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Handout 4: The classical Lie algebras and their root systems

Here we define the classical Lie algebras. Their root systems are given by the classical root systems A_n, B_n, C_n, and D_n. In the following E_{ij} denotes the matrix with 1 at the intersection of the i-th row and the j-th column and 0 everywhere else. The Lie bracket of E_{ij} and E_{kl} is given by

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \tag{1}$$

A_n — the special linear Lie algebras $\mathfrak{sl}_{n+1}\mathbb{C}$

$\mathfrak{sl}_{n+1}\mathbb{C} = \{X \in \mathfrak{gl}_{n+1}\mathbb{C} \mid \text{trace } X = 0\}$. Its Killing form B and Cartan subalgebra $\mathfrak{h}$ are given by

$$\begin{aligned} \mathfrak{h} &:= \text{diagonal matrices in } \mathfrak{sl}_{n+1}\mathbb{C} \\ &= \mathsf{span}(E_{ii} - E_{jj} | i = 1, \dots n\} \\ B(x, y) &= 2(n+1)\text{trace } (x \circ y) \end{aligned}$$

Lets denote by ϵ_i, $i = 1, \dots, n+1$ the linear form on the diagonal matrices which assigns to a diagonal matrix its i'th entry. By (1) one calculates that the roots of $\mathfrak{sl}(n+1, \mathbb{C})$ are given by:

$$\Delta := \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n+1\}$$

with the roots spaces $\mathbb{C} \cdot E_{ij}$. Calculating the dual and normalized dual vectors we get

$$\begin{aligned} h_{\epsilon_i - \epsilon_j} &= \frac{1}{2(n+1)}(E_{ii} - E_{jj}) \\ H_{\epsilon_i - \epsilon_j} &= E_{ii} - E_{jj} \end{aligned}$$

Hence, the root system of $\mathfrak{sl}_{n+1}\mathbb{C}$ is given by

$$\begin{aligned} \mathfrak{h}_0^* &= \mathsf{span}_{\mathbb{R}}(\Delta) \\ \langle \epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \rangle &= B(h_{\epsilon_i - \epsilon_j}, h_{\epsilon_k - \epsilon_l}) = \frac{1}{2(n+1)} (\delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk}) \end{aligned}$$

This root system is equivalent to A_n.

B_n — the odd-dimensional orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$

$\mathfrak{so}_{2n+1}\mathbb{C} = \{X \in \mathfrak{gl}_{2n+1}\mathbb{C} \mid X + X^t = 0\}$. Set $D_{ij} := E_{ij} - E_{ji}$ for $1 \le i \ne j \le 2n+1$. Then $(D_{ij} | 1 \le i < j \le 2n+1)$ is a basis of $\mathfrak{so}_{2n+1}\mathbb{C}$. Then

$$[D_{ij}, D_{kl}] = \delta_{jk} D_{il} + \delta_{il} D_{jk} - \delta_{jl} D_{ik} - \delta_{ik} D_{jl}. \tag{2}$$

The Killing form is given by

$$B(x, y) = (2n-1)\text{trace } (x \circ y).$$

Consider now the following basis of $\mathfrak{so}_{2n+1}\mathbb{C}$:

$$\begin{aligned} H_i &:= \sqrt{-1} D_{2i-1\ 2i}\,, && && i = 1, \dots n \\ K_i^{\pm} &:= D_{2i-1\ 2n+1} \pm \sqrt{-1} D_{2i\ 2n+1}\,, && && i = 1, \dots n \\ L_{ij}^{\pm} &:= (D_{2i-1\ 2j-1} - D_{2i\ 2j}) && \pm \sqrt{-1}\,(D_{2i-1\ 2j} + D_{2i\ 2j-1}) &, & 1 \le i < j \le n \\ M_{ij}^{\pm} &:= (D_{2i-1\ 2j} - D_{2i\ 2j-1}) && \pm \sqrt{-1}\,(D_{2i-1\ 2j-1} + D_{2i\ 2j}) &, & 1 \le i < j \le n. \end{aligned}$$

Then the H_i's span a Cartan subalgebra $\mathfrak{h}$. Lets denote by η_i, $i = 1, \ldots, n$, the linear form on the Cartan subalgebra $\mathfrak{h}$ given by $\eta_k(H_i) = -\delta_{ik}$. A direct calculation using (2) gives that the roots of $\mathfrak{so}(2n+1, \mathbb{C})$ are given by:

$$\Delta \;:=\; \{\pm\eta_i\}_{i=1}^n \;\cup\; \{\pm(\eta_i + \eta_j)\}_{1\le i<j\le n} \;\cup\; \{\pm(\eta_i - \eta_j)\}_{1\le i<j\le n}$$

$$\text{with the root spaces:} \qquad \mathbb{C}\cdot K_i^{\pm} \quad , \qquad \mathbb{C}\cdot L_{ij}^{\pm} \quad , \qquad \mathbb{C}\cdot M_{ij}^{\pm}$$

This root system is equivalent to B_n.

C_n — the symplectic Lie algebras $\mathfrak{sp}_n\mathbb{C}$

$\mathfrak{sp}_n\mathbb{C} = \{X \in \mathfrak{gl}_{2n}\mathbb{C} \mid X^t J_n + J_n X = 0\}$ with $J_n \overset{\text{def}}{=} \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$, where $\mathbf{1}_n$ is the $n \times n$ identity matrix. This Lie algebra has the Killing form

$$B(x, y) \;=\; 2(n+1)\text{trace}\,(x \circ y).$$

A simple calculation shows that

$$\mathfrak{sp}_n\mathbb{C} = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}_{2n}\mathbb{C} \mid B^t = B \text{ and } C^t = C \right\}$$

and a Cartan subalgebra is given by the diagonal matrices in $\mathfrak{sp}_n\mathbb{C}$,

$$\mathfrak{h} \;=\; \mathsf{span}\left(E_{ii} - E_{i+n\ i+n} \mid 1 \le i \le n\right).$$

The roots spaces are spanned by the following matrices

$$\begin{aligned} Q_{ij} &:= E_{ij} - E_{i+n\ j+n},\ 1 \le i \ne j \le n \\ P_{ij}^{+} &:= E_{i\ j+n} + E_{j\ i+n},\ 1 \le i \le j \le n \\ P_{ij}^{-} &:= P_{ij}^{t} \;=\; E_{j+n\ i} + E_{i+n\ j},\ 1 \le i \le j \le n \end{aligned}$$

The roots are given by

$$\Delta \;:=\; \{\pm 2\epsilon_i\}_{i=1}^n \;\cup\; \{\pm(\epsilon_i + \epsilon_j)\}_{1\le i<j\le n} \;\cup\; \{\epsilon_i - \epsilon_j\}_{1\le i,j\le n}$$

$$\text{with the root spaces:} \qquad \mathbb{C}\cdot P_{ii}^{\pm} \quad , \qquad \mathbb{C}\cdot P_{ij}^{\pm} \quad , \qquad \mathbb{C}\cdot Q_{ij}$$

where the ϵ_i's are defined as above. Thus the root system of the symplectic Lie algebra is isomorphic to C_n.

D_n — the even dimensional orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$

$\mathfrak{so}_{2n}\mathbb{C} = \{X \in \mathfrak{gl}_{2n}\mathbb{C} \mid X + X^t = 0\}$. As for the other orthogonal Lie algebras the Killing form is given by

$$B(x, y) \;=\; 2(n-1)\text{trace}\,(x \circ y).$$

As above, the Cartan subalgebra is spanned by

$$H_i \;:=\; \sqrt{-1}D_{2i-1\ 2i}\ , \qquad i = 1, \ldots n$$

and the roots of $\mathfrak{so}(2n, \mathbb{C})$ are given by:

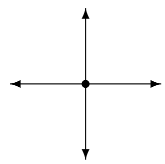
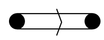
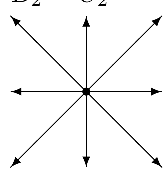
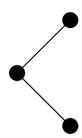
$$\Delta \;:=\; \{\pm(\eta_i + \eta_j)\}_{1\le i<j\le n} \;\cup\; \{\pm(\eta_i - \eta_j)\}_{1\le i<j\le n}$$

$$\text{with the root spaces:} \qquad \mathbb{C}\cdot L_{ij}^{\pm} \quad , \qquad \mathbb{C}\cdot M_{ij}^{\pm}$$

I.e. the root system is equal to D_n.

For small n there are the following **isomorphisms** between these Lie algebras, as it can be seen from the Dynkin diagrams:

(3)

Dynkin diagram	root system	Lie algebra
	$B_1 = C_1 = A_1$	$\mathfrak{so}_3\mathbb{C} \simeq \mathfrak{sp}_1\mathbb{C} \simeq \mathfrak{sl}_2\mathbb{C}$
	$D_2 = A_1 \cup A_1$ 	$\mathfrak{so}_4\mathbb{C} \simeq \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$
	$B_2 = C_2$ 	$\mathfrak{so}_5\mathbb{C} \simeq \mathfrak{sp}_2\mathbb{C}$
	$D_3 = A_3$	$\mathfrak{so}_6\mathbb{C} \simeq \mathfrak{sl}_4\mathbb{C}$

Note that the orthogonal Lie algebras here are defined differently than in Chapter 12 of the book by Erdmann and Wildon, *Introduction to Lie Algebras*, Springer 2006. For us, the orthogonal Lie algebras are given as skew symmetric matrices whereas in the book they are defined in a way that they contain the diagonal matrices as maximal toral subalgebra. Of course, both Lie algebras are isomorphic and the isomorphism can be understood by a change of the basis in $\mathbb{C}^n$.