The Selberg Integral

Adelaide, 5 August 2011
History

The Selberg Integral

Selberg integrals on simple graphs

The Mukhin–Varchenko conjecture
The Wallis formula (1656)

\[
\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots
\]

\[
= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}
\]
Since \[ \frac{\pi}{4} \] Wallis tried to compute the integral

\[
\int_0^1 \sqrt{1 - x^2} \, dx
\]

This proved too hard.

As good mathematicians are wont to do, he therefore considered even harder integrals

\[
I(p, q) = \int_0^1 (1 - x^{1/p})^q \, dx
\]

for \( p, q = -1/2, 0, 1/2, 1, 3/2, 2, \ldots \).
Wallis’ table of $\frac{1}{I(p, q)}$ where $\square = \frac{1}{I(1/2, 1/2)} = \frac{4}{\pi}$:
The Gamma function (Euler 1720s)

\[ \Gamma(x) = \lim_{n \to \infty} \frac{n!n^{x-1}}{x(x+1)\cdots(x+n-1)} \quad x \neq 0, -1, -2, \ldots \]

\[ = \int_0^\infty t^{x-1}e^{-t} \, dt \quad \text{Re}(x) > 0 \]
Wallis’ product formula for $\pi$ is then equivalent to $\Gamma(1/2)\Gamma(3/2) = \pi/2$, and Wallis’ (failed) integral evaluation is

$$\int_0^1 \sqrt{1 - x^2} \, dx = \frac{1}{2} \Gamma(1/2)\Gamma(3/2)$$

Putting $x^2 = t$ this yields

$$\int_0^1 t^{1/2-1}(1 - t)^{3/2-1} \, dt = \Gamma(1/2)\Gamma(3/2)$$

The above observation led Euler to the discovery of a more general integral.
The Euler beta integral (1730s)

\[
\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]

for \(\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0\).

Replacing \((\beta, t) \rightarrow (\zeta, t/\zeta)\) with \(\zeta \in \mathbb{R}\) and letting \(\zeta \rightarrow \infty\) using Stirling's formula returns the integral representation of the gamma function.

Replacing \((\alpha, \beta, t) \rightarrow (\zeta^2 + 1, \zeta^2 + 1, 1/2 - x/(2\zeta))\) and letting \(\zeta \rightarrow \infty\) yields the Gaussian integral

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1
\]

Much more on this later . . .
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The Selberg integral

\[ \int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1} (1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \ dt_1 \cdots dt_n \]

\[ = n! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha + i\gamma)\Gamma(\beta + i\gamma)\Gamma(\gamma + i\gamma)}{\Gamma(\alpha + \beta + (n+i-1)\gamma)\Gamma(\gamma)} \]

for \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > f_n(\alpha, \beta) \).
Unfortunately I have been unable to find formula (11) [The Selberg integral] in the literature. To present a proof here, however, seems inappropriate, as it would make this paper significantly longer. If it turns out that the formula is new, I intend to publish a proof at a later date.
Dear Freeman,

Thanks for your note. Actually I found the formula in (4Y). I had not really planned to publish anything about it, but was later asked to contribute an article to Norwegian to their journal, and thought this might be suitable for that audience. Of course, I did have a bit more along those lines, but did not wish to make the article too lengthy. Obvious are limiting cases like the analog of Euler's integral for the P-function

\[ \int \frac{(\text{e}^{\text{e}} - 1)^{2\text{r}}}{(\text{e}^{\text{e}})^{\text{r}} - 1} \frac{\text{d}t_1 \ldots \text{d}t_n}{(1 + t)^k} \]

which can be derived in a similar way as the other limiting case you mentioned.

Another related (though not so obvious)

formula is

\[ \int_0^{\infty} \frac{\text{e}^{\text{e}}(\text{e}^{\text{e}} - 1)^{2\text{r}}}{(\text{e}^{\text{e}})^{\text{r}} - 1} \frac{\text{d}t_1 \ldots \text{d}t_n}{(1 + t)^k} \]

This is valid for real x, y, z for which the integral converges absolutely (conditions easy to find a bit below). From (2) one can also see no limit at z.

I had not thought about these things when I read Courant's remark about a problem of which I was not so clear how to handle, by using Courant's formula from the 1944 paper. That there hadn't been some integral in analogous integrals among physicists was completely unknown to me until now.

Yours sincerely,
Applications

- Algebra (Coxeter groups, double affine Hecke algebras)
- Conformal field theory (KZ equations)
- Gauge theory (supersymmetry, AGT conjecture)
- Geometry (hyperplane arrangements)
- Number theory (moments $\zeta(s)$)
- Orthogonal polynomials (Generalised Jacobi polynomials)
- Random matrices
- Statistics
- Statistical physics

Below I will discuss one application at the interface between algebra and geometry.
The Macdonald conjectures

Let $G$ be a finite reflection group or finite Coxeter group. That is, $G$ is a finite group of isometries of $\mathbb{R}^n$ generated by reflections in hyperplanes through the origin.

The reflection group $B_2$ of order 8, with 4 reflecting hyperplanes. $B_2$ is isomorphic to the signed permutations on two letters.
Given a set of reflecting hyperplanes, let \( \mathcal{A} \) be the set of normals, with normalisation \( \|a\|^2 = 2 \) for each \( a \in \mathcal{A} \).

For \( x = (x_1, \ldots, x_n) \) form the polynomial \( P(x) = \prod_{a \in \mathcal{A}} a \cdot x \).

Geometrically, \( P(x) \) is the product of the distances from the point \( x \) to each of the hyperplanes—up to a factor \( 2^{\|A\|/2} \) and a possible sign.

To make \( P(x) \) integrable over \( \mathbb{R}^n \) requires the Gaussian measure \( \varphi \):

\[
d\varphi(x) = \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}} \, dx
\]

For each Coxeter group \( G \) Macdonald considered the integral

\[
\int_{\mathbb{R}^n} |P(x)|^{2\gamma} \, d\varphi(x)
\]
By its action on $\mathbb{R}^n$ the reflection group $G$ acts on polynomials in $x$.

The $G$-invariant polynomials form an $\mathbb{R}$-algebra, $\mathbb{R}[f_1, \ldots, f_n]$, generated by $n$ algebraically independent polynomials $f_1, \ldots, f_n$.

The $f_1, \ldots, f_n$ are not unique but their degrees, $d_1, \ldots, d_n$, are.

Macdonald conjectured in 1982 that

$$
\int_{\mathbb{R}^n} |P(x)|^{2\gamma} \, d\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(d_i\gamma + 1)}{\Gamma(\gamma + 1)}
$$

For the trivial group $A_0$ (which maps $\mathbb{R}$ to $\mathbb{R}$ by the identity map) $P(x) = 1$ and the conjecture corresponds to the Gaussian integral

$$
\int_{\mathbb{R}} d\varphi(x) = 1
$$
The reflection group $A_{n-1}$

$A_{n-1}$ is the symmetry group of the $(n-1)$-simplex.

The 3-simplex or tetrahedron.

It is a group of order $n!$ ($\cong \mathfrak{S}_n$) generated by the $\binom{n}{2}$ hyperplanes

$$x_i - x_j = 0 \quad 1 \leq i < j \leq n$$

The polynomial $P(x)$ is given by the Vandermonde product

$$P(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$
The $G$-invariant polynomials are the symmetric polynomials in $x$, generated by the elementary symmetric functions $e_1, \ldots, e_n$:

$$e_r(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Hence the degrees are given by $(d_1, d_2, \ldots, d_n) = (1, 2, \ldots, n)$.

**Macdonald's conjecture for $A_{n-1}$** is thus

$$\int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} \, d\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(i\gamma + 1)}{\Gamma(\gamma + 1)}$$

better known as **Mehta's integral**.

This follows from the **Selberg** integral by setting

$$(\alpha, \beta) = (\zeta + 1, \zeta + 1) \quad t_i = \frac{1}{2} \left( 1 - \frac{x_i}{\sqrt{2\zeta}} \right) \quad \text{and} \quad \zeta \to \infty$$
The reflection groups $B_n$ and $D_n$

$B_n$ and $D_n$ are the symmetry groups of the $n$-cube and $n$-demicube.

In these cases the Macdonald conjecture is

$$
\int_{\mathbb{R}^n} \prod_{i=1}^{n} (2|x_i|^2)^{\gamma} \prod_{1 \leq i < j \leq n} |x_i^2 - x_j^2|^{2\gamma} \, d\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma + 1)}{\Gamma(\gamma + 1)}
$$

and follows again from the Selberg integral:
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\]

\[
\int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} |x_i^2 - x_j^2|^{2\gamma} \, d\varphi(x) = \frac{\Gamma(n\gamma + 1)}{\Gamma(\gamma + 1)} \prod_{i=1}^{n-1} \frac{\Gamma(2i\gamma + 1)}{\Gamma(\gamma + 1)}
\]

and follows again from the Selberg integral:

\[
B_n : \quad (\alpha, \beta) = (\gamma + 1/2, \zeta + 1) \quad t_i = \frac{x_i^2}{2\zeta} \zeta \to \infty
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In these cases the Macdonald conjecture is

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\int_{\mathbb{R}^n} \prod_{i=1}^{n} (2|\mathbf{x}_i|^2)^\gamma \prod_{1 \leq i < j \leq n} |\mathbf{x}_i^2 - \mathbf{x}_j^2|^{2\gamma} \, d\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma + 1)}{\Gamma(\gamma + 1)}
\]

and follows again from the Selberg integral:

\[
D_n : \quad (\alpha, \beta) = (1/2, \zeta + 1) \quad t_i = \frac{x_i^2}{2\zeta} \zeta \to \infty
\]
The dihedral group $l_2(m)$

$l_2(m)$ is the symmetry group of a regular $m$-gon

The 3-gon, 4-gon and pentagon.

Using polar coordinates the Macdonald's conjecture for $l_2(m)$ is entirely elementary.
The exceptional reflection groups

For $E_6$, $E_7$, $E_8$, $F_4$, $G_2$, the proof is hard but follows from a uniform proof for all crystallographic reflection groups due to Opdam.


For the non-crystallographic groups $H_3$ and $H_4$ the proof is hard.


Recall the Selberg integral

$$\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1} (1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq k} |t_i - t_j|^{2\gamma} \, dt_1 \cdots dt_k$$

We write this in shorthand as a labelled graph of a single vertex and no edges:
We can now consider other simple graphs. For example

\[
\begin{array}{cc}
\beta_1 & \beta_2 \\
\alpha_1 & \alpha_2 \\
\{t_i\}_{i=1}^{k_1} & \{s_i\}_{i=1}^{k_2} \\
1 & 2
\end{array}
\]

The rules of the game are as follows:

- A vertex labelled \( p \), \( \{t_i\}_{i=1}^{k_p} \), \( \alpha_p \), \( \beta_p \) contributes a factor
  \[
  \prod_{i=1}^{k_p} t_i^{\alpha_p - 1} (1 - t_i)^{\beta_p - 1} \prod_{1 \leq i < j \leq k_p} |t_i - t_j|^{2\gamma}
  \]

- A pair of vertices labelled \( p \), \( \{t_i\}_{i=1}^{k_p} \), \( \alpha_p \), \( \beta_p \) and \( q \), \( \{s_i\}_{i=1}^{k_q} \), \( \alpha_q \), \( \beta_q \) connected by an edge contributes a factor
  \[
  \prod_{i=1}^{k_p} \prod_{j=1}^{k_q} |t_i - s_j|^{-\gamma}
  \]
Example: The graph $A_2$

\[
\int \prod_{i=1}^{k_1} t_i^{\alpha_1 - 1} (1 - t_i)^{\beta_1 - 1} \prod_{i=1}^{k_2} s_i^{\alpha_2 - 1} (1 - s_i)^{\beta_2 - 1} \\
\times \prod_{1 \leq i < j \leq k_1} |t_i - t_j|^{2\gamma} \prod_{1 \leq i < j \leq k_2} |s_i - s_j|^{2\gamma} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} |t_i - s_j|^{-\gamma} \, dt \, ds
\]
Cartan matrices

The incidence or adjacency matrix $A$ of a (simple) graph $G$ of $r$ vertices labelled $1, 2, \ldots, r$ is a square matrix of size $r$ with entries

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

We call the matrix $C = 2I - A$ the Cartan matrix of $G$.

The rules of the game can now be restated as:

- A vertex labelled $p$, $\{t_i\}_{i=1}^{k_p}$, $\alpha_p$, $\beta_p$ contributes a factor
  $$\prod_{i=1}^{k_p} t_i^{\alpha_p-1} (1 - t_i)^{\beta_p-1} \prod_{1 \leq i < j \leq k_p} |t_i - t_j|^{C_{pp}\gamma}$$

- A pair of vertices labelled $p$, $\{t_i\}_{i=1}^{k_p}$, $\alpha_p$, $\beta_p$ and $q$, $\{s_i\}_{i=1}^{k_q}$, $\alpha_q$, $\beta_q$ contributes a factor
  $$\prod_{i=1}^{k_p} \prod_{j=1}^{k_q} |t_i - s_j|^{C_{pq}\gamma}$$
The BIG question

Can one evaluate the $G$ Selberg integral in closed form?
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For arbitrary choice of $G$, $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ and $k_1, \ldots, k_r$ there is little hope this question has an affirmative answer.
The BIG question

Can one evaluate the \( G \) Selberg integral in closed form?

For arbitrary choice of \( G, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r \) and \( k_1, \ldots, k_r \) there is little hope this question has an affirmative answer.

But representation theory can help, and we restrict attention to the (graphs of the) simply laced Lie algebras

\[
\begin{array}{c}
\text{A}_r \\
\text{D}_r \\
\text{E}_6 \\
\text{E}_7 \\
\text{E}_8
\end{array}
\]
The Mukhin–Varchenko conjecture


Let $a_1, \ldots, a_r$ and $\Lambda_1, \ldots, \Lambda_r$ be the simple roots and fundamental weights of the simple Lie algebra $G$:

$$a_i \cdot a_j = C_{ij} \quad \text{and} \quad a_i \cdot \Lambda_j = \delta_{ij}$$

Let $\Lambda$ and $\Lambda'$ be dominant integral weights:

$$\Lambda = c_1 \Lambda_1 + \cdots + c_r \Lambda_r \quad \text{and} \quad \Lambda' = d_1 \Lambda_1 + \cdots + d_r \Lambda_r$$

where the $c_i, d_i$ are nonnegative integers.
For $V_\Lambda$ and $V_{\Lambda'}$ two highest-weight modules of $G$ of weight $\Lambda$ and $\Lambda'$, compute the tensor product $V_\Lambda \otimes V_{\Lambda'}$.

This tensor product decomposes as

$$V_\Lambda \otimes V_{\Lambda'} = \bigoplus M^{\Lambda''}_{\Lambda\Lambda'} V_{\Lambda''}$$

where the nonnegative integers $M^{\Lambda''}_{\Lambda\Lambda'}$ are known as multiplicities.

All of the nonzero $\Lambda''$ that occur in the tensor product decomposition are of the form

$$\Lambda'' = \Lambda + \Lambda' - \sum_{i=1}^{r} k_i a_i$$
The Mukhin–Varchenko Conjecture.

Let the tensor-product multiplicity $M_{\Lambda \Lambda}^{\Lambda''}$ be equal to 1, where

$$\Lambda'' = \Lambda + \Lambda' - \sum_{i=1}^{r} k_i a_i$$

Write the exponents $\alpha_i$ and $\beta_i$ of the $G$ Selberg integral as

$$\alpha_i = 1 - \gamma \Lambda \cdot a_i = 1 - c_i \gamma$$

$$\beta_i = 1 - \gamma \Lambda' \cdot a_i = 1 - d_i \gamma$$

Then there exists a (real) domain of integration $D$ such that the $G$ Selberg integral with parameters $k_1, \ldots, k_r$, $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_r$ evaluates as a ratio of products of gamma functions.
• **Schur functions**

The understand the representation theory of the Lie algebra $A_n$ all you need to do is understand the theory of Schur functions of $n + 1$ variables subject to the constraint $x_1 \cdots x_{n+1} = 1$. Let $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ be a partition, e.g.,

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

Then the Schur function indexed by $\lambda$ is the symmetric polynomial

\[
s_\lambda(x_1, \ldots, x_{n+1}) = \frac{\det_{1 \leq i, j \leq n+1}(x_i^{\lambda_j + n - j + 1})}{\prod_{i < j}(x_i - x_j)}
\]
The structure constants of the Schur functions are known as the Littlewood–Richardson coefficients:

\[ s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \]

For the Lie algebra \( A_n \) the \( c^\nu_{\lambda\mu} \) are nothing but the multiplicities \( M_{\Lambda\Lambda'}^{\Lambda''} \).

For example, for \( A_3 \) we have the correspondence

\[ \lambda = \begin{array}{ccc}
1 & & 3 \\
2 & & \\
& &
\end{array} \rightarrow \Lambda = 3\Lambda_1 + 1\Lambda_2 + 2\Lambda_3 \]
There are many $c_{\lambda \mu}^\nu$ known to be exactly one. For example

$$s_\lambda s(r) = \sum_{\nu} s_\nu$$

where the sum is over all partitions such that the skew shape $\nu/\lambda$ has at most one box in each column.

In terms of weights of the Lie algebra $A_n$, the above combinatorial rule — due to Pieri in the context of Schubert calculus — corresponds to

$$\Lambda = \lambda_1 \Lambda_1 + \cdots + \lambda_n \Lambda_n, \quad \Lambda' = \mu_1 \Lambda_1, \quad \Lambda'' = \Lambda + \Lambda' - \sum_{i=1}^{n} k_i \alpha_i$$

where $k_1 \geq k_2 \geq \cdots \geq k_n$. 
\[ G = \mathfrak{sl}_{n+1} = \mathfrak{A}_n, \text{ general } k_1 \geq k_2 \geq \cdots \geq k_n \]

SOW, A Selberg integral for the Lie algebra \( \mathfrak{A}_n \), Acta Math. \textbf{203} (2009), 269–304.

Highest weights \( \Lambda = c_1 \Lambda_1 + \cdots + c_n \Lambda_n \) and \( \Lambda' = d_1 \Lambda_1 \) so that

\[
(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = (1 - c_1 \gamma, \ldots, 1 - c_2 \gamma, 1 - d_1 \gamma, 1, \ldots, 1)
\]

and \( D = \{ t \in \mathbb{R}^{k_1+\cdots+k_n}, \text{ chain} \} \) (in the algebraic topology sense)

\[
\int_D \cdots dt = \prod_{1 \leq r \leq s \leq n}^{k_r-k_{r+1}} \prod_{i=1} \frac{\Gamma(\alpha_r + \cdots + \alpha_s + (i + r - s - 1)\gamma)}{\Gamma(\beta_1 \delta_{r,1} + \alpha_r + \cdots + \alpha_s + (i + r - s + k_r - k_{r-1} - 2)\gamma)}
\times \prod_{s=1}^{n} \prod_{i=1}^{k_s} \frac{\Gamma(\beta_1 \delta_{s,1} + (i - k_{s-1} - 1)\gamma)}{\Gamma(\gamma)} \Gamma(i\gamma) \Gamma(\gamma)
\]
\( g = sl_3 = A_2, \) general \( k_1, k_2 \)


\[
\int_{D} \prod_{i=1}^{k_1} t_i^{\alpha_1 - 1}(1 - t_i)^{\beta_1 - 1} \prod_{i=1}^{k_2} s_i^{\alpha_2 - 1}(1 - s_i)^{\beta_2 - 1} \\
\times \prod_{1 \leq i < j \leq k_1} |t_i - t_j|^{2\gamma} \prod_{1 \leq i < j \leq k_2} |s_i - s_j|^{2\gamma} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} |t_i - s_j|^{-\gamma} \, dt \, ds
\]

\[
= \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1 + i\gamma)\Gamma(\beta_1 + (i - k_2)\gamma)\Gamma((i + 1)\gamma)}{\Gamma(\alpha_1 + \beta_1 + (i + k_1 - k_2 - 1)\gamma)\Gamma(\gamma)}
\times \prod_{i=0}^{k_2-1} \frac{\Gamma(\alpha_2 + i\gamma)\Gamma(\beta_2 + i\gamma)\Gamma((i + 1)\gamma)}{\Gamma(\alpha_2 + \beta_2 + (i + k_2 - k_1 - 1)\gamma)\Gamma(\gamma)}
\times \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1 + \alpha_2 + (i - 1)\gamma)}{\Gamma(\alpha_1 + \alpha_2 + (i + k_2 - 1)\gamma)}
\]

where \( \beta_1 + \beta_2 = \gamma + 1. \)
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