Categorification.

Going to infinity.

Homotopy theory and categorification

Kontsevich Swiss-Cheese conjecture

Sketch of a proof.

From group action to Kontsevich Swiss-Cheese conjecture

Michael Batanin

3 June 2011

From *Set* to *Cat*.

Definition

A category C consists of a set of objects of $Ob(C)$ and for any $a, b \in C$ the set $C(a,b)$ (morphisms between a and b). It is equipped with an associative composition map

$$C(a,b) \times C(b,c) \to C(a,c)$$

$a, b, c \in Ob(C)$ and for any $a \in Ob(C)$ a map

$$Id : 1 \to C(a,a),$$

which plays the role of an identity morphism.

From *Set* to *Cat*.

Definition

A functor

$$F : C \to D$$

between two categories is given by a map $F_0 : Ob(C) \to Ob(D)$ and a family of maps $F_{a,b} : C(a,b) \to D(F_0(a), F_0(b))$ which preserve composition and identity morphisms.

From *Set* to *Cat*.

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- - monoids and their homomorphism form a category *Mon*.
- - categories and functors form a category *Cat*.

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Examples:

- \- sets and functions form a category *Set*;
- \- vector spaces and linear operators form a category *Vect*;
- \- groups and their homomorphisms form a category *Gr*;
- \- monoids and their homomorphism form a category *Mon*.
- \- categories and functors form a category *Cat*.

Observe, that *Mon* is a subcategory of *Cat* which consists of categories which have only one object.

Decategorification

For a category C there is a relation $\simeq$ on its set of objects. $a \simeq b$ if there exists $f : a \to b$ and $g : b \to a$ such that $f \cdot g = id$ and $g \cdot f = id$. This relation is an equivalence relation and so we can speak about classes of equivalences of objects of C. Let $D(C)$ be the set of such classes.

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This gives a functor

$$D : Cat \longrightarrow Set$$

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Decategorification.

Example:

- Let *FinSet* be the category of finite sets then $D(FinSet) = \mathbb{N}$.

Categorification

Most of modern mathematics is done inside the category *Set*. The functor of decategorification shows that above *Set* there is a much bigger universe. Any mathematics we do in *Set* is a shadow of some more complicated mathematics in *Cat*.

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It is a difficult task since there are no precise rules for doing it (no "good" inverse functor) . For example, $D(FinSet) = D(Vect_{fd}) = \mathbb{N}$.

Categorification of algebraic structures.

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The source of complication: Cat is unusual category, it is a 2-category. It means that functors from A to B form a category $Cat(A, B)$! So relations should be replaced by isomorphisms not by equalities. And these isomorphism should satisfy some further relations (coherence relations).

Functor categories.

Definition

Let $F, G : A \to B$ be two functors. A natural transformation $\phi : F \to G$ consists of a family of maps $\phi_x : F(x) \to G(x), x \in Ob(A)$ such that for any $f : x \to y$ in A the following square commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{\phi_x} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow{\phi_y} & G(y) \end{array}$$

Monoidal categories

Monoidal category is categorification of the concept of monoid. It is a category M equipped with a functor

$$\otimes : M \times M \to M$$

and a special object $I \in M$, such that $\otimes$ is associative up to coherent isomorphism and I plays the role of unit with respect to $\otimes$ again up to coherent isomorphisms.

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Examples:

- *Set* with cartesian product $\times$ and one element set 1 as unit.
- $Vect_k$ with tensor product $\otimes$ and basic field k as unit.

Categorification of monoid action.

Definition

An action of a monoid M on a set X is a function

$$- \cdot - \; M \times X \to X$$

such that $(mn) \cdot x = m \cdot (n \cdot x)$ and $e \cdot x = x$.

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Theorem

An action of M on X is the same as monoid homomorphism

$$M \to Set(X, X).$$

Categorification of monoid action.

Proof. We use the following property of Set : There is an isomorphism (bijection)

$$Set(X \times Y, Z) \simeq Set(X, Set(Y, Z)).$$

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This property is fundamental. In categorical language: **the category** Set **is closed monoidal category.**

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Other examples of closed monoidal categories:

- $Vect_k$
- Cat with respect to cartesian product of categories.

Categorification of monoid action.

Categorifying monoid action we obtain an action of a monoidal category M on a category V. Since Cat is closed monoidal category we have the following:

Theorem

An action of a monoidal category M on a category X is the same as monoidal functor

$$M \to Cat(X, X).$$

One object case.

In the previous theorem it may happen that monoidal categories M and X has only one object $*$. It means that $M(*,*) = k$ and $X(*,*) = A$ are monoids.

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It means that k is a commutative monoid by classical Eckman-Hilton argument.

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Unpacking the axioms for action of M on X we see that they are equivalent to the statement that A is a monoid in the monoidal category of k-sets.

One object case

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Linear version: replace *Set* by the category of abelian groups and cartesian product by their tensor product. Then k is a commutative ring and A is just a ring. The axiom of action means that A is not just a ring but a k-algebra.

One object case

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A natural transfromation $\phi : F \to G$ is given by a family of morphisms $F(a) \to G(a)$ of X indexed by objects of X. Since X has only one object and morphisms are elements of A such a family is just an element $x \in A$.

One object case

Naturality square

$$\begin{array}{ccc} * & \xrightarrow{\;x\;} & * \\ {\scriptstyle f(y)}\downarrow & & \downarrow{\scriptstyle g(y)} \\ * & \xrightarrow{\;x\;} & * \end{array}$$

means that such an element x satisfies

$$xg(y) = f(y)x$$

for any $y \in C$.

One object case

Any functor $F : M \to Cat(X, X)$ picks up a homomorphism $f = F(*) : A \to A$. Moreover, it maps $k = M(*, *)$ to the set of natural transformations from F to F that is to the submonoid of $y \in A$ such that $xf(y) = f(y)x$.

One object case

Any functor $F : M \to Cat(X,X)$ picks up a homomorphism $f = F(*) : A \to A$. Moreover, it maps $k = M(*,*)$ to the set of natural transformations from F to F that is to the submonoid of $y \in A$ such that $xf(y) = f(y)x$.

If F is also monoidal it must preserve the monoidal unit. Since M has only one object this monoidal unit is the single object of M but the monoidal unit of $Cat(X,X)$ is the identity functor $Id : X \to X$. So, such a monoidal functor amounts to a monoid map from k to

$$Z(C) = \{x \in C \mid xy = yx,\ y \in C\}.$$

Kontsevich Swiss-Cheese conjecture, baby version.

Theorem

Let k be a commutative monoid and A be a monoid. Then a structure of a k-algebra on A is equivalent to a map of commutative monoids:

$$k \longrightarrow Z(A).$$

As we will see later this version is actually a decatigorification of the original Kontsevich Swiss-Cheese conjecture.

From *Cat* to Cat_2

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Analogously, categories form a new kind of structure: the 2-category *Cat*. In general a 2-category is a category whose hom sets are not sets but categories. So, a 2-category itself is a categorification of the notion of category.

From *Cat* to Cat_2

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Analogously, categories form a new kind of structure: the 2-category *Cat*. In general a 2-category is a category whose hom sets are not sets but categories. So, a 2-category itself is a categorification of the notion of category.

The 2-categories form a 2-category Cat_2. And we have a 2-functor of next decategorification

$$D : Cat_2 \to Cat_1 = Cat.$$

We again can ask about categorification as a "lifting" of structures from *Cat* to Cat_2.

From Cat_2 to Cat_∞.

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Sketch of a proof.

Cat_2 is an unusual 2-category (like *Cat* was unusual category). It can be promoted to a 3-category, so we can continue the process of categorification until infinity.

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As in $Set = Cat_0$ and $Cat = Cat_1$ we can try to equipped Cat_n with a product $\times_n$ with respect to which Cat_n is closed monoidal n-category i.e.

$$Cat_n(X \times Y, Z) \simeq Cat_n(X, Cat_n(Y, Z))$$

(but equivalence should be understood in more and more weak sense).

SC-conjecture, n-th version.

Theorem

An action of a monoidal n-category M on an n-category X is the same as monoidal n-functor

$$M \to Cat_n(X, X).$$

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An action of a monoidal n-category M on an n-category X is the same as monoidal n-functor

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Theorem (SC-conjecture, one object case)

Let k be a braided monoidal n-category and A be a monoidal n-category. Then a structure of a k-algebra on A is equivalent to a map of braided monoidal n-categories:

$$k \longrightarrow Z(A).$$

SC-conjecture, more degeneration.

Theorem (SC-conjecture with m degeneration)

Let k be an $m+1$-monoidal n-category and A be an m-monoidal n-category. Then a structure of an k-algebra on A is equivalent to a map of $m+1$-monoidal n-categories:

$$k \to Z(A),$$

where $Z(A)$ is the $m+1$-monoidal center of the m-monoidal n-category A.

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In general it is not very well understood yet for $n > 3$.

Groupoidofication

One can start categorification in a little bit different way. Instead of *Cat* let consider its subcategory of groupoids *Grp*. A category is a groupoid if all its morphisms are isomorphisms.

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One can continue and ask about ∞-groupoidofication. An advantage of this is that the category of ∞-groupoids is much better understood. It turned out that this is equivalent to the classical homotopy category. So, ∞-groupoidofication amounts to introducing topology and constructing homotopy theory analogues of familiar structures ("homotopification" of mathematics).

Examples of homotopification.

- Homotopification of monoids: E_1-algebras (or A_∞-algebras), they are monoids up to all higher homotopies.

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- Homotopification of commutative monoids: E_n-algebras (algebras of little n-disks operad.)
- Homotopification of an associative k-algebras: Algebras of Swiss-Cheese operads.
- Homotopification of the centre of an A_∞-algebra A: Hochschild complex $CH(A)$;

Original Swiss-Cheese conjecture for $n = 2$

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Conjecture (Kontsevich 1999)

- *The homotopy centre $CH(A)$ of an A_∞-algebra A is a E_2-algebra (Deligne's conjecture) (many proofs exist);*

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- *The homotopy centre $CH(A)$ of an A_∞-algebra A is a E_2-algebra (Deligne's conjecture) (many proofs exist);*
- *The algebra A is a $CH(A)$-algebra, that is there is an action of the Swiss-Cheese operad on $(CH(A), A)$ (Dolgushev, Tamarkin, Tsigan 2009)*

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- *The algebra A is a $CH(A)$-algebra, that is there is an action of the Swiss-Cheese operad on $(CH(A), A)$ (Dolgushev, Tamarkin, Tsigan 2009)*
- *For a E_2-algebra k a structure of a k-algebra on A is equivalent (up to homotopy) to a map of E_2-algebras $k \to CH(A)$.*

Original Swiss-Cheese conjecture for arbitrary n.

Conjecture (Kontsevich 1999)

- *For an E_n-algebra A the homotopy category of E_{n+1}-algebras with an action on A has a terminal object $CH(A)$;*

Original Swiss-Cheese conjecture for arbitrary n.

Conjecture (Kontsevich 1999)

- *For an E_n-algebra A the homotopy category of E_{n+1}-algebras with an action on A has a terminal object $CH(A)$;*

This terminal object is unique up to homotopy and called Hochschild complex of E_n-algebra A.

$(\infty, 1)$-categorization

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This is a mixture of ∞-groupoidofication and categorization. The idea is that we first apply ∞-groupoidofication then we categorify. As a result we replace Cat by $Cat_{\infty,1}$ that is categories whose hom-sets are equipped with topology ($(\infty, 1)$-categories otherwise known as A_∞-categories).

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Such categories have all properties necessary to categorify. In particular, they form next category the so called $(\infty, 2)$-category $Cat_{\infty,2}$ (Tamarkin 2006).

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Such categories have all properties necessary to categorify. In particular, they form next category the so called $(\infty, 2)$-category $Cat_{\infty,2}$ (Tamarkin 2006).

Moreover, there is a product of $(\infty, 1)$-categories which is closed (this follows from combination of Tamarkin's technics and Batanin-Cisinski-Weber results). So we can repeat a story with categorification of monoid action.

SC-conjecture $(\infty, 1)$-version.

Theorem

An action of a monoidal $(\infty, 1)$-category M on an $(\infty, 1)$-category X is the same as monoidal $(\infty, 1)$-functor

$$M \to Cat_{\infty,1}(X, X).$$

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Corollary

Kontsevich Sweese-Cheese conjecture for $n = 2$.

Proof. This is one object version of the previous theorem which is obtained immediately by applying Batanin's symmetrization theorem for Swiss-Cheese 2-operad (Batanin 2008).

SC-conjecture (∞, n)-version.

Theorem (Work in progress with Berger, Cisinski, Markl, Weber)

An action of a monoidal (∞, n)-category M on an (∞, n)-category X is the same as monoidal (∞, n)-functor

$$M \to Cat_{\infty,n}(X, X).$$

SC-conjecture (∞, n)-version.

Theorem (Work in progress with Berger, Cisinski, Markl, Weber)

An action of a monoidal (∞, n)-category M on an (∞, n)-category X is the same as monoidal (∞, n)-functor

$$M \to Cat_{\infty,n}(X, X).$$

Corollary

Kontsevich Sweese-Cheese conjecture for arbitrary n

Proof. This is one object , one arrow, ..., one $(n-1)$-arrow version of the previous theorem which is obtained immediately by applying Batanin's symmetrization theorem for Swiss-Cheese n-operad (Batanin 2008).