# Lattices in exotic groups

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# Outline

1. Background: locally compact groups and lattices
2. Question: finite generation of lattices
3. Lattices in Lie groups
4. Lattices in exotic groups

# Topological groups

A topological group is a group $G$ with

- a (Hausdorff) topology, such that
- group operations are continuous.

That is, $G$ has compatible topological and algebraic structures.

# Locally compact groups

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Examples

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2. $G = SL(2, \mathbb{R}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{R}, \; ad - bc = 1 \right\}$

# Haar measure

$G$ locally compact group

### Theorem (Haar, Weil 1930s)

*$\exists$ countably additive measure $\mu$ on the Borel subsets of $G$ s.t.*

- *$\mu$ is left-invariant: $\forall g \in G$, $\forall$ Borel sets $E$*

$$\mu(gE) = \mu(E)$$

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# Examples of Haar measure

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# Haar measure

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2. Compute Haar measure on $G = SL(2, \mathbb{R})$ using Iwasawa decomposition

$$g = kan$$

where $k \in SO(2, \mathbb{R})$, $a = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

# Haar measure

Examples

1. Lebesgue measure on $G = (\mathbb{R}^n, +)$
2. $G = SL(2, \mathbb{R})$ acts on upper half-plane

$$\mathcal{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

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Normalise Haar measure $\mu$ to be compatible with this action.

# Lattices

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  By abuse of notation write $\mu(G/\Gamma) < \infty$.

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A subgroup $\Gamma < G$ is a lattice if

- $\Gamma$ is discrete
- $\mu(G/\Gamma) < \infty$.

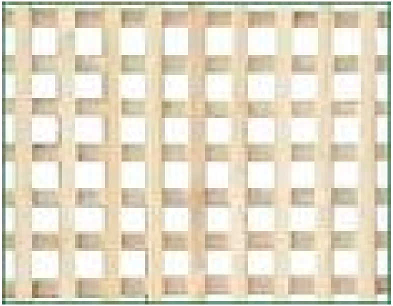
A lattice $\Gamma < G$ is

- uniform (or cocompact) if $G/\Gamma$ is compact
- otherwise, nonuniform (or noncocompact).

# Example of a uniform lattice

### Example

$\mathbb{Z}^n < \mathbb{R}^n$ is discrete



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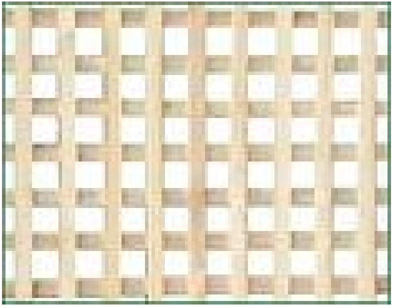

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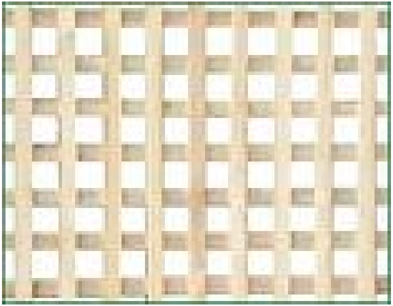

$\mathbb{R}^n/\mathbb{Z}^n$ is $n$–torus, has finite Lebesgue measure, is compact, so $\mathbb{Z}^n$ is uniform lattice in $\mathbb{R}^n$

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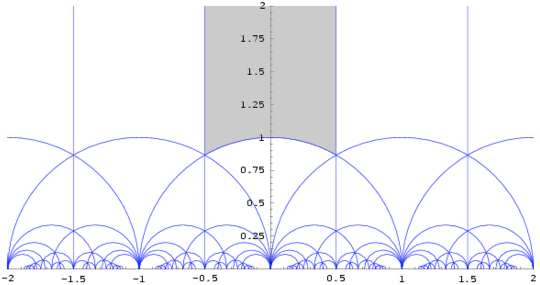
Action of $\Gamma = SL(2, \mathbb{Z})$ on upper half-plane $\mathcal{U}$ induces tessellation



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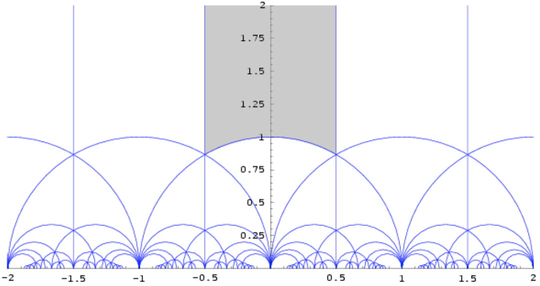


Haar measure $\mu$ on $G = SL(2, \mathbb{R})$ is normalised so that

$$\mu(G/\Gamma) = \text{area of fundamental domain} = \frac{\pi}{3}$$

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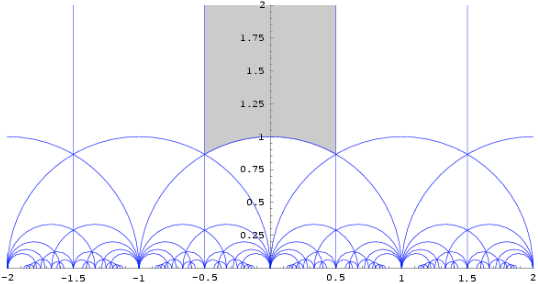


$\mu(G/\Gamma) =$ area of fundamental domain $= \frac{\pi}{3}$

Non-compact fundamental domain $\leftrightarrow$ $SL(2,\mathbb{Z})$ is nonuniform lattice in $SL(2,\mathbb{R})$.

# Question

Given a locally compact group $G$, are lattices in $G$ finitely generated?

# Examples of finitely generated lattices

### Examples

1. Every lattice $\Gamma < \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$, hence is finitely generated.

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2. $SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$ is finitely generated by

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(Euclidean algorithm)

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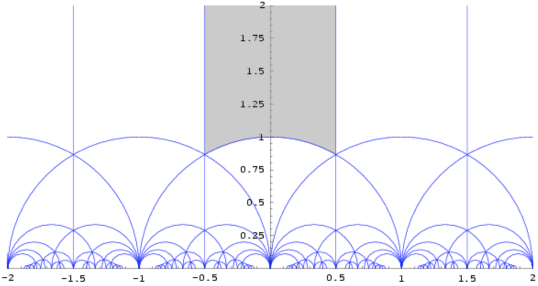
Geometrically, fundamental domain is finite-sided:



# Lattices in Lie groups

$G = SL(2, \mathbb{R})$ is a Lie group, $\Gamma = SL(2, \mathbb{Z})$ is finitely generated.

What about lattices in other Lie groups?

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What about lattices in other Lie groups?

1. Lie groups as locally compact groups
2. Examples of lattices in Lie groups
3. Finite generation of lattices in Lie groups

# Lie groups as locally compact groups

Examples

1. $G = SL(n, \mathbb{R})$ is a real Lie group, hence a connected locally compact group.

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1. $G = SL(n, \mathbb{R})$ is a real Lie group, hence a connected locally compact group.
2. "p-adic Lie groups" such as $G = SL(n, \mathbb{Q}_p)$ or $G = SL(n, \mathbb{F}_q((t^{-1})))$ are locally compact but totally disconnected.

# Lattices in real and "p-adic" Lie groups

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2. $G = SL(n, \mathbb{Q}_p)$ has uniform lattices but no nonuniform lattices.
3. $\Gamma = SL(n, \mathbb{F}_q[t])$ is a nonuniform lattice in $G = SL(n, \mathbb{F}_q((t^{-1})))$.

# Finite generation for lattices in higher-rank Lie groups

**Theorem (Kazhdan 1967)**

*Let $G$ be a higher-rank real or "p-adic" Lie group. Then every lattice $\Gamma < G$ is finitely generated.*

**Examples**

For $n \geq 3$, $SL(n, \mathbb{Z})$ and $SL(n, \mathbb{F}_q[t])$ are finitely generated.

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### Theorem (special case of Margulis Superrigidity, 1970s)

*If $\Gamma$ a lattice in $G$ as above, then any linear representation of $\Gamma$ extends to the whole of $G$.*

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These implications hold for all locally compact groups $G$ and all lattices $\Gamma < G$.

# Kazhdan's Property (T)

$G$ locally compact group

$\pi : G \to U(\mathcal{H})$ unitary representation of $G$ on Hilbert space $\mathcal{H}$.

### Definition

Let $\varepsilon > 0$ and $K \subset G$ be compact. A unit vector $v \in \mathcal{H}$ is $(\varepsilon, K)$–invariant if $\forall g \in K$

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### Definition

$G$ has Kazhdan's Property (T) if any unitary representation of $G$ which almost has invariant vectors has nontrivial *invariant* vectors.

# $\Gamma$ has (T) $\implies$ $\Gamma$ is finitely generated

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$G$ has Kazhdan's Property (T) if any unitary representation of $G$ which almost has invariant vectors has nontrivial invariant vectors.

## Theorem (Kazhdan 1967)

*A discrete group $\Gamma$ with Property (T) is finitely generated.*

## Proof.

Enumerate $\Gamma = \{\gamma_i\}$ and let $\Gamma_n = \langle \gamma_1, \ldots, \gamma_n \rangle$.

Let $\pi_n$ be rep of $\Gamma$ on $L^2(\Gamma/\Gamma_n)$ induced by trivial rep of $\Gamma_n$.
Then $\pi_n$ contains unit vector $\chi_{e\Gamma_n}$ invariant under $\gamma_1, \ldots, \gamma_n$.

Hence $\pi := \oplus \pi_n$ almost has invariant vectors.

Since $\Gamma$ has (T), $\pi$ has a nontrivial invariant vector $f \in \oplus L^2(\Gamma/\Gamma_n)$.

Project $f$ to each factor. Projections are invariant, and for some $n$ nontrivial. So for some $n$, $\pi_n$ has nontrivial invariant vector $f_n$.

Thus $\Gamma/\Gamma_n$ is finite, so $\Gamma$ is finitely generated.

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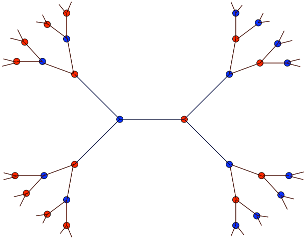
1. Tree lattices
2. Lattices for polygonal complexes

# Automorphism groups of trees

$T$ locally finite tree e.g. $T_3$ the 3–regular tree



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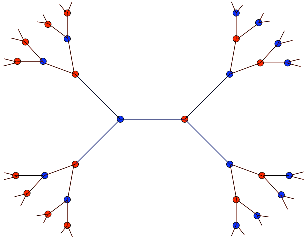

$G = \mathrm{Aut}(T)$ automorphism group of $T$

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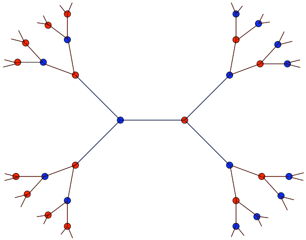

Equip $G = \mathrm{Aut}(T)$ with compact-open topology:
fix basepoint $v_0 \in T$, neighbourhood basis of $1_G$ is

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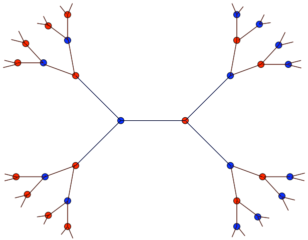

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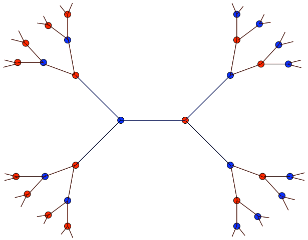

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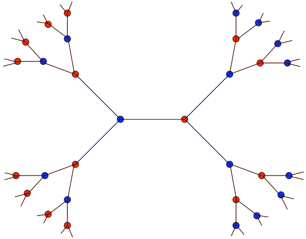

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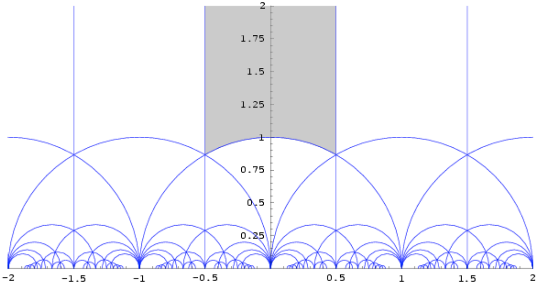
Example

$G = \mathrm{Aut}(T_3)$ is nondiscrete locally compact group.

# Motivation

- Study real Lie groups and their lattices via action on symmetric space
  e.g. upper half-plane is symmetric space for $SL(2, \mathbb{R})$



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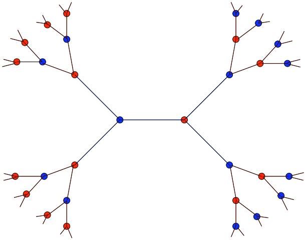
- Study real Lie groups and their lattices via action on symmetric space
  e.g. upper half-plane is symmetric space for $SL(2, \mathbb{R})$
- Study "p–adic" Lie groups and their lattices via action on building
  e.g. $T_{q+1}$ is building for $SL(2, \mathbb{F}_q((t^{-1})))$



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*Can normalise Haar measure $\mu$ on $G$ so that $\forall$ discrete $\Gamma < G$*

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*Moreover $\Gamma$ uniform $\iff$ the graph $T/\Gamma$ is compact (finite).*

So $\Gamma < G$ is

- uniform lattice $\iff$ $\Gamma \curvearrowright T$ with finite stabilisers and finite quotient

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*Can normalise Haar measure $\mu$ on $G$ so that $\forall$ discrete $\Gamma < G$*

$$\mu(G/\Gamma) = \sum_{v \in Vert(T/\Gamma)} \frac{1}{|Stab_\Gamma(\tilde{v})|} \leq \infty$$

*Moreover $\Gamma$ uniform $\iff$ the graph $T/\Gamma$ is compact (finite).*

So $\Gamma < G$ is

- uniform lattice $\iff$ $\Gamma \curvearrowright T$ with finite stabilisers and finite quotient
- nonuniform lattice $\iff$ $\Gamma \curvearrowright T$ with finite stabilisers and *infinite* quotient, so that series above converges

# Lattices in Aut($T$)

$T$ locally finite tree, $G = \mathrm{Aut}(T)$ compact-open topology
$\Gamma < G$ is discrete $\iff$ $\Gamma \curvearrowright T$ with finite stabilisers.

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Applies to all locally compact $G$ acting cocompactly on locally finite tree
e.g. $G = SL_2(\mathbb{F}_q((t^{-1})))$ and its building $T_{q+1}$

# Examples of tree lattices

**Example**

Uniform lattice $\Gamma$ in $G = \mathrm{Aut}(T_3)$:

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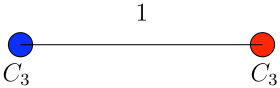

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Uniform lattice in $G = \mathrm{Aut}(T_3)$:

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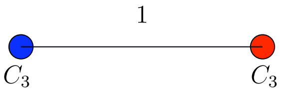
$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * C_3$



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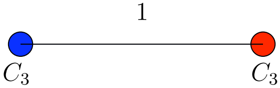

# Examples of tree lattices

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Nonuniform lattice in $G = \mathrm{Aut}(T_3)$:

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$$\mu(G/\Gamma) = \sum \frac{1}{|\mathrm{Stab}_\Gamma(\tilde{v})|} < \infty$$

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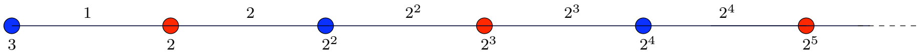

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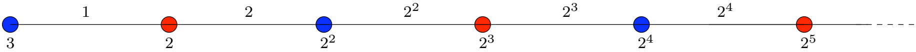

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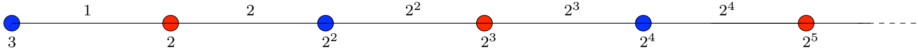

# Finite generation of tree lattices

A uniform tree lattice is always finite generated (fundamental group of finite graph of finite groups).

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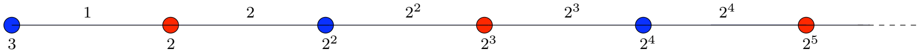
But:

### Theorem (Serre, Bass)

*Let* $\Gamma < \mathrm{Aut}(T)$ *be a* *nonuniform* *tree lattice. Then* $\Gamma$ *is* *not* *finitely generated.*

### Corollary

$SL_2(\mathbb{F}_q[t])$ *is not finitely generated.*



# Polygonal complexes

A polygonal complex is a CW–complex obtained by gluing together convex polygons by isometries along their edges.

All polygons are from the same fixed constant curvature manifold: $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$.

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Tessellations of sphere

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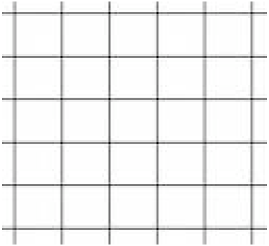
Tessellations of Euclidean plane



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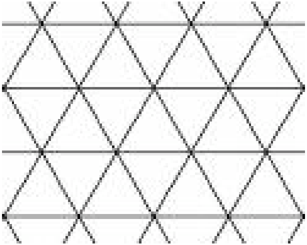

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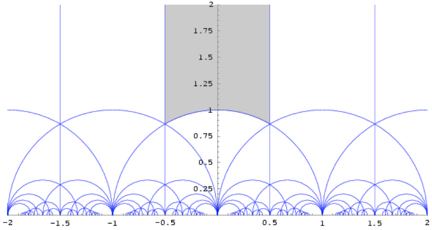

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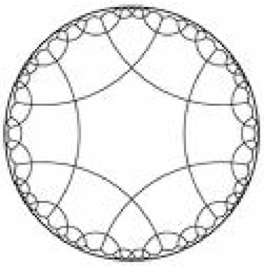
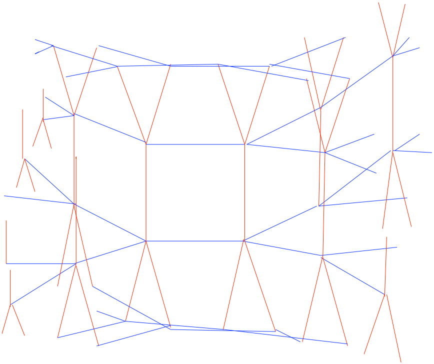


# Product of trees

$T_3 \times T_3$ product of trees



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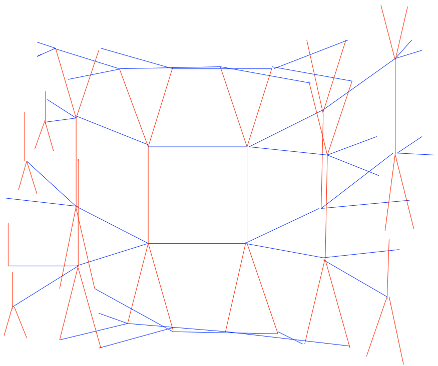

This is the building for $SL_2(\mathbb{F}_2((t))) \times SL_2(\mathbb{F}_2((t^{-1})))$.

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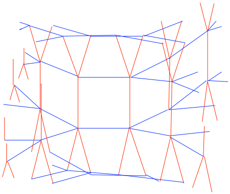

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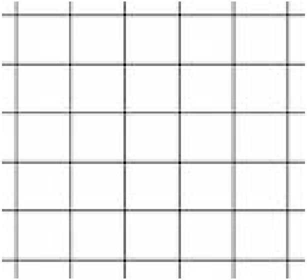
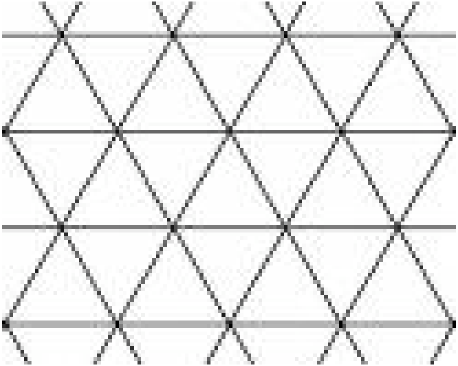


# Buildings

Building for $SL(3, \mathbb{F}_q((t^{-1})))$ has apartments



# Links

$X$ polygonal complex
$v$ vertex of $X$

The link of $v$ in $X$ is the graph $L = \mathrm{Lk}(v, X)$ with

- $\mathrm{Vert}(L) \leftrightarrow$ edges of $X$ containing $v$
- $\mathrm{Edge}(L) \leftrightarrow$ faces of $X$ containing $v$
- Vertices adjacent in $L \iff$ corresp. edges of $X$ share a face

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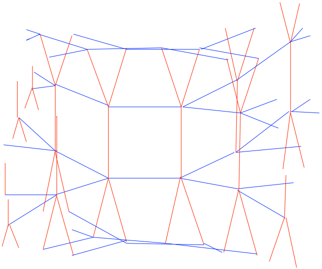
Example

Product of trees $T_3 \times T_3$



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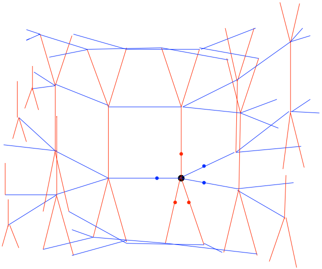

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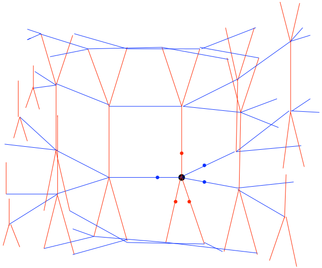
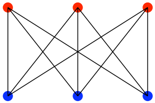
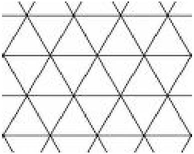


Link is complete bipartite graph $K_{3,3}$

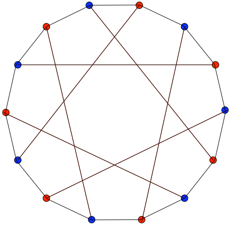


# Examples of links

Building for $SL(3, \mathbb{F}_2((t^{-1})))$ has apartments



and links



# $(k, L)$–complexes

Let $k \geq 3$ and let $L$ be a graph. A $(k, L)$–complex is a polygonal complex such that

- all faces are $k$–gons
- all vertex links are $L$

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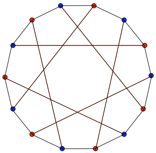
## Examples

1. Product of trees $T_3 \times T_3$ is a $(4, K_{3,3})$–complex.
2. Building for $SL_3(\mathbb{F}_2((t^{-1})))$ is a $(3, L)$–complex where $L$ is



# Bourdon's building $I_{p,q}$

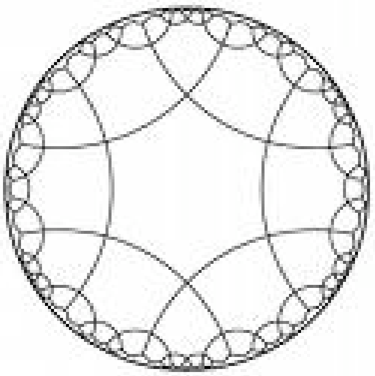
$I_{p,q}$ is a $(p, K_{q,q})$–complex such that:

- all faces are regular right-angled hyperbolic $p$–gons
- all vertex links are $K_{q,q}$

Hyperbolic version of product of trees

# Bourdon's building

$I_{6,2}$: hexagons, links $K_{2,2}$



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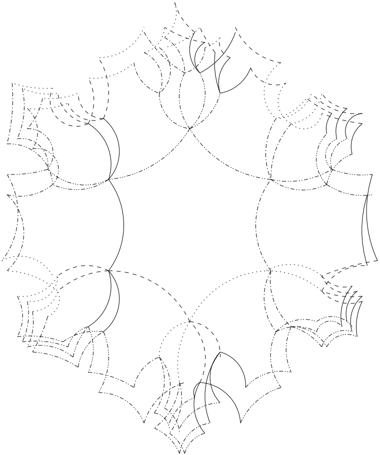

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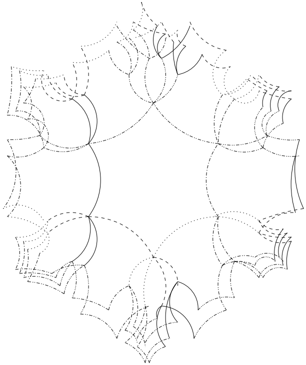
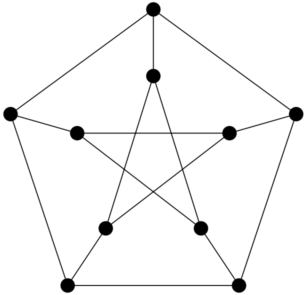


- Hyperbolic building, right-angled building
- Building for certain Kac–Moody groups over finite fields

# A $(k, L)$–complex which is not a building

**Theorem (Swiątkowski, 1999)**

*For $k \geq 4$, there exists a unique simply-connected $(k, L)$–complex where $L$ is Petersen graph*



*or any $s$–arc regular connected trivalent graph, $s \geq 3$.*

# Motivation

### Theorem (Tits)

*Let $\mathcal{G}$ be higher-rank "p-adic Lie group"*
*e.g.* $\mathcal{G} = SL(3, \mathbb{Q}_p), SL(3, \mathbb{F}_q((t^{-1})))$.

*Let $X$ be building for $\mathcal{G}$.*

*Then $\mathcal{G}$ is finite index or cocompact in* $\mathrm{Aut}(X)$.

# Lattices for polygonal complexes

$X$ locally finite polygonal complex
$G$ locally compact group acting cocompactly on $X$

Lattices in $G$ characterised same way as tree lattices: $\Gamma < G$ is

- uniform lattice $\iff$ $\Gamma \curvearrowright X$ with finite stabilisers and finite quotient
- nonuniform lattice $\iff$ $\Gamma \curvearrowright X$ with finite stabilisers and *infinite* quotient, so that

$$\mu(G/\Gamma) = \sum_{v \in X/\Gamma} \frac{1}{|\mathrm{Stab}_\Gamma(\tilde{v})|} < \infty$$

# Finite generation of lattices for polygonal complexes

$G$ has Kazhdan's Property (T)

$\implies$ any lattice $\Gamma < G$ is finitely generated

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$G$ has Kazhdan's Property (T)

$\implies$ any lattice $\Gamma < G$ is finitely generated

**Theorem (Ballmann–Świątkowski, Cartwright–Mantero–Steger–Zappa, Zuk 1990s)**

*Let $L$ be a graph satisfying a certain spectral condition. Let $X$ be a locally finite, simply-connected $(3, L)$–complex.*
*e.g. $X$ = building for $SL_3(\mathbb{F}_q((t^{-1})))$.*
*Then any locally compact group $G$ acting cocompactly on $X$ has Kazhdan's Property (T).*

# Finite generation of lattices for polygonal complexes

If $G$ does *not* have Kazhdan's Property (T), lattices $\Gamma < G$ may or may not be finitely generated.

e.g. all uniform tree lattices are finitely generated, all nonuniform tree lattices are not finitely generated.

# Finite generation of lattices for polygonal complexes

If $G$ does *not* have Kazhdan's Property (T), lattices $\Gamma < G$ may or may not be finitely generated.

e.g. all uniform tree lattices are finitely generated, all nonuniform tree lattices are not finitely generated.

**Theorem (Ballmann–Swiątkowski 1997)**

*Let $k \geq 4$ and let $L$ be a graph of girth $\geq 4$. Let $X$ be a locally finite, simply-connected $(k, L)$–complex. Then a locally compact group $G$ acting cocompactly on $X$ does not have Kazhdan's Property (T).*

**Examples**

Products of trees, Bourdon's building, $(k, L)$–complexes with Petersen graph links . . .

# Irreducible lattices in products

## Definition

Let $G_1$ and $G_2$ be locally compact groups. Let $G = G_1 \times G_2$.
A lattice $\Gamma < G$ is irreducible if it has dense projections to both $G_1$ and $G_2$.

## Examples

1. $\Gamma = SL_2(\mathbb{F}_q[t, t^{-1}])$ is an irreducible lattice in

$$G = SL_2(\mathbb{F}_q((t))) \times SL_2(\mathbb{F}_q((t^{-1})))$$

   $G$ has building the product of trees $T_{q+1} \times T_{q+1}$

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2. Kac–Moody groups over finite fields are irreducible lattices for product of twin buildings.

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## Examples

1. $\Gamma = SL_2(\mathbb{F}_q[t, t^{-1}])$ in $G = SL_2(\mathbb{F}_q((t))) \times SL_2(\mathbb{F}_q((t^{-1})))$.
2. Kac–Moody groups over finite fields.

## Theorem (Raghunathan 1989)

*For $i = 1, 2$, let $G_i$ be a "p-adic Lie group" whose building is a locally finite regular or biregular tree. Then any irreducible lattice in $G = G_1 \times G_2$ is finitely generated.*

## Corollary

$SL_2(\mathbb{F}_q[t, t^{-1}])$ *is finitely generated.*

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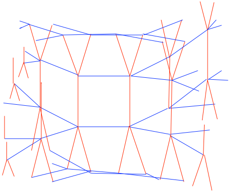
**Question**

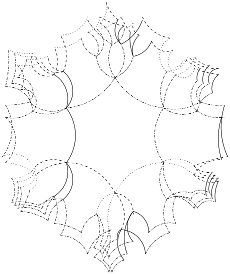
*Let $G_1$ and $G_2$ be any locally compact groups which act distance-transitively on locally finite regular or biregular trees. Is any irreducible lattice in $G = G_1 \times G_2$ finitely generated?*

# Nonuniform lattices for right-angled buildings

## Examples

Right-angled buildings include products of trees, Bourdon's building, . . .





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### Definition

$\Gamma \curvearrowright X$ has a strict fundamental domain if there is a subcomplex $Y \subset X$ containing exactly one point from each $\Gamma$–orbit.

### Theorem (T–Wortman 2010)

*Let $G$ be a locally compact group acting cocompactly on a locally finite right-angled building $X$. Let $\Gamma$ be a nonuniform lattice in $G$. If $\Gamma$ has a strict fundamental domain, then $\Gamma$ is not finitely generated.*

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Proof uses facts about buildings and topological criterion:
$\Gamma$ is finitely generated $\iff \exists\, r > 0$ such that $r$–neighbourhood of a $\Gamma$–orbit is connected.

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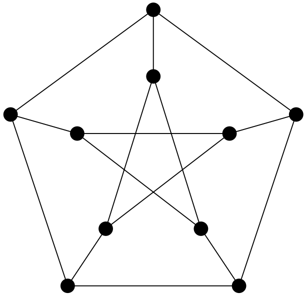
## Question

*If $X$ is "negatively curved" (e.g. Bourdon's building), is every nonuniform lattice not finitely generated?*

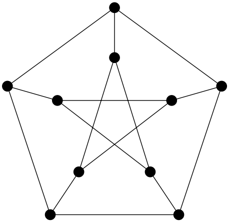
# Nonuniform lattices for other $(k, L)$–complexes



## Partial Result (T 2007)

*Let $k \geq 4$ and let $L$ be the Petersen graph. Let $X$ be the unique simply-connected $(k, L)$–complex. If $k$ has a prime divisor less than 11, then $G = Aut(X)$ admits a nonuniform lattice $\Gamma$, and $\Gamma$ is not finitely generated.*

# Nonuniform lattices for other $(k, L)$–complexes



### Partial Result (T 2007)

*Let $k \geq 4$ and let $L$ be the Petersen graph. Let $X$ be the unique simply-connected $(k, L)$–complex. If $k$ has a prime divisor less than* 11*, then $G = Aut(X)$ admits a nonuniform lattice $\Gamma$, and $\Gamma$ is not finitely generated.*

### Questions

*Do nonuniform lattices exist for other $k$?*
*Are all nonuniform lattices not finitely generated?*