Comparing Einstein to Newton via Post Newtonian expansions

Todd A. Oliynyk
Monash University

Newtonian Gravity

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Spacetime

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$$M = [0, T) \times \Sigma$$

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$$\Sigma = \mathbb{T}^3 \qquad (\text{Cosmological setting})$$

Time and Distance

Time and Distance

Coordinates

(x^i) $i=0,1,2,3$ are Cartesian Coordinates on $M=(0,T)\times\Sigma$

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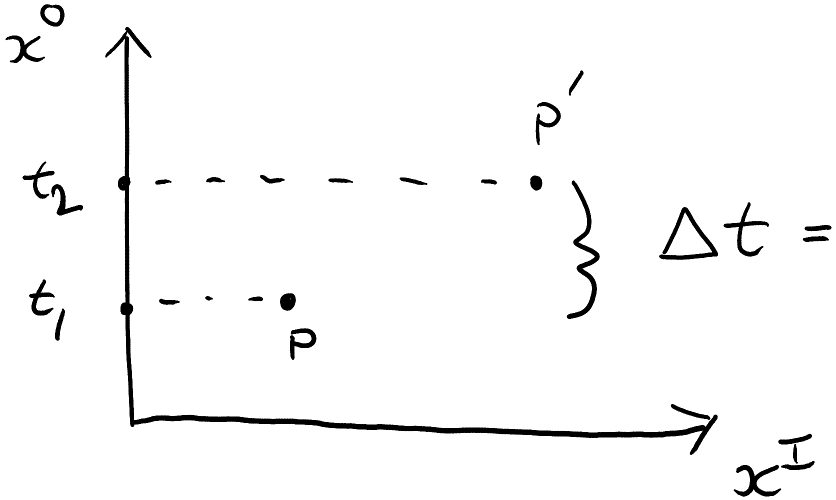
Time

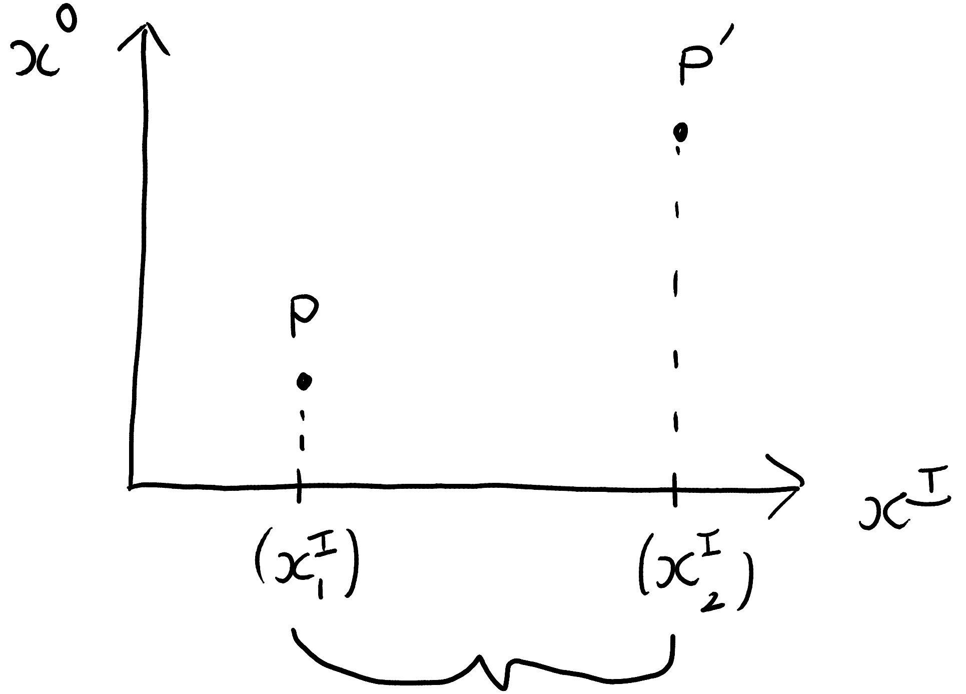
$t = x^0 \in (0,T)$ measures the time between events



$$\Delta t = t_2 - t_1$$

Distance

$ds^2 = \delta_{IJ}\, dx^I dx^J$ measures the distance between events



$$\Delta x = \sqrt{\delta_{IJ}\left(x_2^I - x_1^I\right)\left(x_2^J - x_1^J\right)}$$

Field Equations

Field Equations

Gravitational

$$\Delta \Phi = 4\pi G \rho \quad \left(\Delta = \delta^{IJ} \partial_I \partial_J \right)$$

$\Phi = \Phi(t, x^I)$ is the Newtonian Potential

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Matter Perfect Fluid

$$\partial_t \rho + \partial_I (\rho w^I) = 0$$

$$\rho \left(\partial_t w^J + w^I \partial_I w^J \right) = -\left(\rho \partial^J \Phi + \partial^J p \right)$$

where

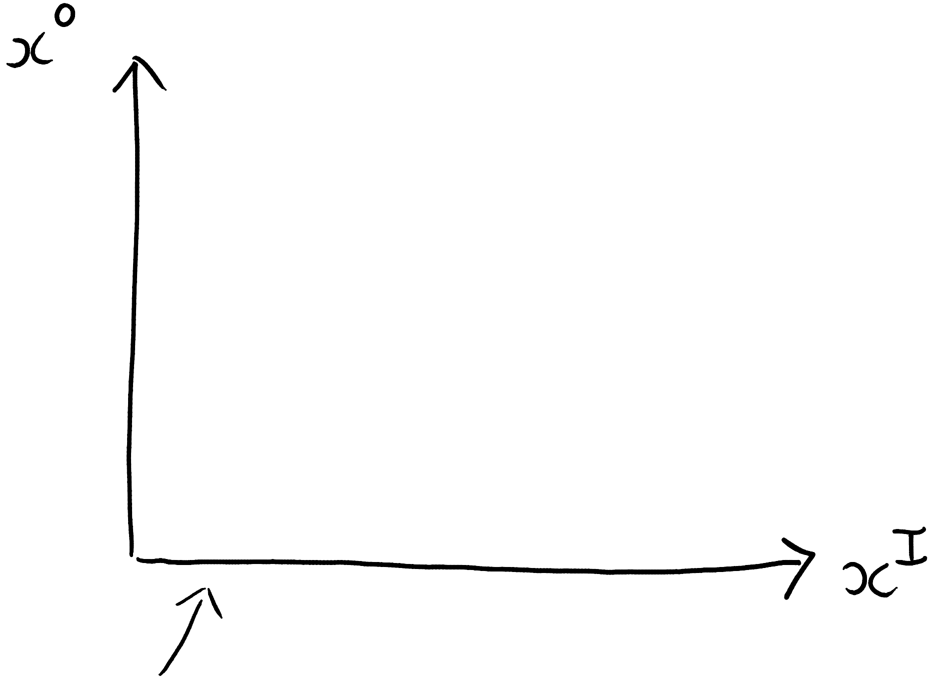
$\rho = \rho(t, x^I)$ is the fluid density

$w^I = w^I(t, x^I)$ is the fluid velocity

$p = f(\rho)$ is the fluid pressure

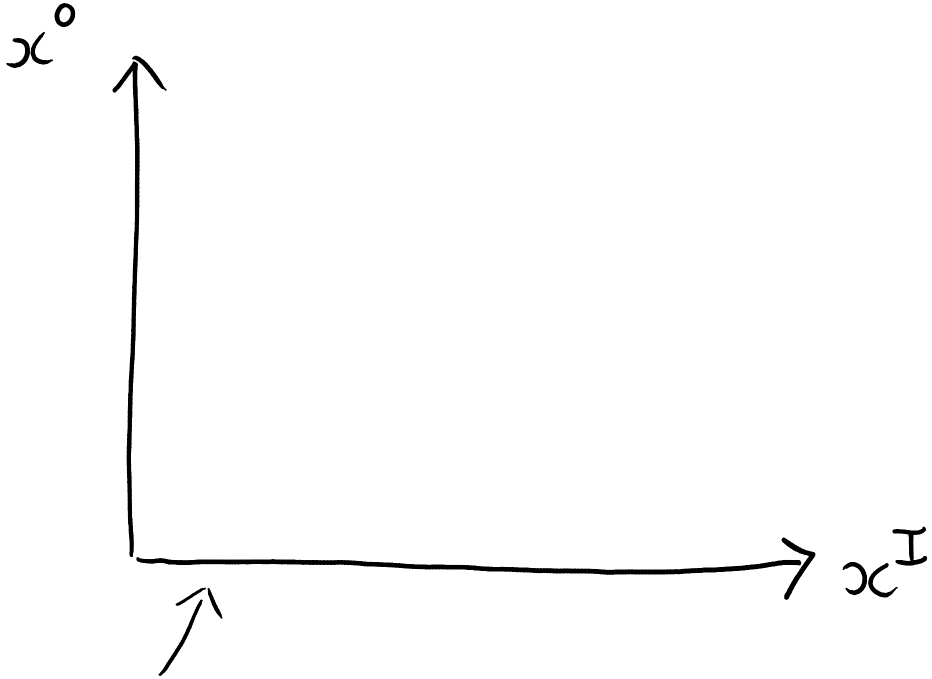
The Initial Value Problem

The Initial Value Problem



(i) Choose the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

The Initial Value Problem



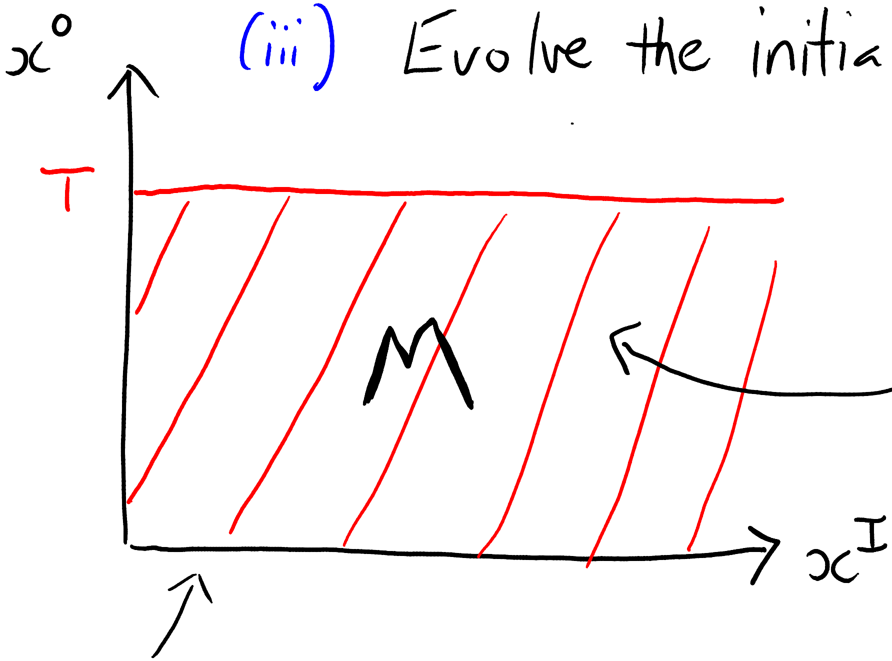
(i) Choose the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

(ii) Specify the initial data on Σ_0

$$\left(\rho|_{\Sigma_0}, w^I|_{\Sigma_0}\right) = \left(\rho_0(x^J), w_0^I(x^J)\right)$$

The Initial Value Problem

(iii) Evolve the initial data using the Field Equations



$\left(\rho(t,x^I), w^J(t,x^J), \Phi(t,x^J)\right)$

for $(t,x^I) \in M = [0,T) \times \Sigma$

(i) Choose the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

(ii) Specify the initial data on Σ_0

$$\left(\rho\big|_{\Sigma_0}, w^I\big|_{\Sigma_0}\right) = \left(\rho_0(x^J), w_0^I(x^J)\right)$$

Einstein Gravity

Spacetime

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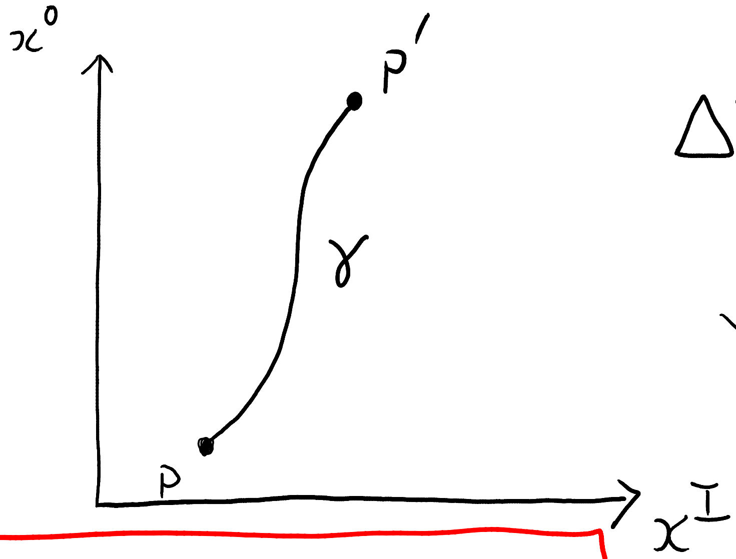
Metric

$$g = g_{ij}\,dx^i dx^j \qquad g_{ij} = g_{ij}(t, x^I)$$

$$g_{ij} = g_{ji}$$

$$(g_{ij}) \underset{\text{at each point}}{\sim} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time



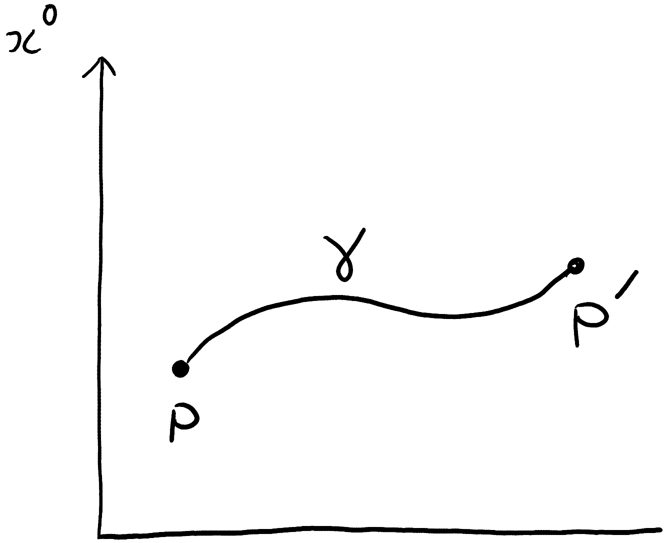
$\gamma : [0,1] \longrightarrow M$

$\gamma(0) = P \,,\ \gamma(1) = P'$

$\gamma(s)$ is a world-line

$$\underbrace{\Delta\tau = \frac{1}{c^2}\int_0^1 \sqrt{-g_{ij}\frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}}\, ds}_{\text{Proper time interval between events } P \text{ and } P' \text{ measured by an observer moving along the world-line } \gamma(s)}$$

Distance



$$\Delta x = \int_0^1 \sqrt{g_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}} \, ds$$

Distance between spacelike separated events P and P'

$\gamma : [0,1] \longrightarrow M$

$\gamma(0) = P, \quad \gamma(1) = P'$

is a geodesic, i.e.

$$\frac{d^2\gamma}{ds^2} + \Gamma^k_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} = 0$$

Field Equations

Field Equations

Gravitational

$$G_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

where

$$G_{ij} = -\frac{1}{2} g^{kl} \partial^2_{kl} g_{ij} + \frac{1}{4} g_{ij} g^{kl} g^{mn} \partial^2_{kl} g_{mn} - g_{k(i} \partial_{j)} H^k$$

$$+ \frac{1}{2} g_{ij} \partial_k H^k + f_{ij}(g, \partial g)$$

$$H^j = \partial_k g^{kj} + \frac{1}{2} g^{kj} g^{lm} \partial_k g_{lm}$$

Matter - Perfect Fluid

Matter - Perfect Fluid

Stress Energy Tensor

$$T^{ij} = \left(\rho + \frac{1}{c^2} p\right) v^i v^j + p g^{ij}$$

where $\rho = \rho(t, x^I)$ is the proper energy density

$v^i = v^i(t, x^I)$ is the fluid four-velocity normalized so that

$$g_{ij} v^i v^j = -c^2$$

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Equations of motion

$$\nabla_i T^{ij} = 0$$

The Initial Value Problem

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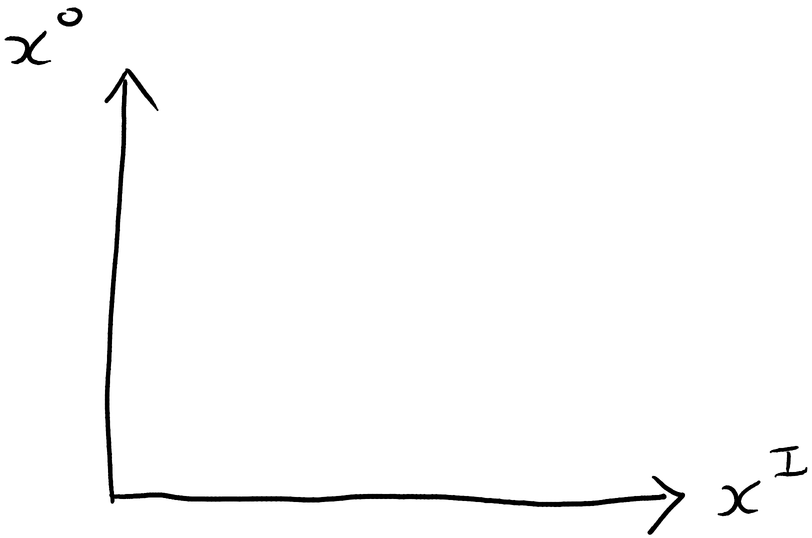

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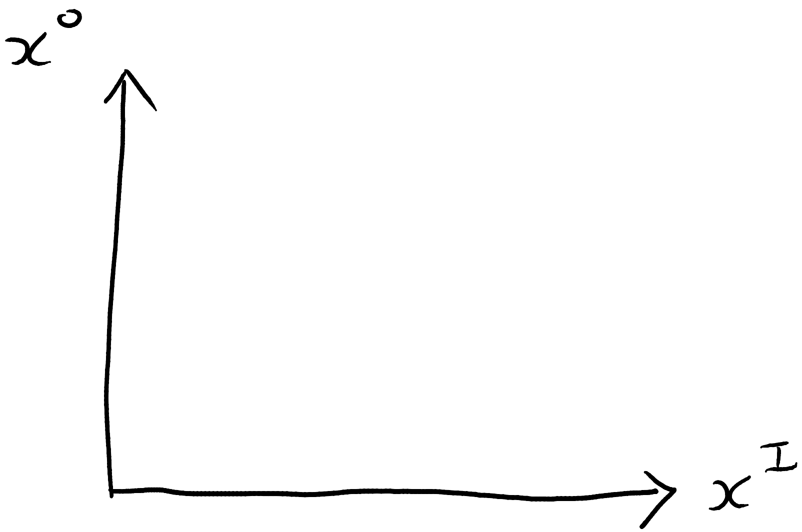
The Initial Value Problem

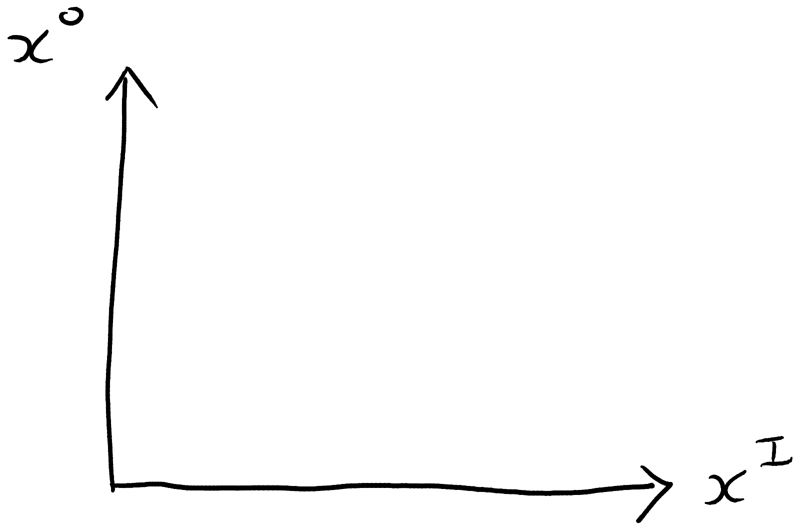


(i) Choose the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

(ii) Solve the constraint equations

$$\left(G^{i0} = \frac{8\pi}{c^4} T^{i0} ,\ g^{ij} \Gamma_{ij}^{k} = 0 ,\ g_{ij} V^i V^j = -c^2 \right) \text{ on } \Sigma_0$$

The Initial Value Problem



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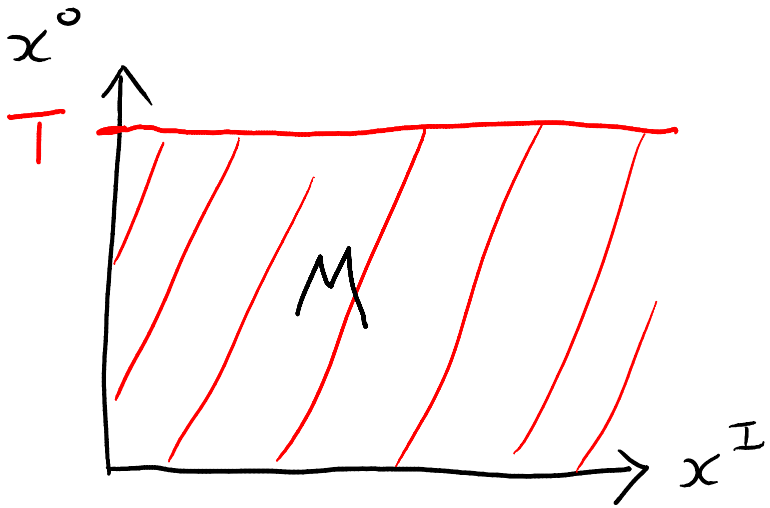
(iii) Specify the initial data

$$\left(g_{ij}\big|_{\Sigma_0} ,\; \partial_t g_{ij}\big|_{\Sigma_0} ,\; \rho\big|_{\Sigma_0} ,\; v^i\big|_{\Sigma_0} \right)$$

The Initial Value Problem



(iv) Use the field equations to evolve the initial data

$$\left(g_{ij}(t, x^I),\ \rho(t, x^I),\ v^i(t, x^I) \right)$$

for $(t, x^I) \in M = [0, T) \times \Sigma$

(i) Choose the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

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$$\left(g_{ij}\big|_{\Sigma_0},\ \partial_t g_{ij}\big|_{\Sigma_0},\ \rho\big|_{\Sigma_0},\ v^i\big|_{\Sigma_0} \right)$$

Newtonian Gravity as an Approximation

Newtonian Gravity as an Approximation

Heuristic Principal

Einstein gravity is well approximated by Newtonian gravity in spacetime regions where

$$\varepsilon = \frac{v_T}{c} \ll 1$$

Here, c is the speed of light and v_T is the typical speed of the matter

Newtonian vs. Einstein gravity

Newtonian vs. Einstein gravity

Newtonian gravity

$$\Delta \Phi = \rho \qquad \left(\Delta = \delta^{IJ} \partial_I \partial_J \right)$$

Newtonian vs. Einstein gravity

Newtonian gravity

$$\Delta \Phi = \rho \qquad \left(\Delta = \delta^{IJ} \partial_I \partial_J \right)$$

Einstein gravity

The following equations are the Einstein field equations with the following simplifying assumptions

(i) no coupling to matter, and

(ii) axi-symmetry.

Numerically generated axisymmetric black hole spacetimes: Numerical methods and code tests

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We develop a flexible computer code to study axisymmetric black hole spacetimes. The code is currently set up to evolve the fully nonlinear Einstein equations in azimuthal and equatorial plane symmetry. The initial data for this code generally consists of a combination of one black hole and an arbitrary amplitude, time symmetric gravitational wave. We present a discussion of the mathematical framework for the problem, various coordinate and time slice choices, and a battery of code tests.

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I. INTRODUCTION

A. Overview

In this paper we report on a computer code developed to study the fully nonlinear Einstein equations for axisymmetric spacetimes. This code represents an essential step in a longer term program to develop codes for solving the Einstein equations in the absence of any symmetries. The motivation for such a project is severalfold. (1) Such codes will be required to perform calculations of fully relativistic sources of gravitational waves. Calculations of this nature will be important for a theoretical understanding of gravitational wave astronomy, which promises to provide a new window on the astrophysical Universe [1]. Currently there exist no analytic techniques for computing waveforms expected from promising sources of strong gravitational waves, such as the coalescence of rotating black holes. Computational methods are currently our only recourse for computing such waveforms. (2) The study of general relativity itself as a fundamental theory of physics is a difficult undertaking, due in part to the complicated, nonlinear nature of the equations. Of all the analytic solutions found in the past 75 years of study only a relatively small number correspond to astrophysically interesting situations and these are usually very idealized, e.g., the Schwarzschild and Kerr solutions. The study of these solutions and their perturbations has been extremely fruitful, helping to shape our understanding of the theory as a whole. However, not being strongly dynamical, they represent only a small part of the "solution space" of general relativity. The study of the strongly dynamical regions should provide new insights into the nature of the Einstein equations. (3) In the past decade the power of the fastest single processor vector computers has increased by perhaps an order of magnitude, while the next 5 years should witness an increase of 1000 times in overall power due to the development of massively parallel machines. If the current state of numerical relativity is used as a guide, this acceleration in the power of supercomputers should make possible the computation of complex, strongly dynamical, astrophysically realistic spacetimes. It is hoped that the study of the Einstein equations, a complicated set of hyperbolic and elliptic equations, can act as a driving force to develop accurate numerical techniques suitable for this new generation of machines.

In this paper we discuss a suite of codes which has been developed at NCSA over the past 5 years. The codes are specialized for the computation of axisymmetric, equatorial plane symmetric spacetimes and have been applied to systems consisting of a single oscillating black-hole and the head-on collision of two equal mass black holes. Overall the metric, numerical methods, and spacetime analysis tools used to compute and analyze the data for these two systems are exactly the same. Where the codes differ is in the boundary conditions, initial conditions, and the computational grid used to match the geometry of the different topologies. Here we will refer to this suite of codes as "the code" with the understanding that the results obtained by one code are not significantly different from those obtained by the other codes. (A modified version of this code has been used to evolve the collision of two equal mass black holes, as described in Ref. [2].) The emphasis in this paper is on the numerical algorithms used and various tests of the code's accuracy, convergence, and stability. Companion papers [3,4] are devoted to other aspects of this system. In [3,5,6], we discuss many details of the initial-value problem for this system, which consists of a time symmetric gravitational wave superimposed on a black hole. For completeness, some details of the distorted black hole initial data are provided in Secs. II C and II D, but a full discussion appears in Refs. [3,5]. In another paper [4] we discuss the evolution of low and moderate amplitude gravitational

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APPENDIX A: THE COMPLETE SET OF EINSTEIN EQUATIONS

This appendix presents the "3+1" form of the Einstein equations used in our code. All the equations are written *explicitly* in terms of the kinematic and dynamical variables and their partial derivatives with respect to the spatial and temporal variables. The equations were derived from a package written in the MACSYMA language.

The most general three-metric for an axisymmetric nonrotating system is given by

$$\gamma_{ij} = \psi^4\,\hat{\gamma}_{ij} = \begin{pmatrix} a\,\psi^4 & c\,\psi^4 & 0 \\ c\,\psi^4 & b\,\psi^4 & 0 \\ 0 & 0 & d\,\psi^4 \sin^2\theta \end{pmatrix}$$

and the most general extrinsic curvature tensor is

$$K_{ij} = \psi^4\,\hat{K}_{ij} = \begin{pmatrix} H_a\,\psi^4 & H_c\,\psi^4 & 0 \\ H_c\,\psi^4 & H_b\,\psi^4 & 0 \\ 0 & 0 & H_d\,\psi^4 \sin^2\theta \end{pmatrix}.$$

The kinematic variables include the lapse function α and the shift vector with two nonzero components:

$$\beta^i = (\beta^\eta, \beta^\theta, 0)\,.$$

With the addition of the notation

$$\delta = ab - c^2$$

the intrinsic Ricci curvature tensor for the three-dimensional spacelike hypersurfaces is determined from the three-metric and has the nonzero components

$$\begin{aligned}
R_{\eta\eta} = {} & -\frac{2\,a^2\,\frac{\partial\psi}{\partial\theta}\,\cot\theta}{\delta\,\psi} + \frac{2\,a\,c\,\frac{\partial\psi}{\partial\eta}\,\cot\theta}{\delta\,\psi} + \frac{a\,\frac{\partial c}{\partial\eta}\,\cot\theta}{\delta} - \frac{\frac{\partial a}{\partial\eta}\,c\,\cot\theta}{2\,\delta} - \frac{a\,\frac{\partial a}{\partial\theta}\,\cot\theta}{2\,\delta} - \frac{2\,a^2\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi} \\
& - \frac{2\,a^2\,(\frac{\partial\psi}{\partial\theta})^2}{\delta\,\psi^2} + \frac{4\,a\,c\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^2} - \frac{a^2\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi} + \frac{a\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi} + \frac{2\,a\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{\frac{\partial a}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} \\
& - \frac{3\,a\,\frac{\partial a}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{2\,a^2\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{2\,a^2\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{a^2\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{a^3\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} \\
& + \frac{a^2\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{2\,a\,b\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi} - \frac{2\,\frac{\partial^2\psi}{\partial\eta^2}}{\psi} + \frac{4\,a\,c\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi} - \frac{2\,a\,b\,(\frac{\partial\psi}{\partial\eta})^2}{\delta\,\psi^2} + \frac{6\,(\frac{\partial\psi}{\partial\eta})^2}{\psi^2} \\
& + \frac{a\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi} - \frac{a\,b\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi} - \frac{2\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{2\,a\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} \\
& + \frac{2\,a^2\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{2\,a\,b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a^2\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{a^2\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{a\,\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} \\
& + \frac{a\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{2\,\delta\,d} - \frac{\frac{\partial a}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d} - \frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d} - \frac{\frac{\partial^2 d}{\partial\eta^2}}{2\,d} + \frac{(\frac{\partial d}{\partial\eta})^2}{4\,d^2} - \frac{c\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{2\,\delta\,d} \\
& + \frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d} + \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d} + \frac{a\,\frac{\partial^2 c}{\partial\eta\partial\theta}}{\delta} - \frac{a\,\frac{\partial^2 b}{\partial\eta^2}}{2\,\delta} - \frac{a\,\frac{\partial^2 a}{\partial\theta^2}}{2\,\delta} + \frac{a\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta^2} \\
& - \frac{a\,\frac{\partial a}{\partial\theta}\,c\,\frac{\partial c}{\partial\theta}}{2\,\delta^2} - \frac{a\,\frac{\partial a}{\partial\eta}\,b\,\frac{\partial c}{\partial\theta}}{2\,\delta^2} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial c}{\partial\eta}}{2\,\delta^2} - \frac{a^2\,\frac{\partial b}{\partial\theta}\,\frac{\partial c}{\partial\eta}}{2\,\delta^2} + \frac{a\,\frac{\partial a}{\partial\eta}\,\frac{\partial b}{\partial\theta}\,c}{4\,\delta^2} - \frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\eta}\,c}{4\,\delta^2} \\
& + \frac{a^2\,\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\theta}}{4\,\delta^2} + \frac{a^2\,(\frac{\partial b}{\partial\eta})^2}{4\,\delta^2} + \frac{a\,\frac{\partial a}{\partial\eta}\,b\,\frac{\partial b}{\partial\eta}}{4\,\delta^2} + \frac{a\,(\frac{\partial a}{\partial\theta})^2\,b}{4\,\delta^2}\,,
\end{aligned}$$

$$
\begin{aligned}
R_{\eta\theta} = &-\frac{2\,a\,c\,\frac{\partial\psi}{\partial\theta}\cot\theta}{\delta\,\psi}+\frac{2\,a\,b\,\frac{\partial\psi}{\partial\eta}\cot\theta}{\delta\,\psi}-\frac{2\,\frac{\partial\psi}{\partial\eta}\cot\theta}{\psi}-\frac{\frac{\partial d}{\partial\eta}\cot\theta}{2\,d}-\frac{\frac{\partial a}{\partial\theta}\,c\cot\theta}{2\,\delta}+\frac{a\,\frac{\partial b}{\partial\eta}\cot\theta}{2\,\delta}\\
&-\frac{2\,a\,c\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi}-\frac{2\,a\,c\,(\frac{\partial\psi}{\partial\theta})^2}{\delta\,\psi^2}+\frac{4\,a\,b\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^2}+\frac{2\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\psi^2}-\frac{a\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi}+\frac{a\,b\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi}\\
&-\frac{\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{d\,\psi}+\frac{2\,a\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{3\,\frac{\partial a}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}+\frac{2\,a\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}+\frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{2\,a^2\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}\\
&+\frac{2\,a\,b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{a^2\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{a\,\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{a^2\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{a\,\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{2\,b\,c\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi}\\
&+\frac{4\,a\,b\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi}-\frac{6\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\psi}-\frac{2\,b\,c\,(\frac{\partial\psi}{\partial\eta})^2}{\delta\,\psi^2}+\frac{a\,b\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi}-\frac{\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{d\,\psi}-\frac{b\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi}\\
&+\frac{2\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}-\frac{3\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{a\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{2\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{2\,a\,b\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{2\,a\,b^2\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}\\
&+\frac{a\,b\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{\frac{\partial a}{\partial\eta}\,b^2\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{a^2\,b\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{a\,\frac{\partial a}{\partial\theta}\,b^2\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{\frac{\partial d}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{4\,d^2}-\frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d}\\
&+\frac{a\,\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d}-\frac{\frac{\partial^2 d}{\partial\eta\partial\theta}}{2\,d}-\frac{\frac{\partial b}{\partial\eta}\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d}+\frac{\frac{\partial a}{\partial\theta}\,b\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d}-\frac{\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta}+\frac{\frac{\partial a}{\partial\theta}\,\frac{\partial c}{\partial\theta}}{2\,\delta}\\
&+\frac{c\,\frac{\partial^2 c}{\partial\eta\partial\theta}}{\delta}+\frac{\frac{\partial b}{\partial\eta}\,\frac{\partial c}{\partial\eta}}{2\,\delta}-\frac{\frac{\partial^2 b}{\partial\eta^2}\,c}{2\,\delta}-\frac{\frac{\partial^2 a}{\partial\theta^2}\,c}{2\,\delta}-\frac{\frac{\partial a}{\partial\eta}\,\frac{\partial b}{\partial\theta}}{4\,\delta}+\frac{\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\eta}}{4\,\delta}\\
&\frac{a\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta^2-\frac{\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial c}{\partial\theta}}{2\,\delta^2}-\frac{a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial c}{\partial\theta}}{2\,\delta^2}-\frac{a\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial c}{\partial\eta}}{2\,\delta^2}-\frac{a\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial c}{\partial\eta}}{2\,\delta^2}+\frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\theta}\,c}{4\,\delta^2}}\\
&+\frac{a\,(\frac{\partial b}{\partial\eta})^2\,c}{4\,\delta^2}+\frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial b}{\partial\eta}\,c}{4\,\delta^2}+\frac{(\frac{\partial a}{\partial\theta})^2\,b\,c}{4\,\delta^2}+\frac{a\,\frac{\partial a}{\partial\eta}\,b\,\frac{\partial b}{\partial\theta}}{4\,\delta^2}-\frac{a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial b}{\partial\eta}}{4\,\delta^2},
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} = &-\frac{2\,a\,b\,\frac{\partial\psi}{\partial\theta}\cot\theta}{\delta\,\psi}+\frac{2\,b\,c\,\frac{\partial\psi}{\partial\eta}\cot\theta}{\delta\,\psi}-\frac{\frac{\partial d}{\partial\theta}\cot\theta}{d}-\frac{c\,\frac{\partial c}{\partial\theta}\cot\theta}{\delta}+\frac{\frac{\partial b}{\partial\eta}\,c\cot\theta}{2\,\delta}+\frac{a\,\frac{\partial b}{\partial\theta}\cot\theta}{2\,\delta}\\
&-\frac{2\,a\,b\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi}-\frac{2\,\frac{\partial^2\psi}{\partial\theta^2}}{\psi}-\frac{2\,a\,b\,(\frac{\partial\psi}{\partial\theta})^2}{\delta\,\psi^2}+\frac{6\,(\frac{\partial\psi}{\partial\theta})^2}{\psi^2}+\frac{4\,b\,c\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^2}-\frac{a\,b\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi}\\
&+\frac{b\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi}-\frac{2\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}+\frac{\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}+\frac{a\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{2\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{2\,a\,b\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}\\
&+\frac{2\,a\,b^2\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{a\,b\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{\frac{\partial a}{\partial\eta}\,b^2\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{a^2\,b\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{a\,\frac{\partial a}{\partial\theta}\,b^2\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{2\,b^2\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi}\\
&+\frac{4\,b\,c\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi}-\frac{2\,b^2\,(\frac{\partial\psi}{\partial\eta})^2}{\delta\,\psi^2}+\frac{b\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi}-\frac{b^2\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi}+\frac{2\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}-\frac{\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}\\
&-\frac{3\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{2\,a\,b^2\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{2\,b^2\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{a\,b\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{\frac{\partial a}{\partial\theta}\,b^2\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{a\,b^2\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}\\
&+\frac{\frac{\partial a}{\partial\eta}\,b^3\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{\frac{\partial^2 d}{\partial\theta^2}}{2\,d}+\frac{(\frac{\partial d}{\partial\theta})^2}{4\,d^2}-\frac{c\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{2\,\delta\,d}+\frac{\frac{\partial b}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d}+\frac{a\,\frac{\partial b}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d}\\
&+\frac{b\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\eta}}{2\,\delta\,d}-\frac{\frac{\partial b}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d}-\frac{b\,\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d}+\frac{b\,\frac{\partial^2 c}{\partial\eta\partial\theta}}{\delta}-\frac{b\,\frac{\partial^2 b}{\partial\eta^2}}{2\,\delta}-\frac{\frac{\partial^2 a}{\partial\theta^2}\,b}{2\,\delta}\\
&+\frac{b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta^2}-\frac{\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial c}{\partial\theta}}{2\,\delta^2}-\frac{\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial c}{\partial\theta}}{2\,\delta^2}-\frac{b\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial c}{\partial\eta}}{2\,\delta^2}-\frac{a\,b\,\frac{\partial b}{\partial\theta}\,\frac{\partial c}{\partial\eta}}{2\,\delta^2}\\
&+\frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial b}{\partial\theta}\,c}{4\,\delta^2}\\
&-\frac{\frac{\partial a}{\partial\theta}\,b\,\frac{\partial b}{\partial\eta}\,c}{4\,\delta^2}+\frac{a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial b}{\partial\theta}}{4\,\delta^2}+\frac{a\,b\,(\frac{\partial b}{\partial\eta})^2}{4\,\delta^2}+\frac{\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial b}{\partial\eta}}{4\,\delta^2}+\frac{(\frac{\partial a}{\partial\theta})^2\,b^2}{4\,\delta^2}+1,
\end{aligned}
$$

$$
\begin{aligned}
\frac{R_{\phi\phi}}{\sin^2\theta} = & -\frac{4\,a\,d\,\frac{\partial\psi}{\partial\theta}\,\cot\theta}{\delta\,\psi} + \frac{4\,c\,d\,\frac{\partial\psi}{\partial\eta}\,\cot\theta}{\delta\,\psi} - \frac{a\,\frac{\partial d}{\partial\theta}\,\cot\theta}{\delta} + \frac{c\,\frac{\partial d}{\partial\eta}\,\cot\theta}{\delta} - \frac{\frac{\partial a}{\partial\theta}\,d\,\cot\theta}{\delta} - \frac{a\,c\,\frac{\partial c}{\partial\theta}\,d\,\cot\theta}{\delta^2} \\
& + \frac{a\,b\,\frac{\partial c}{\partial\eta}\,d\,\cot\theta}{\delta^2} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,d\,\cot\theta}{2\,\delta^2} - \frac{\frac{\partial a}{\partial\eta}\,b\,c\,d\,\cot\theta}{2\,\delta^2} + \frac{a^2\,\frac{\partial b}{\partial\theta}\,d\,\cot\theta}{2\,\delta^2} + \frac{a\,\frac{\partial a}{\partial\theta}\,b\,d\,\cot\theta}{2\,\delta^2} - \frac{2\,a\,d\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi} \\
& - \frac{2\,a\,d\,(\frac{\partial\psi}{\partial\theta})^2}{\delta\,\psi^2} + \frac{4\,c\,d\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^2} - \frac{2\,a\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} + \frac{2\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{2\,\frac{\partial a}{\partial\theta}\,d\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{2\,a\,c\,\frac{\partial c}{\partial\theta}\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} \\
& + \frac{2\,a\,b\,\frac{\partial c}{\partial\eta}\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{\frac{\partial a}{\partial\eta}\,b\,c\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{a^2\,\frac{\partial b}{\partial\theta}\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{a\,\frac{\partial a}{\partial\theta}\,b\,d\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{2\,b\,d\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi} \\
& + \frac{4\,c\,d\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi} - \frac{2\,b\,d\,(\frac{\partial\psi}{\partial\eta})^2}{\delta\,\psi^2} + \frac{2\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{2\,b\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{2\,\frac{\partial b}{\partial\eta}\,d\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{2\,a\,b\,\frac{\partial c}{\partial\theta}\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} \\
& - \frac{2\,b\,c\,\frac{\partial c}{\partial\eta}\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial b}{\partial\theta}\,c\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{\frac{\partial a}{\partial\theta}\,b\,c\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{a\,b\,\frac{\partial b}{\partial\eta}\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{\frac{\partial a}{\partial\eta}\,b^2\,d\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial^2 d}{\partial\theta^2}}{2\,\delta} \\
& + \frac{a\,(\frac{\partial d}{\partial\theta})^2}{4\,\delta\,d} - \frac{c\,\frac{\partial d}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{2\,\delta\,d} - \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{2\,\delta} - \frac{a\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{2\,\delta^2} + \frac{a\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{2\,\delta^2} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta^2} \\
& - \frac{\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta^2} + \frac{a^2\,\frac{\partial b}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{4\,\delta^2} + \frac{a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial d}{\partial\theta}}{4\,\delta^2} - \frac{b\,\frac{\partial^2 d}{\partial\eta^2}}{2\,\delta} + \frac{c\,\frac{\partial^2 d}{\partial\eta\partial\theta}}{\delta} + \frac{b\,(\frac{\partial d}{\partial\eta})^2}{4\,\delta\,d} \\
& - \frac{\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{2\,\delta} + \frac{a\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\eta}}{2\,\delta^2} - \frac{b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{2\,\delta^2} - \frac{a\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta^2} - \frac{\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta^2} + \frac{a\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{4\,\delta^2} + \frac{\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial d}{\partial\eta}}{4\,\delta^2} + \frac{a\,d}{\delta}.
\end{aligned}
$$

The Hamiltonian constraint written explicitly in terms of the extrinsic and intrinsic curvature components is

$$
0 = \frac{R_{\phi\phi}}{d\psi^4\sin^2\theta} - \frac{2R_{\eta\theta}\,c}{\delta\,\psi^4} + \frac{R_{\eta\eta}\,a}{\delta\,\psi^4} + \frac{R_{\theta\theta}\,b}{\delta\,\psi^4} - \frac{2\,H_c^2}{\delta} - \frac{4\,c\,H_d\,H_c}{\delta\,d} + \frac{2\,H_a\,H_b}{\delta} + \frac{2\,a\,H_d\,H_b}{\delta\,d} + \frac{2\,b\,H_d\,H_a}{\delta\,d}.
$$

Explicitly evaluating the curvature scalar R of the 3D hypersurfaces yields

$$
\begin{aligned}
0 = & -\frac{8\,a\,\frac{\partial\psi}{\partial\theta}\,\cot\theta}{\delta\,\psi^5} + \frac{8\,c\,\frac{\partial\psi}{\partial\eta}\,\cot\theta}{\delta\,\psi^5} - \frac{2\,a\,\frac{\partial d}{\partial\theta}\,\cot\theta}{\delta\,d\,\psi^4} + \frac{2\,c\,\frac{\partial d}{\partial\eta}\,\cot\theta}{\delta\,d\,\psi^4} - \frac{2\,\frac{\partial a}{\partial\theta}\,\cot\theta}{\delta\,\psi^4} \\
& - \frac{2\,a\,c\,\frac{\partial c}{\partial\theta}\,\cot\theta}{\delta^2\,\psi^4} + \frac{2\,a\,b\,\frac{\partial c}{\partial\eta}\,\cot\theta}{\delta^2\,\psi^4} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,\cot\theta}{\delta^2\,\psi^4} - \frac{\frac{\partial a}{\partial\eta}\,b\,c\,\cot\theta}{\delta^2\,\psi^4} + \frac{a^2\,\frac{\partial b}{\partial\theta}\,\cot\theta}{\delta^2\,\psi^4} + \frac{a\,\frac{\partial a}{\partial\theta}\,b\,\cot\theta}{\delta^2\,\psi^4} \\
& - \frac{8\,a\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi^5} - \frac{4\,a\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi^5} + \frac{4\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi^5} - \frac{8\,\frac{\partial a}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} - \frac{8\,a\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} + \frac{8\,a\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} \\
& - \frac{4\,a\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} - \frac{4\,\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} + \frac{4\,a^2\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} + \frac{4\,a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi^5} - \frac{8\,b\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi^5} + \frac{16\,c\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi^5} \\
& + \frac{4\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi^5} - \frac{4\,b\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi^5} - \frac{8\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} + \frac{8\,a\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} - \frac{8\,b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} - \frac{4\,a\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} \\
& - \frac{4\,\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} + \frac{4\,a\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} + \frac{4\,\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi^5} - \frac{a\,\frac{\partial^2 d}{\partial\theta^2}}{\delta\,d\,\psi^4} + \frac{a\,(\frac{\partial d}{\partial\theta})^2}{2\,\delta\,d^2\,\psi^4} - \frac{c\,\frac{\partial d}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{\delta\,d^2\,\psi^4} \\
& - \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{\delta\,d\,\psi^4} - \frac{a\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{\delta^2\,d\,\psi^4} + \frac{a\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{\delta^2\,d\,\psi^4} - \frac{a\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{2\,\delta^2\,d\,\psi^4} - \frac{\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial d}{\partial\theta}}{2\,\delta^2\,d\,\psi^4} + \frac{a^2\,\frac{\partial b}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{2\,\delta^2\,d\,\psi^4} \\
& + \frac{a\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial d}{\partial\theta}}{2\,\delta^2\,d\,\psi^4} - \frac{b\,\frac{\partial^2 d}{\partial\eta^2}}{\delta\,d\,\psi^4} + \frac{2\,c\,\frac{\partial^2 d}{\partial\eta\partial\theta}}{\delta\,d\,\psi^4} + \frac{b\,(\frac{\partial d}{\partial\eta})^2}{2\,\delta\,d^2\,\psi^4} - \frac{\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{\delta\,d\,\psi^4} + \frac{a\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial d}{\partial\eta}}{\delta^2\,d\,\psi^4} \\
& - \frac{b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{\delta^2\,d\,\psi^4} - \frac{a\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{2\,\delta^2\,d\,\psi^4} - \frac{\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial d}{\partial\eta}}{2\,\delta^2\,d\,\psi^4} + \frac{a\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{2\,\delta^2\,d\,\psi^4} + \frac{\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial d}{\partial\eta}}{2\,\delta^2\,d\,\psi^4} + \frac{2\,\frac{\partial^2 c}{\partial\eta\partial\theta}}{\delta\,\psi^4} \\
& - \frac{\frac{\partial^2 b}{\partial\eta^2}}{\delta\,\psi^4} - \frac{\frac{\partial^2 a}{\partial\theta^2}}{\delta\,\psi^4} + \frac{2\,a}{\delta\,\psi^4} + \frac{2\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta^2\,\psi^4} - \frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial c}{\partial\theta}}{\delta^2\,\psi^4} - \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial c}{\partial\theta}}{\delta^2\,\psi^4} \\
& - \frac{\frac{\partial b}{\partial\eta}\,c\,\frac{\partial c}{\partial\eta}}{\delta^2\,\psi^4} - \frac{a\,\frac{\partial b}{\partial\theta}\,\frac{\partial c}{\partial\eta}}{\delta^2\,\psi^4} + \frac{\frac{\partial a}{\partial\eta}\,\frac{\partial b}{\partial\theta}\,c}{2\,\delta^2\,\psi^4} - \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\eta}\,c}{2\,\delta^2\,\psi^4} + \frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial b}{\partial\theta}}{2\,\delta^2\,\psi^4} + \frac{a\,(\frac{\partial b}{\partial\eta})^2}{2\,\delta^2\,\psi^4} \\
& + \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial b}{\partial\eta}}{2\,\delta^2\,\psi^4} + \frac{(\frac{\partial a}{\partial\theta})^2\,b}{2\,\delta^2\,\psi^4} - \frac{2\,H_c^2}{\delta} - \frac{4\,c\,H_d\,H_c}{\delta\,d} + \frac{2\,H_a\,H_b}{\delta} + \frac{2\,a\,H_d\,H_b}{\delta\,d} + \frac{2\,b\,H_d\,H_a}{\delta\,d}
\end{aligned}
$$

and the two components of the momentum constraint become, for $H_1 = 0$,

$$
\begin{aligned}
0 = &+\frac{a\,H_c \cot\theta}{\delta} - \frac{c\,H_a \cot\theta}{\delta} + \frac{4\,c\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} + \frac{14\,a\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{8\,a\,b\,c\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} \\
&- \frac{8\,a^2\,b\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{4\,a\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} + \frac{2\,a^2\,c\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{4\,a^2\,b\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{10\,c\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{4\,b\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} \\
&+ \frac{6\,a\,b\,c\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{4\,a\,b^2\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{2\,c\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{8\,b\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{8\,a\,b^2\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{4\,c\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} \\
&- \frac{4\,a\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{4\,a\,b\,c\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{2\,a^2\,b\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{6\,b\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{4\,b^2\,c\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{2\,a\,b^2\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} \\
&- \frac{2\,H_d\,\frac{\partial\psi}{\partial\eta}}{d\,\psi} - \frac{c\,\frac{\partial H_c}{\partial\theta}}{\delta} - \frac{a\,\frac{\partial H_c}{\partial\theta}}{\delta} + \frac{2\,a\,b\,c\,\frac{\partial H_c}{\partial\theta}}{\delta^2} + \frac{2\,a^2\,b\,\frac{\partial H_c}{\partial\theta}}{\delta^2} + \frac{2\,c\,\frac{\partial H_c}{\partial\eta}}{\delta} \\
&+ \frac{2\,b\,\frac{\partial H_c}{\partial\eta}}{\delta} - \frac{2\,a\,b\,c\,\frac{\partial H_c}{\partial\eta}}{\delta^2} - \frac{2\,a\,b^2\,\frac{\partial H_c}{\partial\eta}}{\delta^2} + \frac{a\,\frac{\partial d}{\partial\theta}\,H_c}{2\,\delta\,d} - \frac{c\,\frac{\partial d}{\partial\eta}\,H_c}{2\,\delta\,d} + \frac{2\,\frac{\partial c}{\partial\theta}\,H_c}{\delta} \\
&- \frac{2\,\frac{\partial c}{\partial\eta}\,H_c}{\delta} + \frac{2\,\frac{\partial b}{\partial\eta}\,H_c}{\delta} - \frac{\frac{\partial a}{\partial\theta}\,H_c}{\delta} - \frac{5\,a\,c\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^2} - \frac{10\,a\,b\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^2} + \frac{6\,b\,c\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^2} \\
&+ \frac{3\,a\,\frac{\partial b}{\partial\theta}\,c\,H_c}{\delta^2} - \frac{2\,a\,\frac{\partial b}{\partial\eta}\,c\,H_c}{\delta^2} + \frac{3\,\frac{\partial a}{\partial\theta}\,b\,c\,H_c}{\delta^2} - \frac{2\,\frac{\partial a}{\partial\eta}\,b\,c\,H_c}{\delta^2} + \frac{7\,a^2\,\frac{\partial b}{\partial\theta}\,H_c}{2\,\delta^2} \\
&- \frac{6\,a\,b\,\frac{\partial b}{\partial\eta}\,H_c}{\delta^2} - \frac{4\,\frac{\partial a}{\partial\eta}\,b^2\,H_c}{\delta^2} + \frac{11\,a\,\frac{\partial a}{\partial\theta}\,b\,H_c}{2\,\delta^2} + \frac{8\,a^2\,b\,c\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^3} + \frac{8\,a^2\,b^2\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^3} - \frac{8\,a\,b^2\,c\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^3} \\
&- \frac{4\,a^2\,b^2\,\frac{\partial c}{\partial\eta}\,hc}{\delta^3} - \frac{4\,a^2\,b\,\frac{\partial b}{\partial\theta}\,c\,H_c}{\delta^3} + \frac{2\,a^2\,b\,\frac{\partial b}{\partial\eta}\,c\,H_c}{\delta^3} - \frac{4\,a\,\frac{\partial a}{\partial\theta}\,b^2\,c\,H_c}{\delta^3} + \frac{2\,a\,\frac{\partial a}{\partial\eta}\,b^2\,c\,H_c}{\delta^3} - \frac{4\,a^3\,b\,\frac{\partial b}{\partial\theta}\,H_c}{\delta^3} \\
&+ \frac{4\,a^2\,b^2\,\frac{\partial b}{\partial\eta}\,H_c}{\delta^3} + \frac{4\,a\,\frac{\partial a}{\partial\eta}\,b^3\,H_c}{\delta^3} - \frac{4\,a^2\,\frac{\partial a}{\partial\theta}\,b^2\,H_c}{\delta^3} + \frac{a\,\frac{\partial H_b}{\partial\theta}}{\delta} - \frac{a^2\,c\,\frac{\partial H_b}{\partial\theta}}{\delta^2} - \frac{a^2\,b\,\frac{\partial H_b}{\partial\theta}}{\delta^2} \\
&- \frac{c\,\frac{\partial H_b}{\partial\eta}}{\delta} - \frac{2\,a\,\frac{\partial H_b}{\partial\eta}}{\delta} + \frac{a\,b\,c\,\frac{\partial H_b}{\partial\eta}}{\delta^2} + \frac{a^2\,b\,\frac{\partial H_b}{\partial\eta}}{\delta^2} + \frac{2\,\frac{\partial c}{\partial\eta}\,H_b}{\delta} + \frac{\frac{\partial a}{\partial\theta}\,H_b}{\delta} \\
&- \frac{\frac{\partial a}{\partial\eta}\,H_b}{\delta} + \frac{3\,a\,c\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^2} + \frac{4\,a^2\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^2} - \frac{2\,a\,c\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^2} - \frac{6\,a\,b\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^2} + \frac{2\,a\,\frac{\partial b}{\partial\eta}\,c\,H_b}{\delta^2} \\
&+ \frac{2\,\frac{\partial a}{\partial\eta}\,b\,c\,H_b}{\delta^2} - \frac{5\,a\,\frac{\partial a}{\partial\theta}\,c\,H_b}{2\,\delta^2} - \frac{2\,a^2\,\frac{\partial b}{\partial\theta}\,H_b}{\delta^2} + \frac{3\,a^2\,\frac{\partial b}{\partial\eta}\,H_b}{2\,\delta^2} - \frac{3\,a\,\frac{\partial a}{\partial\theta}\,b\,H_b}{\delta^2} + \frac{2\,a\,\frac{\partial a}{\partial\eta}\,b\,H_b}{\delta^2} \\
&- \frac{4\,a^2\,b\,c\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^3} - \frac{4\,a^3\,b\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^3} + \frac{2\,a^2\,b\,c\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^3} + \frac{4\,a^2\,b^2\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^3} + \frac{2\,a^3\,\frac{\partial b}{\partial\theta}\,c\,H_b}{\delta^3} - \frac{2\,a^2\,b\,\frac{\partial b}{\partial\eta}\,c\,H_b}{\delta^3} \\
&- \frac{2\,a\,\frac{\partial a}{\partial\eta}\,b^2\,c\,H_b}{\delta^3} + \frac{2\,a^2\,\frac{\partial a}{\partial\theta}\,b\,c\,H_b}{\delta^3} + \frac{2\,a^3\,b\,\frac{\partial b}{\partial\theta}\,H_b}{\delta^3} - \frac{a^3\,b\,\frac{\partial b}{\partial\eta}\,H_b}{\delta^3} + \frac{2\,a^2\,\frac{\partial a}{\partial\theta}\,b^2\,H_b}{\delta^3} - \frac{a^2\,\frac{\partial a}{\partial\eta}\,b^2\,H_b}{\delta^3} \\
&+ \frac{b\,\frac{\partial H_a}{\partial\theta}}{\delta} - \frac{a\,b\,c\,\frac{\partial H_a}{\partial\theta}}{\delta^2} - \frac{a\,b^2\,\frac{\partial H_a}{\partial\theta}}{\delta^2} - \frac{b\,\frac{\partial H_a}{\partial\eta}}{\delta} + \frac{b^2\,c\,\frac{\partial H_a}{\partial\eta}}{\delta^2} + \frac{a\,b^2\,\frac{\partial H_a}{\partial\eta}}{\delta^2} \\
&- \frac{c\,\frac{\partial d}{\partial\theta}\,H_a}{2\,\delta\,d} + \frac{b\,\frac{\partial d}{\partial\eta}\,H_a}{2\,\delta\,d} - \frac{2\,\frac{\partial c}{\partial\theta}\,H_a}{\delta} + \frac{\frac{\partial b}{\partial\theta}\,H_a}{\delta} + \frac{3\,b\,c\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^2} + \frac{5\,a\,b\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^2} \\
&- \frac{2\,b\,c\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^2} - \frac{4\,b^2\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^2} - \frac{3\,a\,\frac{\partial b}{\partial\theta}\,c\,H_a}{2\,\delta^2} + \frac{2\,b\,\frac{\partial b}{\partial\eta}\,c\,H_a}{\delta^2} - \frac{\frac{\partial a}{\partial\theta}\,b\,c\,H_a}{\delta^2} - \frac{3\,a\,b\,\frac{\partial b}{\partial\theta}\,H_a}{\delta^2} \\
&+ \frac{3\,a\,b\,\frac{\partial b}{\partial\eta}\,H_a}{2\,\delta^2} - \frac{2\,\frac{\partial a}{\partial\theta}\,b^2\,H_a}{\delta^2} + \frac{\frac{\partial a}{\partial\eta}\,b^2\,H_a}{\delta^2} - \frac{4\,a\,b^2\,c\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^3} - \frac{4\,a^2\,b^2\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^3} + \frac{2\,a\,b^2\,c\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^3} \\
&+ \frac{4\,a\,b^3\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^3} + \frac{2\,a^2\,b\,\frac{\partial b}{\partial\theta}\,c\,H_a}{\delta^3} - \frac{2\,a\,b^2\,\frac{\partial b}{\partial\eta}\,c\,H_a}{\delta^3} - \frac{2\,\frac{\partial a}{\partial\eta}\,b^3\,c\,H_a}{\delta^3} + \frac{2\,a\,\frac{\partial a}{\partial\theta}\,b^2\,c\,H_a}{\delta^3} + \frac{2\,a^2\,b^2\,\frac{\partial b}{\partial\theta}\,H_a}{\delta^3} \\
&- \frac{a^2\,b^2\,\frac{\partial b}{\partial\eta}\,H_a}{\delta^3} + \frac{2\,a\,\frac{\partial a}{\partial\theta}\,b^3\,H_a}{\delta^3} - \frac{a\,\frac{\partial a}{\partial\eta}\,b^3\,H_a}{\delta^3} - \frac{\frac{\partial H_d}{\partial\eta}}{d} + \frac{\frac{\partial d}{\partial\eta}\,H_d}{2\,d^2}
\end{aligned}
$$

and, for $H_2 = 0$,

$$
\begin{aligned}
0 = & -\frac{c\,H_c\,\cot\theta}{\delta}+\frac{a\,H_b\,\cot\theta}{\delta}-\frac{H_d\,\cot\theta}{d}-\frac{6\,c\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{8\,a\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}\\
&+\frac{8\,a\,b\,c\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{8\,a^2\,b\,H_c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{8\,a\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{4\,a^2\,c\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{6\,a^2\,b\,H_b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}+\frac{4\,c\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}\\
&+\frac{2\,b\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi}-\frac{4\,a\,b\,c\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{2\,a\,b^2\,H_a\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi}-\frac{2\,H_d\,\frac{\partial\psi}{\partial\theta}}{d\,\psi}+\frac{4\,c\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{18\,b\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}\\
&-\frac{8\,a\,b\,c\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{16\,a\,b^2\,H_c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}-\frac{10\,c\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}-\frac{4\,a\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{10\,a\,b\,c\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{4\,a^2\,b\,H_b\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}\\
&-\frac{4\,b\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi}+\frac{6\,b^2\,c\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{4\,a\,b^2\,H_a\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi}+\frac{2\,c\,\frac{\partial H_c}{\partial\theta}}{\delta}+\frac{2\,a\,\frac{\partial H_c}{\partial\theta}}{\delta}-\frac{2\,a\,b\,c\,\frac{\partial H_c}{\partial\theta}}{\delta^2}\\
&-\frac{2\,a^2\,b\,\frac{\partial H_c}{\partial\theta}}{\delta^2}-\frac{c\,\frac{\partial H_c}{\partial\eta}}{\delta}-\frac{b\,\frac{\partial H_c}{\partial\eta}}{\delta}+\frac{2\,a\,b\,c\,\frac{\partial H_c}{\partial\eta}}{\delta^2}+\frac{2\,a\,b^2\,\frac{\partial H_c}{\partial\eta}}{\delta^2}-\frac{c\,\frac{\partial d}{\partial\theta}\,H_c}{2\,\delta\,d}\\
&+\frac{b\,\frac{\partial d}{\partial\eta}\,H_c}{2\,\delta\,d}-\frac{3\,\frac{\partial c}{\partial\theta}\,H_c}{\delta}+\frac{2\,\frac{\partial c}{\partial\eta}\,H_c}{\delta}-\frac{\frac{\partial b}{\partial\eta}\,H_c}{\delta}+\frac{2\,\frac{\partial a}{\partial\theta}\,H_c}{\delta}+\frac{6\,a\,c\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^2}\\
&+\frac{11\,a\,b\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^2}-\frac{6\,b\,c\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^2}-\frac{10\,a\,b\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^2}-\frac{7\,a\,\frac{\partial b}{\partial\theta}\,c\,H_c}{2\,\delta^2}+\frac{3\,a\,\frac{\partial b}{\partial\eta}\,c\,H_c}{\delta^2}-\frac{7\,\frac{\partial a}{\partial\theta}\,b\,c\,H_c}{2\,\delta^2}\\
&+\frac{3\,\frac{\partial a}{\partial\eta}\,b\,c\,H_c}{\delta^2}-\frac{4\,a^2\,\frac{\partial b}{\partial\theta}\,H_c}{\delta^2}+\frac{7\,a\,b\,\frac{\partial b}{\partial\eta}\,H_c}{\delta^2}+\frac{5\,\frac{\partial a}{\partial\eta}\,b^2\,H_c}{\delta^2}-\frac{6\,a\,\frac{\partial a}{\partial\theta}\,b\,H_c}{\delta^2}-\frac{8\,a^2\,b\,c\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^3}\\
&-\frac{8\,a^2\,b^2\,\frac{\partial c}{\partial\theta}\,H_c}{\delta^3}+\frac{12\,a\,b^2\,c\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^3}+\frac{8\,a^2\,b^2\,\frac{\partial c}{\partial\eta}\,H_c}{\delta^3}+\frac{4\,a^2\,b\,\frac{\partial b}{\partial\theta}\,c\,H_c}{\delta^3}-\frac{4\,a^2\,b\,\frac{\partial b}{\partial\eta}\,c\,H_c}{\delta^3}+\frac{4\,a\,\frac{\partial a}{\partial\theta}\,b^2\,c\,H_c}{\delta^3}\\
&-\frac{4\,a\,\frac{\partial a}{\partial\eta}\,b^2\,c\,H_c}{\delta^3}+\frac{4\,a^3\,b\,\frac{\partial b}{\partial\theta}\,H_c}{\delta^3}-\frac{6\,a^2\,b^2\,\frac{\partial b}{\partial\eta}\,hc}{\delta^3}-\frac{6\,a\,\frac{\partial a}{\partial\eta}\,b^3\,H_c}{\delta^3}+\frac{4\,a^2\,\frac{\partial a}{\partial\theta}\,b^2\,H_c}{\delta^3}-\frac{a\,\frac{\partial H_b}{\partial\theta}}{\delta}\\
&+\frac{a^2\,c\,\frac{\partial H_b}{\partial\theta}}{\delta^2}+\frac{a^2\,b\,\frac{\partial H_b}{\partial\theta}}{\delta^2}+\frac{a\,\frac{\partial H_b}{\partial\eta}}{\delta}-\frac{a\,b\,c\,\frac{\partial H_b}{\partial\eta}}{\delta^2}-\frac{a^2\,b\,\frac{\partial H_b}{\partial\eta}}{\delta^2}+\frac{a\,\frac{\partial d}{\partial\theta}\,H_b}{2\,\delta\,d}\\
&-\frac{c\,\frac{\partial d}{\partial\eta}\,H_b}{2\,\delta\,d}-\frac{2\,\frac{\partial c}{\partial\eta}\,H_b}{\delta}-\frac{\frac{\partial a}{\partial\theta}\,H_b}{2\,\delta}+\frac{\frac{\partial a}{\partial\eta}\,H_b}{\delta}-\frac{3\,a\,c\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^2}-\frac{4\,a^2\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^2}\\
&+\frac{3\,a\,c\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^2}+\frac{7\,a\,b\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^2}-\frac{3\,a\,\frac{\partial b}{\partial\eta}\,c\,H_b}{2\,\delta^2}-\frac{2\,\frac{\partial a}{\partial\eta}\,b\,c\,H_b}{\delta^2}+\frac{2\,a\,\frac{\partial a}{\partial\theta}\,c\,H_b}{\delta^2}+\frac{2\,a^2\,\frac{\partial b}{\partial\theta}\,H_b}{\delta^2}\\
&-\frac{2\,a^2\,\frac{\partial b}{\partial\eta}\,H_b}{\delta^2}+\frac{5\,a\,\frac{\partial a}{\partial\theta}\,b\,H_b}{2\,\delta^2}-\frac{3\,a\,\frac{\partial a}{\partial\eta}\,b\,H_b}{\delta^2}+\frac{4\,a^2\,b\,c\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^3}+\frac{4\,a^3\,b\,\frac{\partial c}{\partial\theta}\,H_b}{\delta^3}-\frac{4\,a^2\,b\,c\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^3}\\
&-\frac{6\,a^2\,b^2\,\frac{\partial c}{\partial\eta}\,H_b}{\delta^3}-\frac{2\,a^3\,\frac{\partial b}{\partial\theta}\,c\,H_b}{\delta^3}+\frac{3\,a^2\,b\,\frac{\partial b}{\partial\eta}\,c\,H_b}{\delta^3}+\frac{3\,a\,\frac{\partial a}{\partial\eta}\,b^2\,c\,H_b}{\delta^3}-\frac{2\,a^2\,\frac{\partial a}{\partial\theta}\,b\,c\,H_b}{\delta^3}-\frac{2\,a^3\,b\,\frac{\partial b}{\partial\theta}\,H_b}{\delta^3}\\
&+\frac{2\,a^3\,b\,\frac{\partial b}{\partial\eta}\,H_b}{\delta^3}-\frac{2\,a^2\,\frac{\partial a}{\partial\theta}\,b^2\,H_b}{\delta^3}+\frac{2\,a^2\,\frac{\partial a}{\partial\eta}\,b^2\,H_b}{\delta^3}-\frac{c\,\frac{\partial H_a}{\partial\theta}}{\delta}-\frac{2\,b\,\frac{\partial H_a}{\partial\theta}}{\delta}+\frac{a\,b\,c\,\frac{\partial H_a}{\partial\theta}}{\delta^2}\\
&+\frac{a\,b^2\,\frac{\partial H_a}{\partial\theta}}{\delta^2}+\frac{b\,\frac{\partial H_a}{\partial\eta}}{\delta}-\frac{b^2\,c\,\frac{\partial H_a}{\partial\eta}}{\delta^2}-\frac{a\,b^2\,\frac{\partial H_a}{\partial\eta}}{\delta^2}+\frac{2\,\frac{\partial c}{\partial\theta}\,H_a}{\delta}-\frac{3\,\frac{\partial b}{\partial\theta}\,H_a}{2\,\delta}\\
&+\frac{\frac{\partial b}{\partial\eta}\,H_a}{\delta}-\frac{4\,b\,c\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^2}-\frac{6\,a\,b\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^2}+\frac{3\,b\,c\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^2}+\frac{5\,b^2\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^2}+\frac{2\,a\,\frac{\partial b}{\partial\theta}\,c\,H_a}{\delta^2}\\
&-\frac{5\,b\,\frac{\partial b}{\partial\eta}\,c\,H_a}{2\,\delta^2}+\frac{2\,\frac{\partial a}{\partial\theta}\,b\,c\,H_a}{\delta^2}+\frac{7\,a\,b\,\frac{\partial b}{\partial\theta}\,H_a}{2\,\delta^2}-\frac{3\,a\,b\,\frac{\partial b}{\partial\eta}\,H_a}{\delta^2}+\frac{3\,\frac{\partial a}{\partial\theta}\,b^2\,H_a}{\delta^2}-\frac{2\,\frac{\partial a}{\partial\eta}\,b^2\,H_a}{\delta^2}\\
&+\frac{4\,a\,b^2\,c\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^3}+\frac{4\,a^2\,b^2\,\frac{\partial c}{\partial\theta}\,H_a}{\delta^3}-\frac{4\,a\,b^2\,c\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^3}-\frac{6\,a\,b^3\,\frac{\partial c}{\partial\eta}\,H_a}{\delta^3}-\frac{2\,a^2\,b\,\frac{\partial b}{\partial\theta}\,c\,H_a}{\delta^3}+\frac{3\,a\,b^2\,\frac{\partial b}{\partial\eta}\,c\,H_a}{\delta^3}\\
&+\frac{3\,\frac{\partial a}{\partial\eta}\,b^3\,c\,H_a}{\delta^3}-\frac{2\,a\,\frac{\partial a}{\partial\theta}\,b^2\,c\,H_a}{\delta^3}-\frac{2\,a^2\,b^2\,\frac{\partial b}{\partial\theta}\,H_a}{\delta^3}+\frac{2\,a^2\,b^2\,\frac{\partial b}{\partial\eta}\,H_a}{\delta^3}-\frac{2\,a\,\frac{\partial a}{\partial\theta}\,b^3\,H_a}{\delta^3}+\frac{2\,a\,\frac{\partial a}{\partial\eta}\,b^3\,H_a}{\delta^3}\\
&-\frac{\frac{\partial H_d}{\partial\theta}}{d}+\frac{\frac{\partial d}{\partial\theta}\,H_d}{2\,d^2}.
\end{aligned}
$$

The evolution equations for the conformal three-metric components are given as follows. The metric evolution for $\hat{\gamma}_{11}=a$:

$$
\frac{\partial a}{\partial t}=-2\,\alpha\,H_a+\frac{4\,a\,\beta^\theta}{\psi}\,\frac{\partial\psi}{\partial\theta}+\frac{4\,a\,\beta^\eta}{\psi}\,\frac{\partial\psi}{\partial\eta}+2\,\frac{\partial\beta^\theta}{\partial\eta}\,c+\frac{\partial a}{\partial\theta}\,\beta^\theta+2\,a\,\frac{\partial\beta^\eta}{\partial\eta}+\frac{\partial a}{\partial\eta}\,\beta^\eta.
$$

The metric evolution for $\hat{\gamma}_{22} = b$:

$$\frac{\partial b}{\partial t} = -2\,\alpha\,H_b + \frac{4\,b\,\beta^\theta}{\psi}\frac{\partial\psi}{\partial\theta} + \frac{4\,b\,\beta^\eta}{\psi}\frac{\partial\psi}{\partial\eta} + 2\,\frac{\partial\beta^\eta}{\partial\theta}\,c + 2\,b\,\frac{\partial\beta^\theta}{\partial\theta} + \frac{\partial b}{\partial\theta}\,\beta^\theta + \frac{\partial b}{\partial\eta}\,\beta^\eta .$$

The metric evolution for $\hat{\gamma}_{12} = c$:

$$\frac{\partial c}{\partial t} = -2\,\alpha\,H_c + \frac{4\,c\,\beta^\theta}{\psi}\frac{\partial\psi}{\partial\theta} + \frac{4\,c\,\beta^\eta}{\psi}\frac{\partial\psi}{\partial\eta} + \beta^\theta\,\frac{\partial c}{\partial\theta} + \beta^\eta\,\frac{\partial c}{\partial\eta} + \frac{\partial\beta^\theta}{\partial\theta}\,c + \frac{\partial\beta^\eta}{\partial\eta}\,c + b\,\frac{\partial\beta^\theta}{\partial\eta} + a\,\frac{\partial\beta^\eta}{\partial\theta} .$$

The metric evolution for $\hat{\gamma}_{33} = d$:

$$\frac{\partial d}{\partial t} = -2\,\alpha\,H_d + 2\,\beta^\theta\,d\,\cot\theta + \frac{4\,d\,\beta^\theta}{\psi}\frac{\partial\psi}{\partial\theta} + \frac{4\,d\,\beta^\eta}{\psi}\frac{\partial\psi}{\partial\eta} + \beta^\theta\,\frac{\partial d}{\partial\theta} + \beta^\eta\,\frac{\partial d}{\partial\eta} .$$

Finally the equations that govern the evolution of the extrinsic curvature components are the metric evolution for $\hat{K}_{11} = H_a$:

$$\begin{aligned}\frac{\partial H_a}{\partial t} = {} & \frac{4\,\beta^\theta\,H_a\,\frac{\partial\psi}{\partial\theta}}{\psi} + \frac{2\,a\,\frac{\partial\alpha}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} - \frac{2\,a^2\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} + \frac{4\,\beta^\eta\,H_a\,\frac{\partial\psi}{\partial\eta}}{\psi} + \frac{2\,a\,\frac{\partial\alpha}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} - \frac{2\,a\,\frac{\partial\alpha}{\partial\eta}\,b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} \\ & + \frac{4\,\frac{\partial\alpha}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\psi^5} - \frac{\frac{\partial\alpha}{\partial\eta}\,c\,\frac{\partial c}{\partial\eta}}{\delta\,\psi^4} + \frac{a\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial c}{\partial\eta}}{\delta\,\psi^4} - \frac{\frac{\partial a}{\partial\eta}\,\frac{\partial\alpha}{\partial\theta}\,c}{2\,\delta\,\psi^4} + \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial\alpha}{\partial\eta}\,c}{2\,\delta\,\psi^4} + \frac{\frac{\partial a}{\partial\eta}\,\frac{\partial\alpha}{\partial\eta}\,b}{2\,\delta\,\psi^4} \\ & - \frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial\alpha}{\partial\theta}}{2\,\delta\,\psi^4} - \frac{\frac{\partial^2\alpha}{\partial\eta^2}}{\psi^4} - \frac{2\,a\,\alpha\,H_c^2}{\delta} + \frac{2\,\alpha\,c\,H_a\,H_c}{\delta} + 2\,\frac{\partial\beta^\theta}{\partial\eta}\,H_c + \frac{a\,\alpha\,H_a\,H_b}{\delta} \\ & + \beta^\theta\,\frac{\partial H_a}{\partial\theta} + \beta^\eta\,\frac{\partial H_a}{\partial\eta} - \frac{\alpha\,b\,H_a^2}{\delta} + \frac{\alpha\,H_d\,H_a}{d} + 2\,\frac{\partial\beta^\eta}{\partial\eta}\,H_a + \frac{R_{\eta\eta}\,\alpha}{\psi^4} .\end{aligned}$$

The metric evolution for $\hat{K}_{22} = H_b$:

$$\begin{aligned}\frac{\partial H_b}{\partial t} = {} & \frac{4\,\beta^\theta\,H_b\,\frac{\partial\psi}{\partial\theta}}{\psi} + \frac{2\,\frac{\partial\alpha}{\partial\eta}\,b\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} - \frac{2\,a\,\frac{\partial\alpha}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} + \frac{4\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\psi^5} + \frac{4\,\beta^\eta\,H_b\,\frac{\partial\psi}{\partial\eta}}{\psi} + \frac{2\,\frac{\partial\alpha}{\partial\theta}\,b\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} \\ & - \frac{2\,\frac{\partial\alpha}{\partial\eta}\,b^2\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} - \frac{\frac{\partial\alpha}{\partial\theta}\,c\,\frac{\partial c}{\partial\theta}}{\delta\,\psi^4} + \frac{\frac{\partial\alpha}{\partial\eta}\,b\,\frac{\partial c}{\partial\theta}}{\delta\,\psi^4} - \frac{\frac{\partial\alpha}{\partial\eta}\,\frac{\partial b}{\partial\theta}\,c}{2\,\delta\,\psi^4} + \frac{\frac{\partial\alpha}{\partial\theta}\,\frac{\partial b}{\partial\eta}\,c}{2\,\delta\,\psi^4} + \frac{a\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial b}{\partial\theta}}{2\,\delta\,\psi^4} \\ & - \frac{\frac{\partial\alpha}{\partial\eta}\,b\,\frac{\partial b}{\partial\eta}}{2\,\delta\,\psi^4} - \frac{\frac{\partial^2\alpha}{\partial\theta^2}}{\psi^4} - \frac{2\,\alpha\,b\,H_c^2}{\delta} + \frac{2\,\alpha\,c\,H_b\,H_c}{\delta} + 2\,\frac{\partial\beta^\eta}{\partial\theta}\,H_c\beta^\theta\,\frac{\partial H_b}{\partial\theta} \\ & + \beta^\eta\,\frac{\partial H_b}{\partial\eta} - \frac{a\,\alpha\,H_b^2}{\delta} + \frac{\alpha\,b\,H_a\,H_b}{\delta} + \frac{\alpha\,H_d\,H_b}{d} + 2\,\frac{\partial\beta^\theta}{\partial\theta}\,H_b + \frac{R_{\theta\theta}\,\alpha}{\psi^4} .\end{aligned}$$

The metric evolution for $\hat{K}_{12} = H_c$:

$$\begin{aligned}\frac{\partial H_c}{\partial t} = {} & \frac{4\,\beta^\theta\,H_c\,\frac{\partial\psi}{\partial\theta}}{\psi} - \frac{2\,a\,\frac{\partial\alpha}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} + \frac{2\,a\,\frac{\partial\alpha}{\partial\eta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} + \frac{4\,\beta^\eta\,H_c\,\frac{\partial\psi}{\partial\eta}}{\psi} - \frac{2\,\frac{\partial\alpha}{\partial\eta}\,b\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} + \frac{2\,a\,\frac{\partial\alpha}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} \\ & - \frac{\frac{\partial\alpha}{\partial\eta}\,\frac{\partial b}{\partial\eta}\,c}{2\,\delta\,\psi^4} - \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial\alpha}{\partial\theta}\,c}{2\,\delta\,\psi^4} + \frac{a\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial b}{\partial\eta}}{2\,\delta\,\psi^4} + \frac{\frac{\partial a}{\partial\theta}\,\frac{\partial\alpha}{\partial\eta}\,b}{2\,\delta\,\psi^4} - \frac{\frac{\partial^2\alpha}{\partial\eta\partial\theta}}{\psi^4} + \beta^\theta\,\frac{\partial H_c}{\partial\theta} \\ & + \beta^\eta\,\frac{\partial H_c}{\partial\eta} - \frac{a\,\alpha\,H_b\,H_c}{\delta} - \frac{\alpha\,b\,H_a\,H_c}{\delta} + \frac{\alpha\,H_d\,H_c}{d} + \frac{\partial\beta^\theta}{\partial\theta}\,H_c + \frac{\partial\beta^\eta}{\partial\eta}\,H_c \\ & + \frac{2\,\alpha\,c\,H_a\,H_b}{\delta} + \frac{\partial\beta^\theta}{\partial\eta}\,H_b + \frac{\partial\beta^\eta}{\partial\theta}\,H_a + \frac{R_{\eta\theta}\,\alpha}{\psi^4} .\end{aligned}$$

The metric evolution for $\hat{K}_{33} = H_d$:

$$\begin{aligned}\frac{\partial H_d}{\partial t} = {} & \frac{\frac{\partial\alpha}{\partial\eta}\,c\,d\,\cot\theta}{\delta\,\psi^4} - \frac{a\,\frac{\partial\alpha}{\partial\theta}\,d\,\cot\theta}{\delta\,\psi^4} + 2\,\beta^\theta\,H_d\,\cot\theta + \frac{4\,\beta^\theta\,H_d\,\frac{\partial\psi}{\partial\theta}}{\psi} + \frac{2\,\frac{\partial\alpha}{\partial\eta}\,c\,d\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} - \frac{2\,a\,\frac{\partial\alpha}{\partial\theta}\,d\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^5} \\ & + \frac{4\,\beta^\eta\,H_d\,\frac{\partial\psi}{\partial\eta}}{\psi} + \frac{2\,\frac{\partial\alpha}{\partial\theta}\,c\,d\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} - \frac{2\,\frac{\partial\alpha}{\partial\eta}\,b\,d\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi^5} + \frac{\frac{\partial\alpha}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{2\,\delta\,\psi^4} - \frac{a\,\frac{\partial\alpha}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{2\,\delta\,\psi^4} + \frac{\frac{\partial\alpha}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{2\,\delta\,\psi^4} \\ & - \frac{\frac{\partial\alpha}{\partial\eta}\,b\,\frac{\partial d}{\partial\eta}}{2\,\delta\,\psi^4} - \frac{2\,\alpha\,c\,H_d\,H_c}{\delta} + \frac{a\,\alpha\,H_d\,H_b}{\delta} + \frac{\alpha\,b\,H_d\,H_a}{\delta} + \beta^\theta\,\frac{\partial H_d}{\partial\theta} + \beta^\eta\,\frac{\partial H_d}{\partial\eta} - \frac{\alpha\,H_d^2}{d} + \frac{R_{\phi\phi}\,\alpha}{\psi^4\,\sin^2\theta} .\end{aligned}$$

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Newtonian Limit

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The Newtonian limit refers to the study of one parameter families of solutions

$$\{ g^{\varepsilon}_{ij}, \rho_{\varepsilon}, v^{i}_{\varepsilon} \} \qquad 0 < \varepsilon < \varepsilon_0$$

to the Einstein-Euler equations

$$G^{ij} = 2\epsilon^4 T^{ij} \quad \& \quad \nabla_i T^{ij} = 0$$

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Heuristics

In the limit $\varepsilon \searrow 0$, $\{ g^\varepsilon_{ij}, \rho_\varepsilon, v^i_\varepsilon \}$ should reduce to a solution of the Poisson-Euler equations

Post-Newtonian Expansions

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Terminology

The n^{th} Post-Newtonian expansion corresponds to the ε^{2n} order.

$\frac{1}{2}$ PN equations

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Metric expansion

$$g^{\varepsilon}_{00} = -\frac{1}{\varepsilon^2} - 2\underset{-}{\overset{0}{\phi}} - 2\underset{-}{\overset{1}{\phi}}\epsilon + O(\varepsilon^2)$$

$$g^{\varepsilon}_{0I} = O(\varepsilon^2)$$

$$g^{\varepsilon}_{IJ} = \delta_{IJ} - 2\underset{-}{\overset{0}{\phi}}\delta_{IJ}\varepsilon^2 - 2\underset{-}{\overset{1}{\phi}}\delta_{IJ}\varepsilon^3 + O(\varepsilon^4)$$

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Matter field expansion

$$v^0_{\varepsilon} = 1 + \left(\underset{-}{\overset{0}{\Phi}} + \frac{1}{2}\delta_{IJ}\overset{0}{v}{}^I\overset{0}{v}{}^J\right)\varepsilon^2 + O(\varepsilon^3)$$

$$v^I_{\varepsilon} = \overset{0}{v}{}^I + \overset{1}{v}{}^I\varepsilon + O(\varepsilon^2)$$

$$\rho_{\varepsilon} = \overset{0}{\rho} + \overset{1}{\rho}\varepsilon + O(\varepsilon^2)$$

Order ε^0 equations

$$\partial_t \overset{\circ}{\rho} + \partial_I \left(\overset{\circ}{\rho} \overset{\circ}{v}{}^I \right) = 0$$

$$\partial_t \left(\overset{\circ}{\rho} \overset{\circ}{v}{}^I \right) + \partial_J \left(\overset{\circ}{\rho} \overset{\circ}{v}{}^J \overset{\circ}{v}{}^I \right) + \delta^{IJ} \partial_J \overset{\circ}{p} = - \overset{\circ}{\rho} \, \delta^{IJ} \partial_J \overset{\circ}{\underline{\Phi}}$$

$$\Delta \overset{\circ}{\underline{\phi}} = \overset{\circ}{\rho}$$

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Order ε^1 equations ($\frac{1}{2}$ PN)

$$\partial_t \overset{1}{\rho} + \partial_I (\overset{0}{\rho} \overset{1}{u}{}^I + \overset{1}{\rho} \overset{0}{u}{}^I) = 0$$

$$\partial_t (\overset{0}{\rho} \overset{1}{u}{}^I + \overset{1}{\rho} \overset{0}{u}{}^I) + \partial_J (\overset{0}{\rho} \overset{0}{u}{}^I \overset{1}{u}{}^J$$

$$\overset{0}{\rho} \overset{1}{u}{}^I \overset{0}{u}{}^J + \overset{1}{\rho} \overset{0}{u}{}^I \overset{0}{u}{}^J) + \delta^{IJ} \partial_J \overset{1}{p} = -\overset{1}{\rho} \delta^{IJ} \partial_J \overset{0}{\Phi} - \overset{0}{\rho} \delta^{IJ} \partial_J \overset{1}{\Phi}$$

$$\Delta \overset{1}{\Phi} = \overset{1}{\rho}$$

Empirical Observations

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Evidence

(i) Numerical simulations of clustering properties using the Newtonian field equations are in agreement with observations over a huge range of scales.

(ii) Heuristic arguments support the contention that Newtonian gravity is a good approximation for almost all regions of the universe.

Beyond Newtonian Gravity

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Formal Post Newtonian Expansions

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(i) Assumes the existence of a 1-parameter family

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of solutions to the Einstein-Euler equations

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$$\boxed{\text{This step often requires making more assumptions about the one parameter family of solutions that are difficult to justify.}}$$

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Precession of the Perihelion

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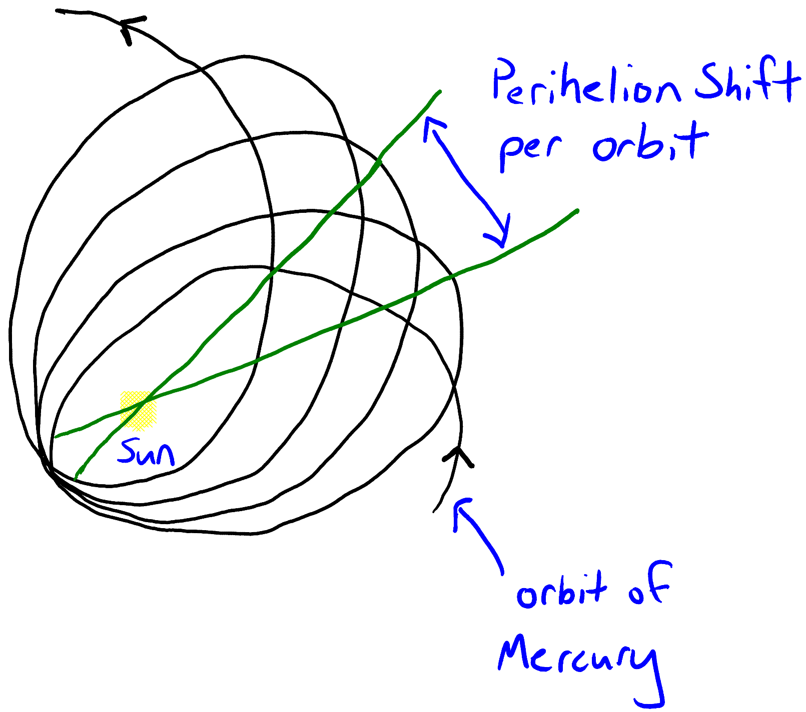
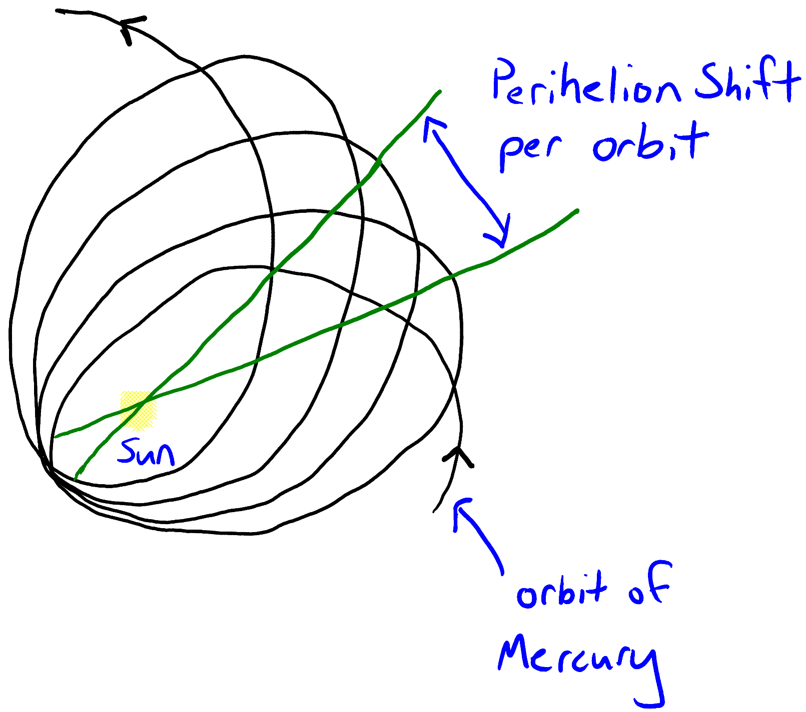


Formal Post-Newtonian Expansion Highlights

$\left(1 \text{ arcsecond} = \frac{1}{3600} \text{ deg} \right)$

Precession of the Perihelion

(i) Mercury's perihelion is observed to advance about 574 $\frac{\text{arcseconds}}{\text{century}}$.



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(i) Mercury's perihelion is observed to advance about 574 $\frac{\text{arcseconds}}{\text{century}}$.

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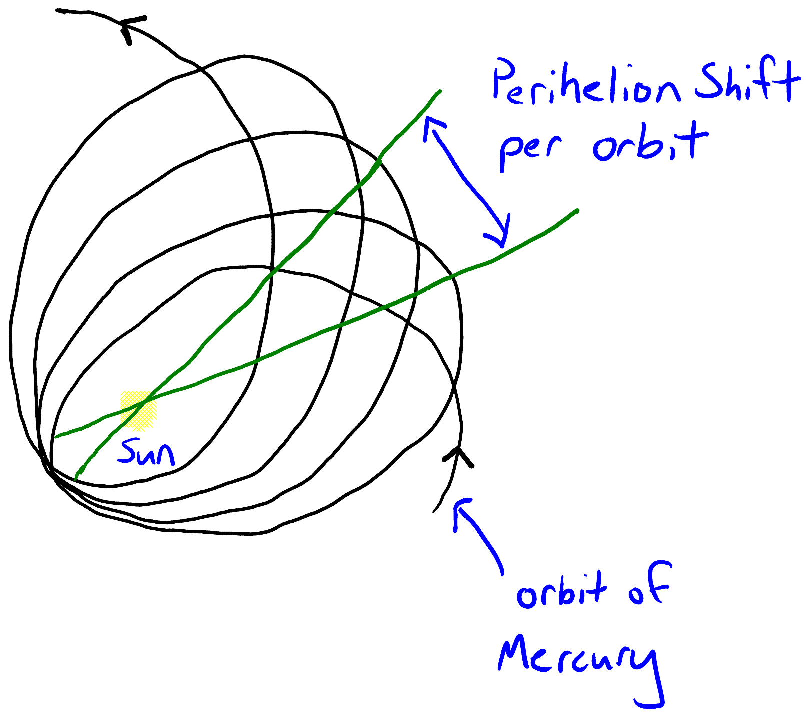

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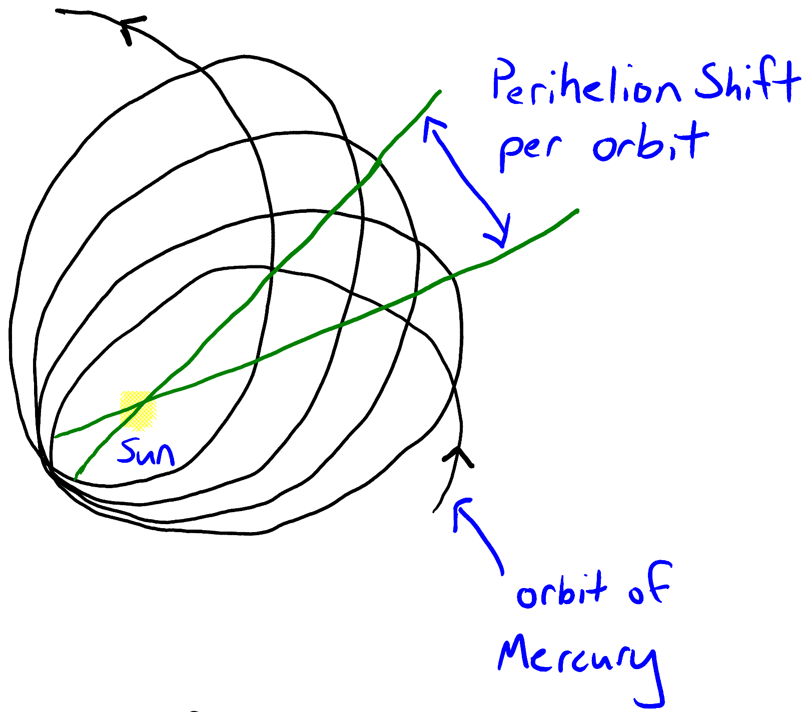

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(ii) Theoretical calculations using classical mechanics and Newtonian gravity predict an advance of approximately 531 $\frac{\text{arcseconds}}{\text{century}}$.

(iii) In 1915, Einstein showed using a formal PN expansion that general relativistic effect accounted for the missing 43 $\frac{\text{arcseconds}}{\text{century}}$.

<u>Ref</u>: A. Einstein, "Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity", Sitzunger. Preuss. Akad. Wiss (1915), 831–839

Hulse-Taylor Binary

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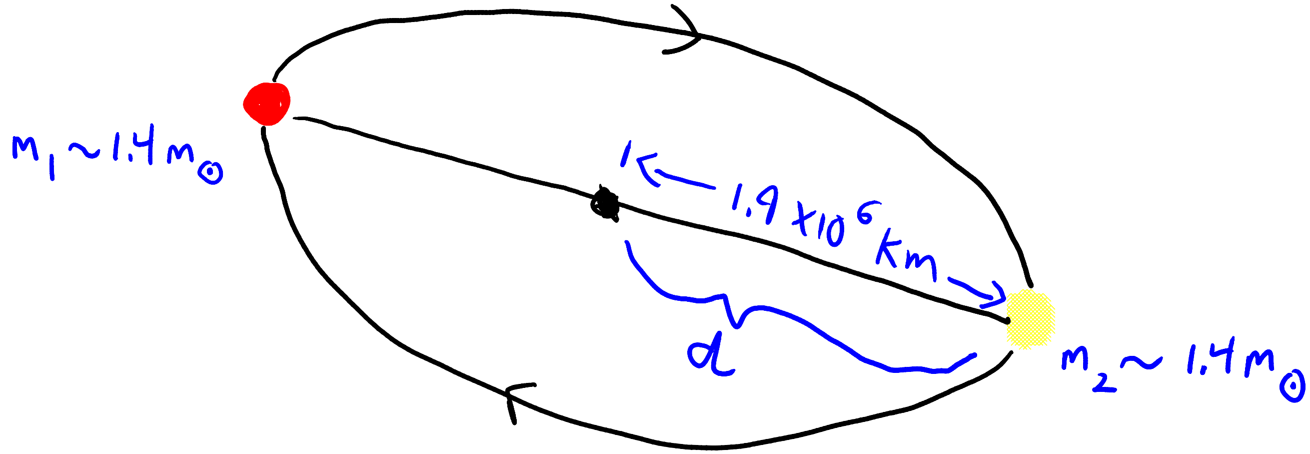


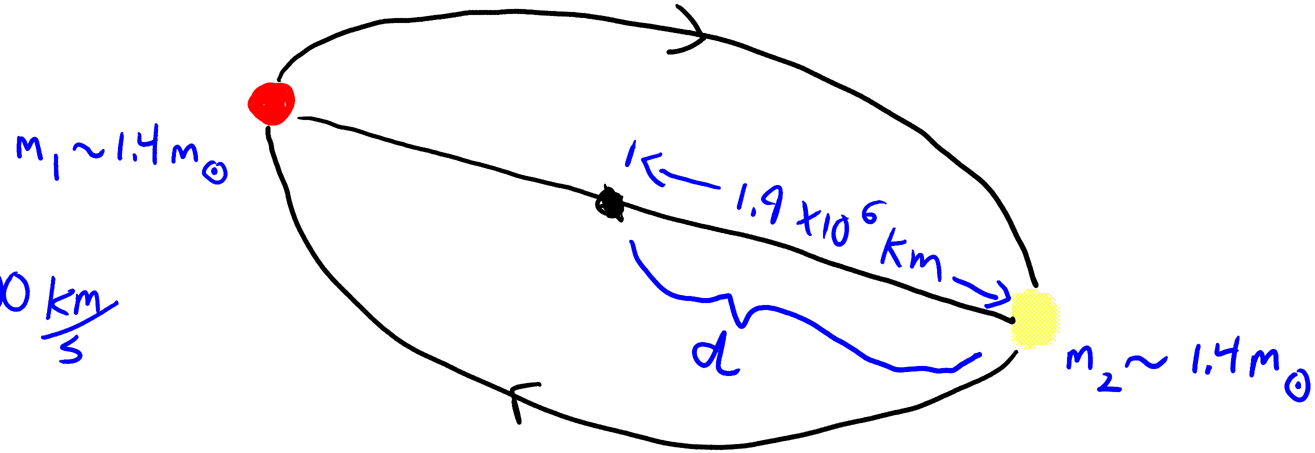
Hulse-Taylor Binary

Orbital period ~ 7.55 hr

min seperation ~ 1.1 $r_\odot$

max seperation ~ 4.8 $r_\odot$

average orbital speed ~ 400 $\frac{km}{s}$



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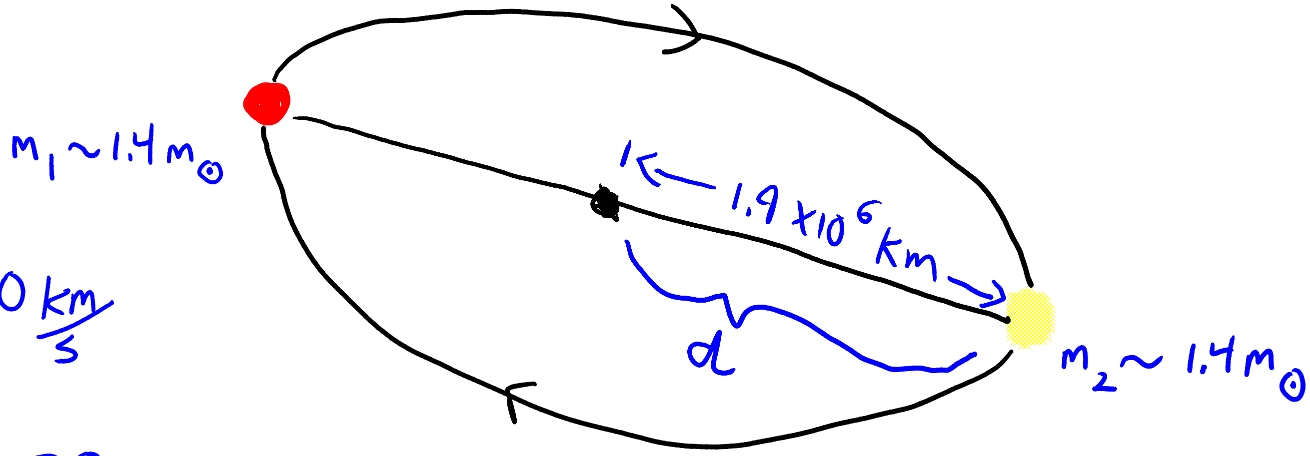

$$\varepsilon = \frac{V_I}{c} \sim 0.00133$$

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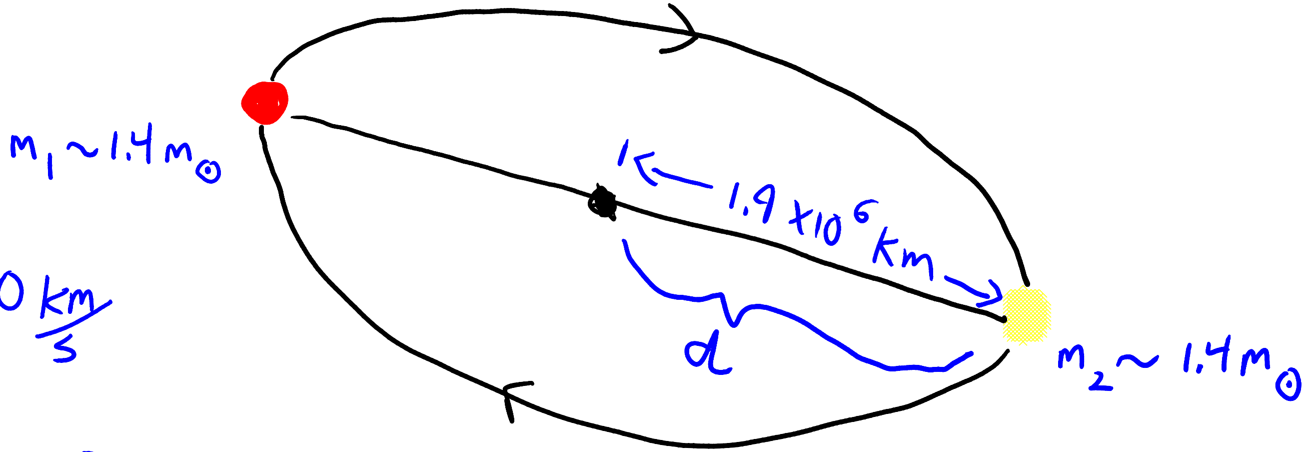

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Energy loss due to Gravitational Radiation

$$\Delta\tau \sim -76.5\ \frac{\mu s}{yr} \qquad \Delta d \sim -3.5\ \frac{m}{yr}$$

[Ref: R.A. Hulse and J.H. Taylor, "Discovery of a pulsar in a binary system", Astrophys J. Lett. 195 (1975), L51-L53]

Rigorous Post-Newtonian Expansions

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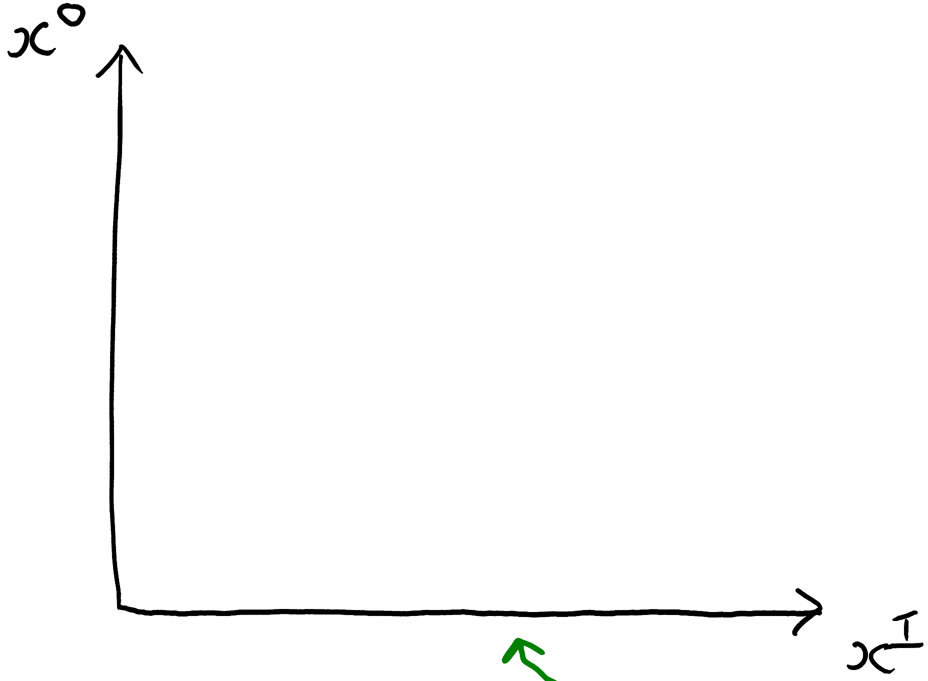

Step I Select the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

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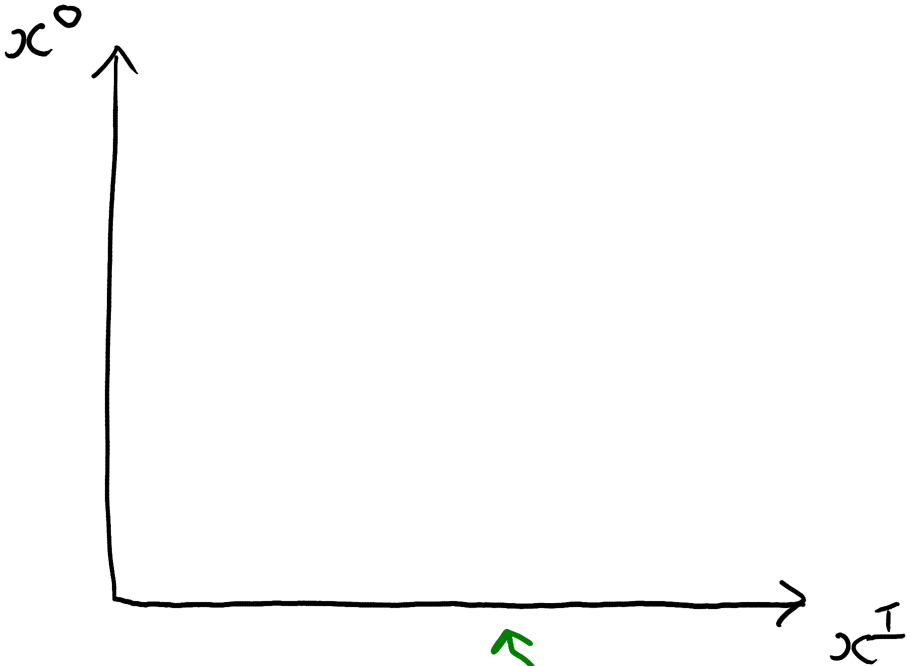


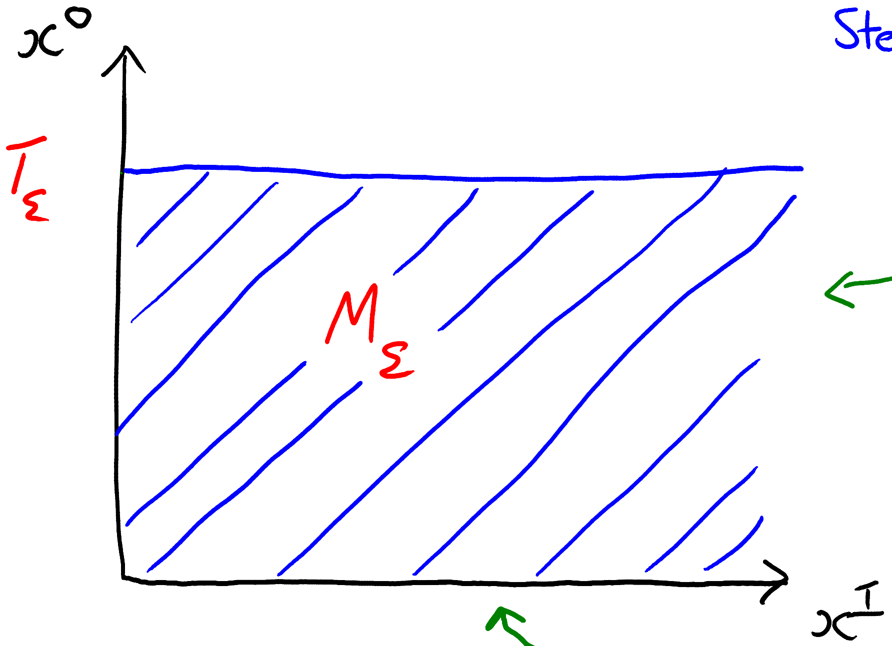
Step I Select the initial hypersurface $\Sigma_0 = \{0\} \times \Sigma$

Step II Construct initial data $\left\{ g^{\varepsilon}_{ij}\big|_{\Sigma_0},\ \partial_t g_{ij}\big|_{\Sigma_0},\ \rho_{\varepsilon}\big|_{\Sigma_0},\ v^i_{\varepsilon}\big|_{\Sigma_0} \right\}$

satisfying the constraint equations

$$\left(G^{0I} - 2\varepsilon^4 T^{0I}\right)\Big|_{\Sigma_0} = 0 \qquad g^{ij}_{\varepsilon}\,\Gamma^{k}_{ij}\Big|_{\Sigma_0} = 0 \qquad v^i_{\varepsilon} v^{\varepsilon}_i\Big|_{\Sigma_0} = \frac{1}{\varepsilon^2}$$

Rigorous Post-Newtonian Expansions



Step III Use the Einstein-Euler equations

$$G^{ij} = 2\varepsilon^4 T^{ij} \qquad \nabla_i T^{ij} = 0$$

to evolve the initial data and generate a 1-parameter family

$$\{ g^\varepsilon_{ij} ,\ \rho_\varepsilon ,\ v^i_\varepsilon \}$$

of solutions on $M_\varepsilon = [0, T_\varepsilon) \times \Sigma$

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Step II Construct initial data $\left\{ g^\varepsilon_{ij}\big|_{\Sigma_0} ,\ \partial_t g_{ij}\big|_{\Sigma_0} ,\ \rho_\varepsilon\big|_{\Sigma_0} ,\ v^i_\varepsilon\big|_{\Sigma_0} \right\}$

satisfying the constraint equations

$$\left(G^{0I} - 2\varepsilon^4 T^{0I}\right)\Big|_{\Sigma_0} = 0 \qquad g^{ij}_\varepsilon \Gamma^k_{ij}\Big|_{\Sigma_0} = 0 \qquad v^i_\varepsilon v^\varepsilon_i\Big|_{\Sigma_0} = \frac{1}{\varepsilon^2}$$

Step IV Show that there exists a $T > 0$ such that

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Step VI Try to expand $\{ g^\varepsilon_{ij} , \rho_\varepsilon , v^i_\varepsilon \}$ in ε by suitably restricting the initial data.

Literature on the rigorous PN-expansions

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a) A.D. Rendall, "The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system", Commun. Math. Phys. 163 (1994), 89-112

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a) A.D. Rendall, "The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system", Commun. Math. Phys. 163 (1994), 89 - 112

b) T.A. Oliynyk, "The Newtonian limit for perfect fluids", Commun. Math. Phys. 276 (2007), 131 - 188

" , "Post-Newtonian Expansions for perfect fluids", Commun. Math. Phys 288 (2009), 847 - 886

" , "Cosmological post-Newtonian expansions to arbitrary order", Commun. Math. Phys. 295 (2010), 431-463

" , "A rigorous formulation of the cosmological Newtonian limit without averaging", JHDE 7 (2010), 405-431

" , "The fast Newtonian limit", preprint

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$\Sigma = \mathbb{R}^3$ (Isolated Systems)

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PN-expansions to the 2PN order (i.e. ε^4)

$\Sigma = \mathbb{T}^3$ (Cosmological)

PN-expansions to arbitrary PN order (i.e. ε^ℓ for any $\ell \in \mathbb{N}$)

A canonical form for the Einstein-Euler equations

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New Variables

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$$g^{ij} = \frac{\epsilon}{-\det(Q)} Q^{ij}$$

where

$$Q^{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{IJ} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -1 & 0 \\ 0 & \bar{u}^{IJ} \end{pmatrix} + \varepsilon^3 \begin{pmatrix} 0 & \bar{u}^{J0} \\ \bar{u}^{I0} & 0 \end{pmatrix} + \varepsilon^4 \begin{pmatrix} \bar{u}^{00} & 0 \\ 0 & 0 \end{pmatrix}$$

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$$v^i = (1 + \varepsilon w^0)\delta^i_0 + \delta^i_J w^J$$

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The Newtonian fluid 3-velocity

A 1^{st} order formulation

$$u_0^{ij} = \varepsilon \partial_t \bar{u}^{ij} \qquad u_I^{ij} = \partial_I \bar{u}^{ij} \qquad u^{ij} = \epsilon \bar{u}^{ij}$$

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Subtracting the Newtonian Potential

$$W = \left(u_0^{ij}, w_I^{ij}, u^{ij}, \rho, w^i \right)^T$$

where

$$w_I^{ij} = u_I^{ij} - \delta_0^i \delta_0^j \partial_I \Phi$$

and

$$\Delta \Phi = \rho$$

Gauge Reduced Field Equations

$$A^0(\varepsilon, W)\partial_t W = \frac{1}{\varepsilon} C^I \partial_I W + A^I(\varepsilon, W)\partial_I W + F(\varepsilon, W)$$

where

$$C^I = \begin{pmatrix} C_G^I & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_G^I = \begin{pmatrix} 0 & \delta^{IJ} & 0 \\ \delta^{IJ} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^0 \geq \gamma \mathbb{1}, \qquad (A^i)^T = A$$

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There exists a well developed theory pioneered by Klainerman, Majda, Kreiss and Schochet for handling the limit $\varepsilon \searrow 0$ for equations of this form.

If $W_\varepsilon|_{t=0} = O(1)$, then there exists a $T, \varepsilon_0 > 0$ and and a 1-parameter family of solutions $W_\varepsilon(t,x)$ defined for all $(\varepsilon,t,x) \in (0,\varepsilon_0) \times (0,T) \times \Sigma$.

A model Equation

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Consider

$$\partial_t u_\varepsilon = \frac{1}{\varepsilon} C^I \partial_I u_\varepsilon \qquad u_\varepsilon(0) = u_{0\varepsilon} \in L^2(\mathbb{R}^3).$$

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$$\boxed{\therefore \quad \| u_\varepsilon(t) \|_{L^2} = \| u_{0\varepsilon} \|_{L^2} \quad \forall (t, \varepsilon) \in (0,\infty) \times (0,\infty)}$$

The Limit Equation

The Limit Equation

If $\partial_t W_\varepsilon|_{t=0} = O(1)$ as $\varepsilon \searrow 0$, then the solutions $W_\varepsilon(t,x)$ will converge on $(0,T)\times M$ to a solution of

$$A^0(0,W)\partial_t W = \partial_I W + F(0,W) + C^I\partial_I W$$

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$$A^0(0,W)\partial_t W = \partial_I W + F(0,W) + c^I \partial_I \omega$$

$$c^I \partial_I W = 0$$

These are just the Poisson-Euler equations of Newtonian gravity!

$$\mathring{W} = (0,0,0,\mathring{\rho},0,\mathring{w}^I) \quad \text{and} \quad \omega = (\delta_0^i \delta_0^j \partial_t \mathring{\Phi},0,0,0,0,0)^T$$

where

$$\partial_t \mathring{\rho} + \partial_I(\mathring{w}^I \mathring{\rho}) = 0$$

$$\mathring{\rho}(\partial_t \mathring{w}^J + \mathring{w}^I \partial_I \mathring{w}^J) = -\mathring{\rho}\,\partial^J \mathring{\Phi}$$

$$\Delta \mathring{\Phi} = \mathring{\rho}$$

There also exists an error estimate of the form

$$\| W_\varepsilon(t) - \overset{\circ}{W}(t) \| \lesssim \varepsilon \qquad \text{for all } (\varepsilon, t) \in (0, \varepsilon_0) \times (0, T)$$

There also exists an error estimate of the form

$$\| w_\varepsilon(t) - \overset{0}{w}(t) \| \lesssim \varepsilon \qquad \text{for all } (\varepsilon, t) \in (0, \varepsilon_0) \times (0, T)$$

This quantification of the error is what seperates the rigorous from the formal.

Kreiss's Bounded Derivative Principle

Kreiss's Bounded Derivative Principle

If $\partial_t^p W_\varepsilon \big|_{t=0} = O(1)$ as $\varepsilon \searrow 0$ for $p = 1, 2, \ldots, \ell+1$

then the solutions $W_\varepsilon(t,x)$ admit a convergent expansion of the form

$$W_\varepsilon(t,x) = \sum_{p=0}^{\ell} \varepsilon^p \overset{p}{W}(t,x) + \sum_{p=\ell+1}^{\infty} \varepsilon^p W_\varepsilon(t,x)$$

on $(0,T) \times \Sigma$ where:

(i) For $p = 1, 2, \cdots, \ell$, $\overset{p}{W}$ satisfies a linear (non-local) symmetric hyperbolic system that depends only on $\{ \overset{q}{W} \mid q = 0, 1, \dots, p-1 \}$.

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(ii) For $p \geq \ell + 1$, $\overset{p}{W}_{\varepsilon}$ satisfies a linear (non-local) symmetric hyperbolic system that depends only on $\{ \overset{q}{W} \mid q = 0, 1, \dots, p-1 \} \cup \{ \overset{q}{W}_{\varepsilon} \mid q = \ell+1, \ell+2, \dots, p-1 \}$