \in \mathcal{M}\ \exists$ local Killing vector fields $K_1, \ldots, K_n$ such that

$$\mathrm{span}(K_1|_p, \ldots, K_m|_p) = T_p\mathcal{M}.$$

Let V be a parallel vector field on $\mathcal{M}$.

- Then the distribution $V^\perp$ is parallel and hence integrable with leafs $\mathcal{N}$ of codim 1.
- $(\mathcal{M}, \mathrm{g})$ is locally $V^\perp$-homogeneous $\iff \forall\, p, q \in \mathcal{N}$ *in the same leaf* of $V^\perp$ we have (2).
- Consequence: $\forall\, p \in \mathcal{M}\ \exists$ local Killing vector fields $K_1, \ldots, K_{m-1}$ such that

$$\mathrm{span}(K_1|_p, \ldots, K_{m-1}|_p) = V^\perp|_p.$$

Killing vector fields (infinitesimal isometries)

$\mathfrak{iso} :=$ Lie algebra of Killing vector fields. $K \in \mathfrak{iso} \iff \phi := \nabla K \in \mathfrak{so}(T\mathcal{M})$, i.e.,

$$\mathfrak{iso} := \left\{ K \in \Gamma(T\mathcal{M}) \mid \boxed{\mathrm{g}(\nabla_X K, Y) + \mathrm{g}(\nabla_Y K, X) = 0} \ \forall\, X, Y \in T\mathcal{M} \right\}$$

Differential consequences: $\boxed{\nabla_X \phi = -\mathrm{R}(K, X), \quad \nabla_K \mathrm{R} = \phi \cdot \mathrm{R}}$

- Killing vf's define parallel sections a vector bundle

$$\mathfrak{iso} \ \simeq \ \left\{ \begin{pmatrix} K \\ \phi \end{pmatrix} \in \Gamma\left(\mathcal{K} := \begin{array}{c} T\mathcal{M} \\ \oplus \\ \mathfrak{so}(T\mathcal{M}) \end{array}\right) \ \middle| \ \nabla^{\mathcal{K}}_X \begin{pmatrix} K \\ \phi \end{pmatrix} := \begin{pmatrix} \nabla_X K - \phi(X) \\ \nabla_X \phi + \mathrm{R}(K, X) \end{pmatrix} = 0 \right\}$$

- $\dim(\mathfrak{iso}) \leq \mathrm{rk}(\mathcal{K}) = \frac{1}{2} m(m+1)$.
- Evaluation map at a fixed point p:

$$\begin{array}{rccl} \kappa \ : \ \mathfrak{iso} & \hookrightarrow & \mathfrak{so}(T_p\mathcal{M}, \mathrm{g}_p) \ltimes T_p\mathcal{M} \ \simeq \ \mathfrak{so}(r, s) \ltimes \mathbb{R}^{r,s} \\ K & \mapsto & -(\nabla K, K)\,|_p \end{array}$$

  This is <u>not</u> a Lie algebra homomorphism!
  E.g., $\mathfrak{iso}(\mathbb{S}^m) = \mathfrak{so}(m+1) \not\simeq \mathfrak{so}(m) \ltimes \mathbb{R}^m = \mathfrak{iso}(\mathbb{R}^m)$.

- locally homogeneous $\implies \mathrm{pr}_{\mathbb{R}^{r,s}} \circ \kappa$ is surjective.

The Killing equation for pp-waves

The coordinates in which $\mathrm{g} = 2du(dv + H(u, \mathbf{x})\, du) + d\mathbf{x}^\top d\mathbf{x}$ can be chosen:

$$H(u, 0) \equiv 0, \quad \frac{\partial H}{\partial x^i}(u, 0) \equiv 0.$$

(normal Brinkmann coordinates centred at $p \mapsto 0$).

Proposition

Let $(\mathcal{M}^{n+2}, \mathrm{g})$ be as indecomposable pp-wave. K is Killing $\iff$

$$K = (c - av - \dot{\Psi} \cdot \mathbf{x})\, \partial_v + (\Psi + F\mathbf{x})^i\, \partial_i + (au + b)\, \partial_u, \tag{3}$$

where $a, b, c \in \mathbb{R}$, $F \in \mathfrak{so}(n)$ *and* $\Psi : u \mapsto \Psi(u) \in \mathbb{R}^n$ *satisfying*

$$\boxed{\ddot{\Psi} \cdot \mathbf{x} - \mathrm{grad}(H) \cdot (\Psi + F\mathbf{x}) - (au + b)\dot{H} - 2aH = 0.} \tag{4}$$

Consequences: Differentiating w.r.t. $\mathbf{x} \implies$

$$\ddot{\Psi} + F\mathrm{grad}(H) - \mathrm{Hess}(H)(\Psi + F\mathbf{x}) - (a\,u + b)\mathrm{grad}(\dot{H}) - 2a\mathrm{grad}(H) = 0$$

At $\mathbf{x} = 0$: $\quad \ddot{\Psi}(u) - \mathrm{Hess}(H)(u, 0)\Psi(u) = 0.$

Count: dim of Killing vf's $\leq 2n + 3 + \frac{1}{2}n(n-1)$

Motivation: classical results in dim 4

Jordan-Ehlers-Kundt '60:

- Complete solution of the Killing equation on 4-dim pp-wave $(\mathbb{R}^4, \mathrm{g})$ with $\mathrm{Ric} = \Delta H\, du^2 = 0$ (dimensions of solution space $1, 2, 3, 5, 6$).
- Observation: $(\mathbb{R}^4, \mathrm{g})$ locally $\partial_v^\perp$-homogeneous, then g is a plane wave metric.

Sippel-Goenner '86:

- Solution *without assuming* $\mathrm{Ric} = 0$ (dim's of Killing vf's $\ldots, 7$)
- Only one homogeneous example that is *not a plane wave*: $H(x^1, x^2) := \mathrm{e}^{a_1 x^1 - a_2 x^2}$ with $a_1^2 + a_2^2 \neq 0$ (5 Killing vf's)
- Coordinate transformation $x = a_1 x^1 - a_2 x^2,\ y = a_2 x^1 + a_1 x^2$ shows that this is a product metric (= *decomposable*).

Conclusion for dim 4: *Indecomposable* locally homogeneous pp-waves in dimension 4 are plane waves.

An example in dim 3

Consider $\mathrm{R}^3 \ni (u, v, x)$ with pp-wave metric

$$\mathrm{g} = 2du\mathrm{d}v + 2\mathrm{e}^{2x}du^2 + dx^2,$$

i.e., $H(x) = \mathrm{e}^{2ax}$.
Killing vf's:

$$\partial_v, \quad \partial_u, \quad K := \partial_x + v\,\partial_v - u\,\partial_u.$$

I.e., g is locally homogeneous, indecomposable, *but not a plane wave*.

$$\mathrm{R}(\partial_x, \partial_u) = 2\begin{pmatrix} 0 & \mathrm{e}^{2x} & 0 \\ 0 & 0 & -\mathrm{e}^{2x} \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

i.e. $\underline{\mathit{rank}(\mathrm{R} : \Lambda^2 \to \Lambda^2) = 1}$.

Results in arbitrary dimensions [Globke & L '14]

Theorem 1

Let $(\mathcal{M}^m, \mathrm{g})$ be a pp-wave that is $V^\perp$-homogeneous. Assume that

- $(\mathcal{M}, \mathrm{g})$ is *strongly indecomposable*, i.e., no nbhd is decomposable,
- $\mathrm{rk}(\mathrm{R} : \Lambda^2 T\mathcal{M} \to \Lambda^2 T\mathcal{M}) > 1$ on an open dense subset of $\mathcal{M}$.
  (Recall, $\Lambda^2 V^\perp \subset \mathrm{Ker}(\mathrm{R})$, i.e., $\mathrm{rk}(R) < m - 1$).

Then $(\mathcal{M}, \mathrm{g})$ is a plane wave.

Corollary

A pp-wave is a plane wave if

- strongly indecomposable, locally $V^\perp$-homogeneous, $\mathrm{Ric} = 0$, or
- indecomposable, locally homogeneous & $\exists p$ with $\mathrm{rk}(\mathrm{R}|_p) > 1$, or
- indecomposable, locally homogeneous & $\mathrm{Ric} = 0$ (as in dim 4).

Killing vector fields on plane waves

- For plane waves: $H = \frac{1}{2}\,\mathbf{x}^\top S(u)\mathbf{x}$ for a symmetric u-dep. matrix S. Hence

$$\mathrm{grad}(H) = S\mathbf{x}\,, \quad \mathrm{Hess}(H) = S.$$

- Multiplying the differentiated equation

$$\ddot{\Psi} + F\mathrm{grad}(H) - \mathrm{Hess}(H)(\Psi + F\mathbf{x}) - (a\,u + b)\mathrm{grad}(\dot{H}) - 2a\mathrm{grad}(H) = 0.$$

  by $\mathbf{x}$ implies the Killing equ. (4).

- For plane waves: many solutions with $F = a = b = 0$, as it becomes

$$\ddot{\Psi} - S\Psi = 0. \tag{5}$$

- Hence $\mathfrak{iso}(V)$ contains the Heisenberg algebra $\mathfrak{he}_n(\mathbb{R})$:

$$\partial_v, \quad L_i := \phi_i^k \partial_k - \mathbf{x}\cdot\dot{\Phi}_i\,\partial_v, \quad K_i := \psi_i^k \partial_k - \mathbf{x}\cdot\dot{\Psi}_i\,\partial_v,$$

  where $\Phi_i = (\phi_i^k)_{k=1,\dots,n}$ and $\Psi_i = (\psi_i^k)_{k=1,\dots,n}$ solutions to (5) with

$$\Phi_i(0) = 0, \;\; \dot{\Phi}_i(0) = \mathbf{e}_i \quad \text{and} \quad \Psi_i(0) = \mathbf{e}_i, \;\; \dot{\Psi}_i(0) = 0,$$

- Plane waves have *commuting* Killing vf's $\partial_v, K_1, \dots, K_n$ that span $V^\perp$.

Homogeneous plane waves [Blau & O'Loughlin '03]

- For homogeneous plane waves, we need an additional Killing vector field K_+ such that $K_+|_p = \partial_+$, i.e. with $b \neq 0$.
- Killing equation is a matrix ODE:

$$[S(u), F] + (au + b)\dot{S}(u) + 2a\,S(u) = 0. \quad (6)$$

Two cases:

$a = 0$: W.l.o.g. assume $b = 1$ and (6) becomes

$$[S(u), F] + \dot{S}(u) = 0.$$

Solution:

$$S(u) = \mathrm{e}^{uF} S_0 \mathrm{e}^{-uF} \quad \text{with } F \in \mathfrak{so}(n) \text{ and } S_0 \text{ symmetric.}$$

$a \neq 0$: W.l.o.g. $a = 1$ and (6) becomes ODE with singularity at $u = -b$,

$$(u + b)\dot{S}(u) + [S(u), F] + 2\,S(u) = 0.$$

Solution:

$$S(u) = \tfrac{1}{(u+b)^2}(\mathrm{e}^{\log(u+b)F} S_0 \mathrm{e}^{\log(-(u+b))F}), \quad u > b.$$

- By our corollary, this also gives a classification of indecomposable homogeneous pp-waves satisfying the rank condition.

Further results

Recall that plane waves have commuting Killing vector fields everywhere spanning $V^\perp$. We prove the converse:

Theorem 2

Let $(\mathcal{M}, \mathrm{g})$ be a strongly indecomposable pp-wave: $\forall\ p \in \mathcal{M}\ \exists$ nbhd $\mathcal{U}_p$ with Killing vf's on $\mathcal{U}_p$ that span $V^\perp|_{\mathcal{U}_p}$. Then $(\mathcal{M}, \mathrm{g})$ is a plane wave.

Theorem 3

Let $(\mathcal{M}^m, \mathrm{g})$ be a semi-Riemannian manifold with *commuting* Killing vector fields that span a null distribution of rank $m-1$. Then $\exists$ parallel null vector field V and

$$\mathrm{R}(X, Y)Z = 0 \ \text{ and } \ \nabla_X \mathrm{R} = 0, \quad \forall\ X, Y, Z \in V^\perp.$$

In particular, if $(\mathcal{M}, \mathrm{g})$ is Lorentzian, then it is a plane wave.

Proof of Theorem 1: the setting

Fix $p \in \mathcal{M}$ and normal Brinkmann coordinates centred at p. Consider

$$\mathfrak{iso}_p(V) := \{K \in \mathfrak{iso} \mid \mathrm{g}(K, V)|_p = 0\}.$$

Observe: For pp-waves, $\mathfrak{iso}_p(V)$ is a Lie algebra.
Under the assumption that there are Killing vector fields $K_i \in \mathfrak{iso}_p(V)$ such that $\mathrm{span}(\partial_v, K_1, \ldots, K_n)|_p = V^\perp|_p$ we have to show $\nabla_X \mathrm{R} = 0 \ \forall X \in V^\perp$.

$$\nabla_{\partial_v} \mathrm{R} = 0, \quad \text{and} \quad \nabla_{K_i} \mathrm{R} = -\phi_i \cdot \mathrm{R}, \quad \text{with } \phi_i := \nabla K_i.$$

I.e., we have to determine ϕ_i and its action on R, which on the other hand satisfies $\mathrm{R}(X, Y)Z = 0$ whenever $X, Y \in V^\perp|_p$.

Proof of Theorem 1: the evaluation map

Recall

$$K = (c - av - \dot{\Psi} \cdot \mathbf{x})\,\partial_v + (\Psi + F\mathbf{x})^i\,\partial_i + (au + b)\,\partial_u,$$

Then, with $\mathbf{e}_- = \partial_v|_p, \mathbf{e}_i = \partial_i|_p, \mathbf{e}_+ = \partial_+|_p$, we have

$$\begin{aligned} K|_p &= c\,\mathbf{e}_- + X^i\mathbf{e}_i + b\,\mathbf{e}_+, && X := (X^i)_{i=1}^n = \Psi(0), \\ \nabla_{\mathbf{e}_-} K|_p &= -a\,\mathbf{e}_- \\ \nabla_{\mathbf{e}_i} K|_p &= -Y_i\mathbf{e}_- + F_i{}^k\mathbf{e}_k, && Y := (Y^i)_{i=1}^n = \dot{\Psi}(0) \\ \nabla_{\mathbf{e}_+} K|_p &= Y^i\mathbf{e}_i + a\,\mathbf{e}_+. \end{aligned} \tag{7}$$

$$\Longrightarrow \quad \begin{aligned} \kappa : \mathfrak{iso} &\hookrightarrow \mathfrak{sim}(n) \ltimes \mathbb{R}^{1,n+1} = ((\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n) \ltimes \mathbb{R}^{1,n+1} \\ K &\mapsto \left(\begin{pmatrix} a & Y & 0 \\ 0 & -F & -Y^\top \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} -c \\ -X \\ -b \end{pmatrix} \right) \end{aligned}$$

Observation: When restricted to $\mathfrak{iso}_p(V) = \{b = 0\}$ this is a Lie algebra hom!

Proof of Theorem 1: relation to Euclidean motions

- Hence, we have an injective Lie algebra homomorphism

$$\begin{array}{rcccc} \kappa : \mathfrak{iso}_p(V) & \hookrightarrow & \mathfrak{sim}(n) \ltimes \mathbb{R}^{n+1} & \simeq & \mathfrak{co}(n) \ltimes \mathfrak{he}(n) \\ K & \mapsto & \left(\begin{pmatrix} a & Y & 0 \\ 0 & -F & -Y^\top \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} -c \\ -X \\ 0 \end{pmatrix} \right) & \mapsto & \begin{pmatrix} a & Y^\top & c \\ 0 & F & X \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

- $\mathfrak{co}(n) \ltimes \mathfrak{he}(n)$ contains the Abelian ideal $\mathfrak{a} := \left\{ \begin{pmatrix} a & Y^\top & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$
- Lie algebra homomorphism

$$\begin{array}{rcl} \lambda : \mathfrak{iso}_p(V) & \longrightarrow & (\mathfrak{co}(n) \ltimes \mathfrak{he}(n))/\mathfrak{a} \simeq \mathfrak{euc}(n) = \mathfrak{so}(n) \ltimes \mathbb{R}^n \\ K & \longmapsto & \begin{pmatrix} 0 & X & 0 \\ 0 & F & -X^\top \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Proposition

If $(\mathcal{M}, \mathrm{g})$ is locally $V^\perp$-homogeneous, then $\mathrm{pr}_{\mathbb{R}^n} \circ \lambda : \mathfrak{iso}_p(V) \to \mathbb{R}^n$ is a *surjetive* Lie algebra homomorphism, i.e., the image $\mathfrak{g} := \lambda(\mathfrak{iso}_p(V))$ is an *indecomposable* subalgebra of the Euclidean motions $\mathfrak{euc}(n)$.

Proof of Theorem 1: Indecomposable subalgebras of $\mathfrak{euc}(n)$

Berard-Bergery & Ikemakhen '93:

Let $\mathfrak{g} \subset \mathfrak{euc}(n) = \mathfrak{so}(n) \ltimes \mathbb{R}^n$ act indecomposably on $\mathbb{R}^{1,n+1}$, i.e., without non-degenerate invariant subspaces. Then either

(A) $\mathfrak{g}$ contains the translations $\mathbb{R}^n$, or

(B) $\mathfrak{g}$ contains $\mathbb{R}^q$ for $1 < q < n$, in which case there is a subalgebra $\mathfrak{h} \subset \mathfrak{so}(q)$ and a surjective linear map $\varphi : \mathfrak{h} \to \mathbb{R}^{n-q}$ such that $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} 0 & X & \varphi(F) & 0 \\ 0 & F & 0 & -X \\ 0 & 0 & 0 & -\varphi(F) \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; F \in \mathfrak{h}, X \in \mathbb{R}^q \right\}. \tag{8}$$

Consequences for the Killing vector fields of locally $V^\perp$-homogeneous pp-waves:

(A) $\exists$ KVFs $(K_i)_{i=1,\dots,n}$ with $K_i|_p = \mathbf{e}_i$ and $\phi_i := \nabla K_i|_p : \mathbf{e}_j \to \mathbb{R} \cdot \mathbf{e}_-$.

(B) $\exists$ KVFs $(K_A)_{A=1,\dots,N}$ as in (A) and $(K_a)_{a=N+1,\dots,n}$ with $K_a|_p = \mathbf{e}_a$ &

$$\phi_a : \begin{array}{ccc} \mathbf{e}_A & \mapsto & \overset{(a)}{F}{}_A{}^B \mathbf{e}_B + \mathbb{R} \cdot \mathbf{e}_- \\ \mathbf{e}_b & \mapsto & \mathbb{R} \cdot \mathbf{e}_- \end{array}$$

Proof of Theorem 1: case A

- From the previous slide we know that, in the basis $(\mathbf{e}_-, \mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_+)$,

$$\phi_i := \nabla K_i|_p = \begin{pmatrix} -a_i & -X_i & 0 \\ 0 & 0 & X_i^\top \\ 0 & 0 & a_i \end{pmatrix} \tag{9}$$

- W.l.o.g. we may assume that $a_1, \dots, a_{n-1} = 0$.

Set $\mathrm{R}_{ij} := \mathrm{R}(\mathbf{e}_+, \mathbf{e}_i, \mathbf{e}_+, \mathbf{e}_j)$ and $\nabla_k \mathrm{R}_{ij} := \nabla_{\mathbf{e}_k} \mathrm{R}(\mathbf{e}_+, \mathbf{e}_i, \mathbf{e}_+, \mathbf{e}_j)$. Then

- $\nabla_k \mathrm{R}_{ij}$ is symmetric in all its indices,
- With equation (9) the integrability condition $\nabla_{K_i}\mathrm{R} = -\phi_i \cdot \mathrm{R}$ implies that

$$\nabla_k \mathrm{R}_{ij} = \begin{array}{c} \mathrm{R}(\phi_k(\mathbf{e}_+), \mathbf{e}_i, \mathbf{e}_+, \mathbf{e}_j) + \mathrm{R}(\mathbf{e}_+, \phi_k(\mathbf{e}_i), \mathbf{e}_+, \mathbf{e}_j) \\ +\mathrm{R}(\phi_k(\mathbf{e}_+), \mathbf{e}_j, \mathbf{e}_+, \mathbf{e}_i) + \mathrm{R}(\mathbf{e}_+, \phi_k(\mathbf{e}_j), \mathbf{e}_+, \mathbf{e}_i) \end{array} = 2a_k \mathrm{R}_{ij},$$

- Hence, $\nabla_k \mathrm{R}_{ij} = 0$ with the possible exception of $\nabla_n \mathrm{R}_{nn} = 2a_n \mathrm{R}_{nn}$.
- Then $a_n \neq 0$ implies $\mathrm{R}_{ij} = 0$ apart from R_{nn} contradicting the rank assumption.

Proof of Theorem 1: case B

The proof of case B is very similar, but with the technical difficulty of dealing with the rotational component F. This can be overcome by the specific shape of F which allows to derive an equation

$$\begin{aligned} \mathrm{R}_{bc} &= 0, \quad \text{for all } (b,c) \neq (n,n). \\ \mathrm{R}_{cB} &= 0 \\ -2a_n \mathbf{R} &= [\overset{(n)}{F}, \mathbf{R}], \quad \text{where } \mathbf{R} = (\mathrm{R}_{AB}), \end{aligned}$$

The last equation implies $\mathbf{R} = 0$, since F is skew and $\mathbf{R}$ is symmetric, yielding a contradiction to the rank assumption. □

Many thanks for your attention!