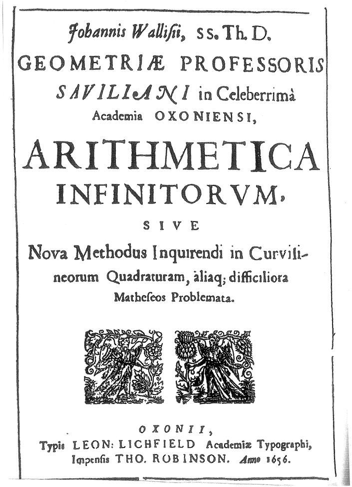
The Selberg Integral

Adelaide, 5 August 2011

History



Johannis Wallisii, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academia OXONIENSI,

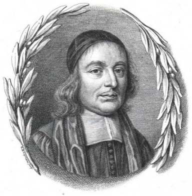
ARITHMETICA
INFINITORVM,

SIVE

Nova Methodus Inquirendi in Curvilineorum Quadraturam, àliaq; difficiliora Matheseos Problemata.

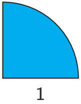
OXONII,
Typis LEON: LICHFIELD Academiæ Typographi,
Impensis THO. ROBINSON. Anno 1656.

- The Wallis formula (1656)



$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots$$

$$= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$$

Since 

$= \frac{\pi}{4}$ Wallis tried to compute the integral

$$\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$$

This proved too hard.

As good mathematicians are wont to do, he therefore considered even harder integrals

$$I(p, q) = \int_0^1 (1 - x^{1/p})^q\,\mathrm{d}x$$

for $p, q = -1/2, 0, 1/2, 1, 3/2, 2, \ldots$.

Wallis' table of $\dfrac{1}{I(p,q)}$ where $\square = \dfrac{1}{I(1/2,1/2)} = \dfrac{4}{\pi}$:

∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	$\frac{15}{48}$	$\frac{8}{35}\square$	$\frac{105}{384}$
1	1	1	1	1	1	1	1	1	1
$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{5}\square$	$\frac{105}{48}$	$\frac{64}{35}\square$	$\frac{945}{384}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$\frac{315}{48}$	$\frac{128}{21}\square$	$\frac{3465}{384}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15
$\frac{4}{15}\square$	1	$\frac{8}{5}\square$	$\frac{7}{2}$	$\frac{64}{15}\square$	$\frac{63}{8}$	$\frac{128}{15}\square$	$\frac{693}{48}$	$\frac{512}{35}\square$	$\frac{9009}{384}$
$\frac{15}{48}$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20	$\frac{1287}{48}$	35
$\frac{8}{35}\square$	1	$\frac{64}{35}\square$	$\frac{9}{2}$	$\frac{128}{21}\square$	$\frac{99}{8}$	$\frac{512}{35}\square$	$\frac{1287}{48}$	$\frac{1024}{35}\square$	$\frac{19305}{384}$
$\frac{105}{384}$	1	$\frac{945}{384}$	5	$\frac{3465}{384}$	15	$\frac{9009}{384}$	35	$\frac{19305}{384}$	70

- The Gamma function (Euler 1720s)

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^{x-1}}{x(x+1)\cdots(x+n-1)} \qquad x \neq 0, -1, -2, \dots$$

$$= \int_0^\infty t^{x-1} \mathrm{e}^{-t} \,\mathrm{d}t \qquad \mathrm{Re}(x) > 0$$

Wallis' product formula for π is then equivalent to $\Gamma(1/2)\Gamma(3/2)=\pi/2$, and Wallis' (failed) integral evaluation is

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\,\Gamma(1/2)\Gamma(3/2)$$

Putting $x^2=t$ this yields

$$\int_0^1 t^{1/2-1}(1-t)^{3/2-1}\mathrm{d}t = \Gamma(1/2)\Gamma(3/2)$$

The above observation led Euler to the discovery of a more general integral.

- The Euler beta integral (1730s)

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Replacing $(\beta, t) \to (\zeta, t/\zeta)$ with $\zeta \in \mathbb{R}$ and letting $\zeta \to \infty$ using Stirling's formula returns the integral representation of the gamma function.

Replacing $(\alpha, \beta, t) \to (\zeta^2+1, \zeta^2+1, 1/2 - x/(2\zeta))$ and letting $\zeta \to \infty$ yields the Gaussian integral

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = 1$$

Much more on this later . . .

niſi ſi ponatur loco n numerus integer affirmatiuus, vt ſerici termini principales fiant algebraici: et inde progreſſiones formaui. Earum itaque termini generales in promtu erunt, et a quanam quadratura quique eius termini intermedii pendeant definire licebit. Hic quidem non plures eiusmodi formulas percurram; ſed vnicam tantum aliquanto generaliorem pertractabo, quae valde late patet, et ad omnes progreſſiones, quarum quilibet termini ſunt facta conſtantia ex numero factorum ab indice pendente accommodari poteſt; quiq; factores ſunt fractiones, quarum numeratores et denominatores in progreſſione quacunque arithmetica progrediuntur, vt: $\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+$ etc.

§. 8. Sit propoſita haec formula $\int x^e dx(1-x)^n$ vicem termini generalis ſubiens, quae integrata ita, vt fiat $=0$, ſi ſit $x=0$; et tum poſito $x=1$, det terminum ordine n progreſſionis inde ortae. Videamus ergo qualem ea ſuppeditet progreſſionem. Eſt $(1-x)^n=1-\frac{n}{1}x+\frac{n.n-1}{1.2}x^2-\frac{n.n-1.n-2}{1.2.3.}x^3$ etc. Et propterea $x^e dx(1-x)^n=x^e dx-\frac{n}{1}x^{e+1}dx+\frac{n.n-1}{1.2}x^{e+2}dx-\frac{n.n-1.n-2}{1.2.3}x^{e+3}dx$ etc. Quare $\int x^e dx(1-x)^n=\frac{x^{e+1}}{e+1}-\frac{n.x^{e+2}}{1.(e+2)}+\frac{n.n-1..x^{e+1}}{1.2.(e+3)}-\frac{n.n-1.n-2.x^{e+4}}{1.2.3.(e+4)}$ etc. Ponatur $x=1$, quia conſtantis additione non eſt opus, et habebitur

$\frac{1}{e+1}$

The Selberg Integral

- The Selberg integral
  Uber einen Satz von A. Gelfond,
  Arch. Math. Naturvid. **44** (1941), 159–171;
  Bemerkninger om et multipelt integral,
  Norsk. Mat. Tidsskr. **24** (1944), 71–78.

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma} \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

$$= n! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma\big(\alpha+\beta+(n+i-1)\gamma\big)\Gamma(\gamma)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > f_n(\alpha,\beta)$.

Leider habe ich die Formel (11) [The Selberg integral] nirgends in der Litteratur finden können, ein Beweis hier zu bringen scheint aber nicht angebracht, da die Arbeit sonst zu sehr anschwellen würde; sollte sich aber herausstellen, dass die Formel neu wäre, beabsichtige ich später ein Beweis zu veröffentlichen.

Unfortunately I have been unable to find formula (11) [The Selberg integral] in the literature. To present a proof here, however, seems inappropriate, as it would make this paper significantly longer. If it turns out that the formula is new, I intend to publish a proof at a later date.

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2/27 – 79

Dear Freeman,

Thanks for your note. Actually I found the formula in 1941. I had not really planned to publish anything about it, but was later asked to contribute an article in Norwegian to this journal and thought this might be suitable for that audience.

Of course, I did have a bit more along these lines, but did not wish to make the article to bulky. Obvious are limiting cases like the analog of Eulers integral for the Γ-function

$$(1)\quad \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_n)^{x-1} e^{-(t_1 + \cdots t_n)} |\Delta(t)|^{2z} dt_1 \dots dt_n,$$

which can be derived in a similar way as the other limiting case you mention.

Another related (though not so obviously) formula is

$$(2)\quad \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{|\Delta(t)|^{2z}}{\prod_{j=1}^n (1+it_j)^x (1-it_j)^y} dt_1 \dots dt_n =$$

$$= (4\pi)^n 2^{-n(x+y-(n-1)z)} \prod_{\nu=1}^n \frac{\Gamma(1+\nu z)\,\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(1+z)\,\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)}$$

This is valid for complex x, y, z for which the integral converges absolutely (conditions easy to find a bit tedious to write down). From (2) one can even (using it for the case $x = y$) again obtain the formula you refer to in your letter as a limiting case.

I had not thought about these things now for more than thirty years when Bombieri consulted me about a problem of his which it seemed to me could be handled by using my formula from the 1944 paper. That there had been some interest in analogous integrals among physicists was completely unknown to me until now.

Yours sincerely

Atle

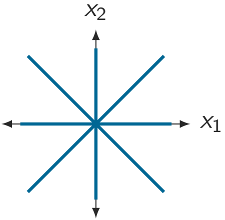
- Applications

  – Algebra (Coxeter groups, double affine Hecke algebras)
  – Conformal field theory (KZ equations)
  – Gauge theory (supersymmetry, AGT conjecture)
  – Geometry (hyperplane arrangements)
  – Number theory (moments $\zeta(s)$)
  – Orthogonal polynomials (Generalised Jacobi polynomials)
  – Random matrices
  – Statistics
  – Statistical physics

Below I will discuss one application at the interface between algebra and geometry.

- The Macdonald conjectures
  Some conjectures for root systems, SIAM J. Math. Anal. **13** (1982), 988–1007.

Let G be a finite reflection group or finite Coxeter group.
That is, G is a finite group of isometries of $\mathbb{R}^n$ generated by reflections in hyperplanes through the origin.



The reflection group B_2 of order 8, with 4 reflecting hyperplanes. B_2 is isomorphic to the signed permutations on two letters.

Given a set of reflecting hyperplanes, let $\mathcal{A}$ be the set of normals, with normalisation $\|\mathbf{a}\|^2 = 2$ for each $\mathbf{a} \in \mathcal{A}$.

For $\mathbf{x} = (x_1, \dots, x_n)$ form the polynomial $P(\mathbf{x}) = \prod_{\mathbf{a}\in\mathcal{A}} \mathbf{a}\cdot\mathbf{x}$

Geometrically, $P(\mathbf{x})$ is the product of the distances from the point $\mathbf{x}$ to each of the hyperplanes—up to a factor $2^{|\mathcal{A}|/2}$ and a possible sign.

To make $P(\mathbf{x})$ integrable over $\mathbb{R}^n$ requires the Gaussian measure φ:

$$\mathrm{d}\varphi(\mathbf{x}) = \frac{\mathrm{e}^{-\|\mathbf{x}\|^2/2}}{(2\pi)^{n/2}}\,\mathrm{d}\mathbf{x}$$

For each Coxeter group G Macdonald considered the integral

$$\int_{\mathbb{R}^n} |P(\mathbf{x})|^{2\gamma}\,\mathrm{d}\varphi(\mathbf{x})$$

By its action on $\mathbb{R}^n$ the reflection group G acts on polynomials in $\mathbf{x}$.

The G-invariant polynomials form an $\mathbb{R}$-algebra, $\mathbb{R}[f_1, \dots, f_n]$, generated by n algebraically independent polynomials $f_1, \dots, f_n$.

The $f_1, \dots, f_n$ are not unique but their degrees, $d_1, \dots, d_n$, are.

Macdonald conjectured in 1982 that

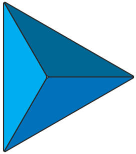
$$\int_{\mathbb{R}^n} |P(\mathbf{x})|^{2\gamma} \, \mathrm{d}\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(d_i\gamma + 1)}{\Gamma(\gamma + 1)}$$

For the trivial group A_0 (which maps $\mathbb{R}$ to $\mathbb{R}$ by the identity map) $P(x) = 1$ and the conjecture corresponds to the Gaussian integral

$$\int_{\mathbb{R}} \mathrm{d}\varphi(x) = 1$$

- The reflection group A_{n-1}

A_{n-1} is the symmetry group of the $(n-1)$-simplex.



The 3-simplex or tetrahedron.

It is a group of order $n!$ ($\cong \mathfrak{S}_n$) generated by the $\binom{n}{2}$ hyperplanes

$$x_i - x_j = 0 \qquad 1 \le i < j \le n$$

The polynomial $P(\mathbf{x})$ is given by the Vandermonde product

$$P(\mathbf{x}) = \prod_{1 \le i < j \le n} (x_i - x_j)$$

The G-invariant polynomials are the symmetric polynomials in $\mathbf{x}$, generated by the elementary symmetric functions $e_1, \dots, e_n$:

$$e_r(\mathbf{x}) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Hence the degrees are given by $(d_1, d_2, \dots, d_n) = (1, 2, \dots, n)$.

Macdonald's conjecture for A_{n-1} is thus

$$\int_{\mathbb{R}^n} \prod_{1 \le i < j \le n} |x_i - x_j|^{2\gamma} \, \mathrm{d}\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(i\gamma + 1)}{\Gamma(\gamma + 1)}$$

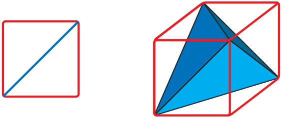
better known as Mehta's integral.

This follows from the Selberg integral by setting

$$(\alpha, \beta) = (\zeta + 1, \zeta + 1) \qquad t_i = \frac{1}{2}\Big(1 - \frac{x_i}{\sqrt{2\zeta}}\Big) \quad \text{and} \quad \zeta \to \infty$$

- The reflection groups B_n and D_n

B_n and D_n are the symmetry groups of the n-cube and n-demicube.



In these cases the Macdonald conjecture is

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} (2|x_i|^2)^{\gamma} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma} \, \mathrm{d}\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

$$\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma} \, \mathrm{d}\varphi(\mathbf{x}) = \frac{\Gamma(n\gamma+1)}{\Gamma(\gamma+1)} \prod_{i=1}^{n-1} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

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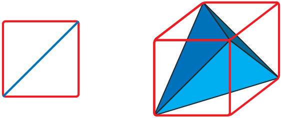

In these cases the Macdonald conjecture is

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} (2|x_i|^2)^{\gamma} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma} \, \mathrm{d}\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

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and follows again from the Selberg integral:

$$\mathrm{B}_n : \quad (\alpha,\beta) = (\gamma+1/2, \zeta+1) \qquad t_i = \frac{x_i^2}{2\zeta}\zeta \to \infty$$

- The reflection groups B_n and D_n

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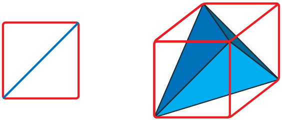


In these cases the Macdonald conjecture is

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} (2|x_i|^2)^{\gamma} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma}\, \mathrm{d}\varphi(\mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

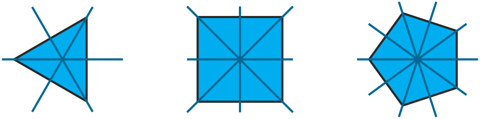
$$\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} |x_i^2 - x_j^2|^{2\gamma}\, \mathrm{d}\varphi(\mathbf{x}) = \frac{\Gamma(n\gamma+1)}{\Gamma(\gamma+1)} \prod_{i=1}^{n-1} \frac{\Gamma(2i\gamma+1)}{\Gamma(\gamma+1)}$$

and follows again from the Selberg integral:

$$D_n: \quad (\alpha,\beta) = (1/2, \zeta+1) \qquad t_i = \frac{x_i^2}{2\zeta}\zeta \to \infty$$

- The dihedral group $I_2(m)$

$I_2(m)$ is the symmetry group of a regular m-gon



The 3-gon, 4-gon and pentagon.

Using polar coordinates the Macdonald's conjecture for $I_2(m)$ is entirely elementary.

- The exceptional reflection groups

For E_6, E_7, E_8, F_4, G_2, the proof is hard but follows from a uniform proof for all crystallographic reflection groups due to Opdam.

Some applications of hypergeometric shift operators, Invent. Math. **98** (1989), 1–18.

For the non-crystallographic groups H_3 and H_4 the proof is hard.

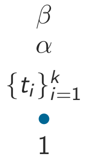
Garvan, Some Macdonald–Mehta integrals by brute force, IMA Vol. Math. Appl. **18**, (1989), 77–98.
Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, Compositio Math. **85** (1993), 333–373.

Selberg integrals on simple graphs

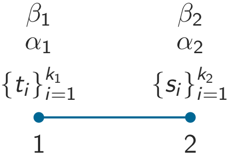
Recall the Selberg integral

$$\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} |t_i - t_j|^{2\gamma} \, \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

We write this in shorthand as a labelled graph of a single vertex and no edges:



We can now consider other simple graphs. For example



The rules of the game are as follows:

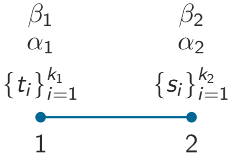
- A vertex labelled p, $\{t_i\}_{i=1}^{k_p}$, α_p, β_p contributes a factor

$$\prod_{i=1}^{k_p} t_i^{\alpha_p-1}(1-t_i)^{\beta_p-1} \prod_{1\le i<j\le k_p} |t_i-t_j|^{2\gamma}$$

- A pair of vertices labelled p, $\{t_i\}_{i=1}^{k_p}$, α_p, β_p and q, $\{s_i\}_{i=1}^{k_q}$, α_q, β_q connected by an edge contributes a factor

$$\prod_{i=1}^{k_p}\prod_{j=1}^{k_q} |t_i-s_j|^{-\gamma}$$

- Example: The graph A_2



$$\int \prod_{i=1}^{k_1} t_i^{\alpha_1-1}(1-t_i)^{\beta_1-1} \prod_{i=1}^{k_2} s_i^{\alpha_2-1}(1-s_i)^{\beta_2-1}$$
$$\times \prod_{1\le i<j\le k_1} |t_i-t_j|^{2\gamma} \prod_{1\le i<j\le k_2} |s_i-s_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |t_i-s_j|^{-\gamma}\, \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s}$$

- Cartan matrices

The incidence or adjacency matrix A of a (simple) graph $\mathcal{G}$ of r vertices labelled $1, 2, \ldots, r$ is a square matrix of size r with entries

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

We call the matrix $C = 2I - A$ the Cartan matrix of $\mathcal{G}$.

The rules of the game can now be restated as:

- A vertex labelled p, $\{t_i\}_{i=1}^{k_p}$, α_p, β_p contributes a factor

$$\prod_{i=1}^{k_p} t_i^{\alpha_p - 1}(1 - t_i)^{\beta_p - 1} \prod_{1 \le i < j \le k_p} |t_i - t_j|^{C_{pp}\gamma}$$

- A pair of vertices labelled p, $\{t_i\}_{i=1}^{k_p}$, α_p, β_p and q, $\{s_i\}_{i=1}^{k_q}$, α_q, β_q contributes a factor

$$\prod_{i=1}^{k_p} \prod_{j=1}^{k_q} |t_i - s_j|^{C_{pq}\gamma}$$

- The BIG question

Can one evaluate the $\mathcal{G}$ Selberg integral in closed form?

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For arbitrary choice of $\mathcal{G}$, $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_r$ and $k_1, \ldots, k_r$ there is little hope this question has an affirmative answer.

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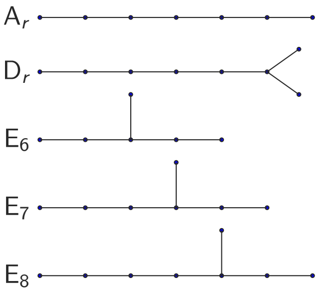
But representation theory can help, and we restrict attention to the (graphs of the) simply laced Lie algebras



The Mukhin–Varchenko conjecture

Remarks on critical points of phase functions and norms of Bethe vectors, Adv. Stud. Pure Math. **27** (2000), 239–246.

Let $\mathbf{a}_1, \dots, \mathbf{a}_r$ and $\mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_r$ be the simple roots and fundamental weights of the simple Lie algebra $\mathcal{G}$:

$$\mathbf{a}_i \cdot \mathbf{a}_j = C_{ij} \quad \text{and} \quad \mathbf{a}_i \cdot \mathbf{\Lambda}_j = \delta_{ij}$$

Let $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ be dominant integral weights:

$$\mathbf{\Lambda} = c_1\mathbf{\Lambda}_1 + \cdots + c_r\mathbf{\Lambda}_r \quad \text{and} \quad \mathbf{\Lambda}' = d_1\mathbf{\Lambda}_1 + \cdots + d_r\mathbf{\Lambda}_r$$

where the c_i, d_i are nonnegative integers.

For $V_{\mathbf{\Lambda}}$ and $V_{\mathbf{\Lambda}'}$ two highest-weight modules of $\mathcal{G}$ of weight $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$, compute the tensor product $V_{\mathbf{\Lambda}} \otimes V_{\mathbf{\Lambda}'}$.

This tensor product decomposes as

$$V_{\mathbf{\Lambda}} \otimes V_{\mathbf{\Lambda}'} = \bigoplus M_{\mathbf{\Lambda}\mathbf{\Lambda}'}^{\mathbf{\Lambda}''} V_{\mathbf{\Lambda}''}$$

where the nonnegative integers $M_{\mathbf{\Lambda}\mathbf{\Lambda}'}^{\mathbf{\Lambda}''}$ are known as multiplicities.

All of the nonzero $\mathbf{\Lambda}''$ that occur in the tensor product decomposition are of the form

$$\mathbf{\Lambda}'' = \mathbf{\Lambda} + \mathbf{\Lambda}' - \sum_{i=1}^{r} k_i \mathbf{a}_i$$

- The Mukhin–Varchenko Conjecture.

Let the tensor-product multiplicity $M_{\boldsymbol{\Lambda}\boldsymbol{\Lambda}'}^{\boldsymbol{\Lambda}''}$ be equal to 1, where

$$\boldsymbol{\Lambda}'' = \boldsymbol{\Lambda} + \boldsymbol{\Lambda}' - \sum_{i=1}^{r} k_i \mathbf{a}_i$$

Write the exponents α_i and β_i of the $\mathcal{G}$ Selberg integral as

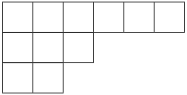
$$\alpha_i = 1 - \gamma \boldsymbol{\Lambda} \cdot \mathbf{a}_i = 1 - c_i \gamma$$

$$\beta_i = 1 - \gamma \boldsymbol{\Lambda}' \cdot \mathbf{a}_i = 1 - d_i \gamma$$

Then there exists a (real) domain of integration D such that the $\mathcal{G}$ Selberg integral with parameters $k_1, \dots, k_r$, $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_r$ evaluates as a ratio of products of gamma functions.

- Schur functions

The understand the representation theory of the Lie algebra A_n all you need to do is understand the theory of Schur functions of $n+1$ variables subject to the constraint $x_1 \cdots x_{n+1} = 1$. Let $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ be a partition, e.g.,



Then the Schur function indexed by λ is the symmetric polynomial

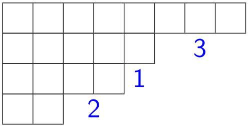
$$s_\lambda(x_1, \dots, x_{n+1}) = \frac{\det_{1 \le i,j \le n+1}\left(x_i^{\lambda_j + n - j + 1}\right)}{\prod_{i<j}(x_i - x_j)}$$

The structure constants of the Schur functions are known as the Littlewood–Richardson coefficients:

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$$

For the Lie algebra A_n the $c_{\lambda\mu}^{\nu}$ are nothing but the multiplicities $M_{\mathbf{\Lambda}\mathbf{\Lambda}'}^{\mathbf{\Lambda}''}$

For example, for A_3 we have the correspondence

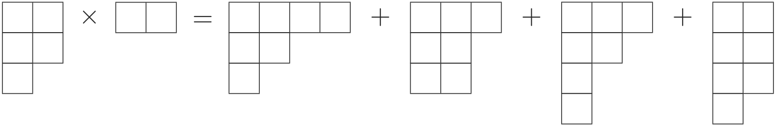
$\lambda =$ 

$\longrightarrow \mathbf{\Lambda} = 3\mathbf{\Lambda}_1 + 1\mathbf{\Lambda}_2 + 2\mathbf{\Lambda}_3$

There are many $c^{\nu}_{\lambda\mu}$ known to be exactly one. For example

$$s_\lambda s_{(r)} = \sum_{\nu} s_\nu$$

where the sum is over all partitions such that the skew shape ν/λ has at most one box in each column.



In terms of weights of the Lie algebra A_n, the above combinatorial rule — due to Pieri in the context of Schubert calculus — corresponds to

$$\mathbf{\Lambda} = \lambda_1 \mathbf{\Lambda}_1 + \cdots + \lambda_n \mathbf{\Lambda}_n, \quad \mathbf{\Lambda}' = \mu_1 \mathbf{\Lambda}_1, \quad \mathbf{\Lambda}'' = \mathbf{\Lambda} + \mathbf{\Lambda}' - \sum_{i=1}^{n} k_i \alpha_i$$

where $k_1 \geq k_2 \geq \cdots \geq k_n$.

- $\mathcal{G} = \mathfrak{sl}_{n+1} = \mathrm{A}_n$, general $k_1 \geq k_2 \geq \cdots \geq k_n$

  SOW, A Selberg integral for the Lie algebra A_n, Acta Math. **203** (2009), 269–304.

Highest weights $\mathbf{\Lambda} = c_1\mathbf{\Lambda}_1 + \cdots + c_n\mathbf{\Lambda}_n$ and $\mathbf{\Lambda}' = d_1\mathbf{\Lambda}_1$ so that

$$(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \big(1 - c_1\gamma, \dots, 1 - c_2\gamma, 1 - d_1\gamma, 1, \dots, 1)$$

and $D = \{\mathbf{t} \in \mathbb{R}^{k_1+\cdots+k_n}, \text{ chain}\}$ (in the algebraic topology sense)

$$\int_D \cdots \, \mathrm{d}\mathbf{t}$$

$$= \prod_{1\leq r\leq s\leq n} \prod_{i=1}^{k_s-k_{s+1}} \frac{\Gamma\big(\alpha_r + \cdots + \alpha_s + (i+r-s-1)\gamma\big)}{\Gamma\big(\beta_1\delta_{r,1} + \alpha_r + \cdots + \alpha_s + (i+r-s+k_r-k_{r-1}-2)\gamma\big)}$$

$$\times \prod_{s=1}^{n}\prod_{i=1}^{k_s} \frac{\Gamma\big(\beta_1\delta_{s,1} + (i-k_{s-1}-1)\gamma\big)\Gamma(i\gamma)}{\Gamma(\gamma)}$$

- $\mathfrak{g} = \mathfrak{sl}_3 = \mathrm{A}_2$, general k_1, k_2

  SOW, The $\mathfrak{sl}_3$ Selberg integral, Adv. Math. **224** (2010), 499–524.

$$\int_D \prod_{i=1}^{k_1} t_i^{\alpha_1-1}(1-t_i)^{\beta_1-1} \prod_{i=1}^{k_2} s_i^{\alpha_2-1}(1-s_i)^{\beta_2-1}$$

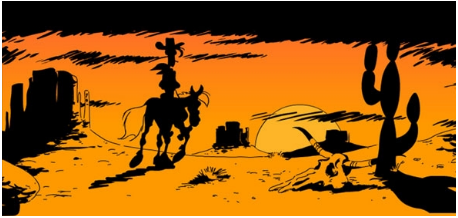
$$\times \prod_{1\le i<j\le k_1} |t_i - t_j|^{2\gamma} \prod_{1\le i<j\le k_2} |s_i - s_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |t_i - s_j|^{-\gamma}\, \mathrm{d}t\mathrm{d}s$$

$$= \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1+i\gamma)\Gamma(\beta_1+(i-k_2)\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_1+\beta_1+(i+k_1-k_2-1)\gamma)\Gamma(\gamma)}$$

$$\times \prod_{i=0}^{k_2-1} \frac{\Gamma(\alpha_2+i\gamma)\Gamma(\beta_2+i\gamma)\Gamma((i+1)\gamma)}{\Gamma(\alpha_2+\beta_2+(i+k_2-k_1-1)\gamma)\Gamma(\gamma)}$$

$$\times \prod_{i=0}^{k_1-1} \frac{\Gamma(\alpha_1+\alpha_2+(i-1)\gamma)}{\Gamma(\alpha_1+\alpha_2+(i+k_2-1)\gamma)}$$

where $\beta_1 + \beta_2 = \gamma + 1$.



The End