Journal of Mathematical Analysis and Applications **235,** 274–294 (1999)
Article ID jmaa.1999.6398, available online at http://www.idealibrary.com on IDEAL®

# Lifetime and Illness in Almost Deterministic Maps

Sanjeeva Balasuriya

*Institute of Fundamental Studies, Kandy, Sri Lanka*

*Submitted by Ravi P. Agarwal*

Received August 31, 1998

This paper presents a completely new approach to analysing the effect of stochasticity upon the stability of sets in deterministic maps. Such stability is quantified via the definition of an expected lifetime. More relevant to dynamical systems modelling is the case where the stochasticity is small; the almost deterministic case. Formal arguments are employed to quantify the illness, through which the effect on the lifetime expectancy due to small stochasticity, is obtained. In the spirit of perturbative analysis, such equations are derived in terms of the deterministic dynamics, and the leading order stochasticity alone; the stochastic trajectories need not be computed. Of help is the definition of the ageing (for mortal trajectories) and the ageing exponent (for the immortal case). Special cases (attracting sets, Hamiltonian maps, etc.) are analysed in some detail, leading to a better understanding of the equations derived. © 1999 Academic Press

## 1. INTRODUCTION

Many phenomena in the physical, chemical, and biological world can be modelled via discrete dynamical systems, i.e., by repeatedly iterating maps defined on some subset of $\mathbb{R}^n$. However, a variety of assumptions are implicit in the development of any such model. Of interest is the fact that it is only the dominant effects which are generally considered in a mathematical model; the effect of other factors is assumed to be small, and hence disregarded. The model is considered successful if its predictions bear some semblance of reality, which moreover provides empirical evidence that the ignored factors are indeed unimportant. This procedure has some shortcomings, particularly in the case where there is chaos present in the system. The inclusion of additional, albeit small, effects in a dynamical system may result in completely different trajectories, and the behaviour of the system may change qualitatively and well as quantitatively. For example, if there is a positively invariant set in a system, there is



0022-247X/99 $30.00

no guarantee that it persists in any guise under a small perturbation. Furthermore, because it is often impossible to quantify the precise nature of the perturbation, even numerical investigations have limited value.

An attractive alternative is to assume that the combined effects of *all* the ignored factors can be modelled by introducing a random term into the dynamical system. Thus, the perturbation will be stochastic, and the resulting mathematical model would form a stochastic dynamical system; in the case of a map, this will form an *almost deterministic map*. The random perturbation may be achieved, say, by incorporating white noise into a parameter. The resulting trajectory of the dynamical system is no longer deterministic; one can attribute a probability for each possible trajectory. Determining the probability is difficult in general, but if a certain type of trajectory is frequently observed in nature, then that trajectory must be a highly probable orbit of the stochastic dynamical system. On the other hand, suppose that something close to an invariant set is observed in a natural system, while a deterministic model predicts precisely such an invariant set. This implies that, under the introduction of a small noise, remaining close to the (deterministic) invariant set would be a high probability event; i.e., the set is stable in some sense. Obtaining some sort of measure of this probability (and hence stability of the set) will be the main focus of this paper.

Assuming that a deterministic model containing an invariant set is formulated for some system in nature, the stability of this set under the inclusion of stochasticity (which models the myriad ignored factors) is examined. If this set is not stable under small random fluctuations in this sense, then it could not possibly be observable in nature. Here, ‘‘stability’’ of the set is in a broad sense: that an (almost) invariant set, which is (almost) identical to the original set, persists with probability (almost) one. A precise definition of stability is clearly prohibitive, but can be assumed intimately linked with what is defined as the ‘‘expected lifetime’’ of a trajectory. Loosely speaking, this is the average number of iterates spent within the (previously invariant) set, before exiting the set. A long expected lifetime implies that the set is very stable under stochasticity, whereas a short expected lifetime suggests instability. This paper explores this concept, and obtains several different expressions for the expected lifetime of trajectories of stochastic maps.

The expected lifetime developed in this paper therefore provides a method of testing whether an invariant set predicted by a purely deterministic dynamical system has a chance of being observable in the real world. This concept is developed in Section 2 in terms of one-step probabilities. This is, the quantification of the stability of the set under many iterations of a map, is obtained assuming that information concerning just *one* iterate is known. This section does not insist on the map being in any sense close to a deterministic map, and indeed the results derived are valid for any stochastic map with stationary transition probabilities. In Sections 3 and 4, the case in which the map is close to a deterministic map is examined. The deterministic dynamics are assumed known, and the implications of stochastic perturbations is investigated. The analysis in these sections is purely formal; certain linearisations are assumed without justification, and highest order effects on the stability of a set is determined. The leading order depreciation of lifetime at each age is defined to be the *illness*, and the resulting reduction in the expected lifetime is calculated. Certain special cases (neighbourhoods of attracting cycles, Hamiltonian maps, finite and infinite deterministic lifetimes, etc.) are handled in some detail. Although the development is formal, the results of these sections are of immediate relevance to nature's dynamical systems, because the maps considered are "almost deterministic" in some sense. Given a certain type of stochastic perturbation, the results of Section 4 provide an immediate quantification of the (leading order) effect on the stability of a set, as measured via the lifetime expectancy. It is therefore possible to analyse, using these results, the suitability of using a given model to emulate invariant sets observable in nature, by estimating the probability of these remaining "invariant" under small external noise; the ageing *exponent* developed in Section 4 provides an immediate quantification of this probability.

The formal arguments of Sections 3 and 4 provide a stepping stone to a new approach of quantifying stability of sets under small external noise. It is hoped that these theories can be further developed, possibly with rigorous analysis, to help extend our understanding of the stability of special sets in phase space, under the influence of nature's stochastic interruptions.

## 2. LIFETIME EXPECTANCY

In this section, a mathematical formalism for the investigation of the stability of sets under iterated mappings with stochasticity is developed, via the expected lifetime of trajectories. Consider the stochastic mapping defined for $x \in \Omega$, where $\Omega$ is a subset of $\mathbb{R}^n$, given by

$$x_{i+1} = M(x_i, \sigma), \qquad i \in \mathbb{N}, \tag{1}$$

for a *fixed* initial condition $x_0$. Here, $M$ is a smooth nonlinear mapping, and $\sigma$ is a measure of the stochasticity inherent at each time-step $i$. That is, $M(\cdot, 0)$ would be a purely deterministic mapping and, for example, $\sigma^2$ could be the variance of a noise introduced into a parameter of the mapping. Notice also that, at this point, $M(x_i, \sigma)$ is permitted to be nonlinear in both $x_i$ and in $\sigma$.

The consecutive iterates of Eq. (1) are called a *trajectory*, and can be represented by the random variables $X_0, X_1, X_2, \dots,$ each of which takes values in $\Omega$. This sequence is in fact a Markov process, where $X_0 = x_0$ is specified, while $X_1$, $X_2$, etc., are probabilistically determined [1].

Let $B$ be a Borel-measurable subset [2] of $\Omega$, and suppose that the initial condition $x_0 \in B$. The stability of the set $B$ under the mapping (1) will be examined, via the definition of the *expected lifetime* of the set $B$. Let $L$ be the lifetime of a given trajectory, defined by

$$L = \sup_{i \in \mathbb{N}} \{ i \colon X_j \in B \text{ for } j = 0, 1, 2, \dots, i-1, i \}. \tag{2}$$

That is, if the variables $X_0, X_1, \dots, X_i$ are each in $B$, but $X_{i+1}$ is not in $B$, then the lifetime of that trajectory is $i$. Notice that the imposition $x_0 \in B$ ensures that the lifetime is nonnegative. The lifetime is clearly a random variable; for it to be of use in quantifying the stability of the set $B$, its expected value needs to be calculated. This is first accomplished by defining the probability that the lifetime of a trajectory is precisely $i$ as $q_i$, i.e.,

$$q_i = \Pr\big( X_{i+1} \in \Omega \setminus B, \text{but } X_j \in B, \, j = 0, 1, \dots, i \big). \tag{3}$$

Therefore, it is clear that the expected lifetime (or *lifetime expectancy*) $\langle L \rangle$ (with respect to the set $B$) of trajectories of (1) can be expressed as

$$\langle L \rangle = \sum_{i=0}^{\infty} i q_i . \tag{4}$$

It is necessary to evaluate $q_i$ in order to use the above expression to determine the lifetime expectancy $\langle L \rangle$ of trajectories, and hence estimate the stability of the set $B$. Of interest is obtaining an expression for $q_i$ in terms of one-step transition probabilities, that is, by considering only one iteration of the map (1). Such one-step probabilities are often relatively easily estimated for a given form of the function $M(x_i, \sigma)$ in (1). Thus, if $q_i$ can be adequately described by one-step probabilities, then $\langle L \rangle$, and hence a measure of the stability of the set $B$, can be computed. With this in mind, define the (one-step) *conditional probability density* $D(x \mid y)$, for $y \in \Omega$ and almost all $x \in \Omega$, by

$$D(x \mid y) = \frac{d}{dx} \{ \Pr( M(y, \sigma) = x ) \}, \tag{5}$$

where $\frac{d}{dx}$ denotes the Radon–Nikodým derivative with respect to the Lebesgue measure on $\Omega$ [2, 3]. This represents the probability density of mapping $y \in \Omega$ to a neighbourhood of the point $x \in \Omega$ under one iteration of the map (1). Furthermore, for each $y \in B$, define $P(y)$ to be the (one-step) probability that $M(y, \sigma) \in B$, i.e.,

$$P(y) = \Pr(M(y, \sigma) \in B). \tag{6}$$

Notice that both $D(x \mid y)$ and $P(y)$ implicitly depend on the stochasticity parameter $\sigma$, but this fact is suppressed in the notation for convenience. Hence, for $y \in B$,

$$P(y) = \int_B D(x \mid y)\, dx, \tag{7}$$

where the integral is with respect to Lebesgue measure, and is over the set $B$. An expression for the lifetime expectancy now is developed in terms of the functions $D(x \mid y)$ and $P(y)$, which represent probabilities associated with one iteration of the map (1). Notice first that the probability that the lifetime is exactly zero is given by $q_0 = 1 - P(x_0)$. However, $q_0$ makes no contribution to the sum in (4), and is therefore irrelevant. For the lifetime to be exactly one, $X_1$ must remain within $B$, and $X_2$ must escape. Thus,

$$q_1 = \int_B [1 - P(x_1)]\, D(x_1 \mid x_0)\, dx_1.$$

This expression is obtained by realising that $D(x_1 \mid x_0)\, dx_1$ represents the probability of mapping $x_0$ to a neighbourhood of $x_1$ with vanishingly small measure $dx_1$, and that the next iterate from $x_1$ leaves $B$ with probability $1 - P(x_1)$. By integrating over all possible values of $x_1$ in $B$, the relevant probability $q_1$ is found. By iteratively proceeding in this fashion, it can be seen that, for $i = 1, 2, \ldots,$

$$q_i = \int_{B^i} [1 - P(x_i)] \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}. \tag{8}$$

Here, the symbol $\int_{B^i}$ is used to denote the multiple Lebesgue integral over $B$ crossed with itself $i$ times. It will be convenient to develop the sequence $q_i$ in terms of another sequence $f_i$, which will be defined for $i \in \mathbb{N}$, by

$$f_i = \int_{B^i} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}. \tag{9}$$

It is clear that $f_i$ is the probability that the trajectory beginning at $x_0$ is still within $B$ by the $i$th step; i.e.,

$$f_i = \Pr\{X_0, X_1, \ldots, X_i \in B\}.$$

Thus, the connection between the sequences $q_i$ and $f_i$ can be expressed as

$$q_i = f_i - f_{i+1}, \qquad i \in \mathbb{N}, \tag{10}$$

which may be verified by direct computation using Eqs. (8) and (9) as well. It is also obvious that $f_i$ is a nonincreasing sequence. In fact, under a certain hypothesis, it can be shown that $f_i$ decreases to zero geometrically. The following definition for the quantity $P_\infty$ is required to state the hypothesis.

DEFINITION 1. $P_\infty$ is defined to be the essential supremum of the function $P(y)$ in $B$, with respect to Lebesgue measure; i.e., it is the maximum value achieved by the function $P$ in all sets of the form $B \setminus E$, where $E$ is any subset of $B$ with Lebesgue measure zero. Thus

$$P_\infty := \operatorname{ess\,sup}\{P(y): y \in B\}. \tag{11}$$

HYPOTHESIS 1. *The essential supremum of $P$ (with respect to Lebesgue measure) in the set $B$ is strictly less than one, i.e., $P_\infty < 1$.*

This hypothesis ensures that there is (almost) always a positive probability of escape from the set $B$. In fact, the quantity $P_\infty$ would appear to be a good measure of the stability of the set $B$ in some sense. However, this is only a measure under *one* iteration of the map (1), and does not provide information on the lifetime expectancy of *trajectories*, which is now developed with the help of the following lemma dealing with the decay properties of the sequence $f_i$.

LEMMA 1. *Under Hypothesis 1, for each $i \in \mathbb{N}$,*

$$f_i \le P(x_0) P_\infty^{i-1},$$

*and hence the sequence $f_i$ is geometrically decreasing.*

*Proof.* Assume that $P_\infty$ is chosen in accordance with Hypothesis 1, and thus $P(y) \leq P_\infty$, for almost all $y \in B$. This permits the estimate

$$\begin{aligned} \frac{f_{i+1}}{f_i} &= \frac{\int_{B^{i+1}} D(x_{i+1} \mid x_i)\, dx_{i+1} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}}{\int_{B^i} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}} \\ &= \frac{\int_{B^i} P(x_i) \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}}{\int_{B^i} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}} \\ &\leq \frac{P_\infty \int_{B^i} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}}{\int_{B^i} \prod_{j=1}^{i} \left\{ D(x_j \mid x_{j-1})\, dx_j \right\}} \\ &= P_\infty. \end{aligned}$$

Therefore, $f_{i+1} - P_\infty f_i \leq 0$, with the initial condition $f_1 = \int_B D(x_1 \mid x_0)\, dx_1 = P(x_0)$. Thus, $f_i \leq P(x_0) P_\infty^{i-1}$, and $f_i$ is a geometrically decreasing sequence. ∎

These properties of the sequence $f_i$ enable an alternative expression for the lifetime expectancy, as is expressed in the following proposition.

PROPOSITION 1. *Under Hypothesis* 1, *the expected lifetime of trajectories of* (1) *can also be written as*

$$\langle L \rangle = \sum_{i=1}^{\infty} f_i, \tag{12}$$

*where* $f_i$ *is as defined in* (9). *Moreover,*

$$\langle L \rangle \leq \frac{P(x_0)}{1 - P_\infty}. \tag{13}$$

*Proof.* Incorporating (10) into the expected lifetime definition (4), one obtains

$$\langle L \rangle = \sum_{i=1}^{\infty} i(f_i - f_{i+1}).$$

This summation can be evaluated by computing the partial sum

$$\begin{aligned} \sum_{i=1}^{k} i(f_i - f_{i+1}) &= (f_1 - f_2) + (2f_2 - 2f_3) + (3f_3 - 3f_4) \\ &\quad + \cdots ((k-1)f_{k-1} - (k-1)f_k) + (kf_k - kf_{k+1}) \\ &= (f_1 + f_2 + \cdots + f_k) - kf_{k+1} \\ &= \sum_{i=1}^{k} f_i - kf_{k+1}. \end{aligned}$$

In the limit as $k \to \infty$ of the above, note that $kf_{k+1} \to 0$, because $f_k$ is geometrically decreasing to zero by Lemma 1. Thus, the lifetime expectancy can be expressed very simply in terms of the sequence $f$ as in (12). Moreover, the estimate $f_i \leq P(x_0)P_\infty^{i-1}$ as given in Lemma 1 permits the upper bound

$$\langle L \rangle \leq \sum_{i=1}^{\infty} P(x_0) P_\infty^{i-1} = \frac{P(x_0)}{1 - P_\infty},$$

as required. ∎

The alternative formula for $\langle L \rangle$ derived above is pleasing in that it is a straightforward infinite sum of terms for which the relatively simple definition (9) exists. The explicit evaluation of the sequence $f_i$ may be difficult in general, but (12) provides a compact theoretical tool to estimate the stability of $B$ under repeated iterations of (1). Note also that the upper bound on $\langle L \rangle$ is consistent with the association of $P_\infty \in [0, 1)$ with the one-step stability of the set $B$: a small $P_\infty$ obviously curtails the lifetime, whereas as $P_\infty$ approaches 1, trajectories of (1) reach immortality. Also worthy of note from the above proof is that, even if Hypothesis 1 were not enforced, the formula (12) remains suitable to some extent. As long as the summation on the right of the expression (12) is convergent, it must converge to the lifetime expectancy. Convergence can also be assured by weaker, though more convoluted and less realistic, conditions than Hypothesis (1), which will not be detailed in the interests of clarity.

## 3. ALMOST DETERMINISTIC MAPS

So far, the discussion has been fairly general in that the development is valid for any Markov process with stationary transition probabilities, given by a known conditional probability distribution $D(x \mid y)$. This (and the next) section specialises on stochastic maps which are almost deterministic, in a sense which will be made precise. Such maps are of great relevance in the field of dynamical systems; the persistence of an invariant set of a deterministic map under small external noise is of extreme importance. Nature's inevitable fickle influences can be modelled by such noise, and hence if a certain form of invariant set is observable in nature, it must persist under small random perturbations of a deterministic map. This section will lay the groundwork for the analysis of almost deterministic maps in this sense; directly meaningful results will follow in Section 4. With this in mind, think of the map (1) as a small perturbation on the *deterministic* map

$$x_{i+1} = H(x_i), \tag{14}$$

with the fixed initial condition $x_0 \in B \subset \mathbb{R}^n$. One possible way to imagine a stochastic perturbation on (14) is to enforce the conditional probability distribution function

$$D(x \mid y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[ -\frac{(x - H(y)) \cdot (x - H(y))}{2\sigma^2} \right], \quad (15)$$

where $\sigma$ is a small positive quantity. This particular form models the point $y$ being mapped to points $x$ distributed normally around the point $H(y)$ with variance $\sigma^2$. Such a Gaussian distribution would be a natural assumption from probability theory and the central limit theorem [1]. In the nonanalytical limit as $\sigma \to 0$, this distribution collapses to the Dirac delta distribution $\delta(x - H(y))$, in which case the point $y$ is mapped to the point $H(y)$ with probability 1. Thus, the limit $\sigma \to 0$ of (15) represents the deterministic map (14), and hence (15) with small $\sigma$ is an appropriate expression for the inclusion of small noise into (14).

Analysis using (15) proves to be difficult, so consider instead the conditional probability distribution

$$D(x \mid y) = \delta[x - H(y)] - \sigma g[x - H(y)] + \mathcal{O}(\sigma^2), \quad (16)$$

for almost all $x, y \in B$. Here, $\sigma$ is a small positive parameter which is a measure of the imposed noise (for example, $\sigma$ may be any positive monotone increasing function of the variance of the noise). As before, $\delta$ is the Dirac delta distribution, and $g$ is a function which is nonnegative, thus ensuring that the resulting probabilities do not exceed 1. The function $g$ is assumed known: the distribution of the noise imposed is known in this linearised fashion. The argument of $g$ is taken to be $x - H(y)$ because that is the vector between the expected position $H(y)$ and the actual mapped position $x$, and it is reasonable to imagine that the probability density is associated with precisely this vector. In fact, the length of this vector may be even more appropriate as an argument for $g$; this special case is discussed later. The expression (16) is a perturbative expansion in the parameter $\sigma$, and it is apparent that $\sigma = 0$ corresponds to the purely deterministic map (14). Equation (16) is a purely formal expression; in practise, such an expansion is not always available for a given form of $D(x \mid y)$. However, it is possible to furnish a *formal* derivation of the first-order correction to the lifetime expectancy, by using the ansatz (16). That is, the reduction in the expected lifetime of a deterministic trajectory under the influence of vanishingly small external noise is derived. This result provides an estimate on the leading order stability of the set $B$ under the almost deterministic map described by the distribution (16).

First, note that $H$ is a purely deterministic map, and hence for a given initial condition $x_0$, the trajectory of (14) is a deterministic sequence in $\mathbb{R}^n$. This sequence will have a deterministic lifetime $\bar{l}$, given by

$$\bar{l} = \sup\left\{j \geq 0: H^k(x_0) \in B \text{ for } k = 0, 1, \ldots, j\right\}. \tag{17}$$

In the event that the trajectory remains within $B$ for all time, $\bar{l}$ will be infinity. In any case, under the influence of small stochasticity in the map, the lifetime will in general reduce, because there could be a nonzero probability that each iterate $i \leq \bar{l}$ may exit $B$. Such a reduction depends on the particular trajectory followed, and hence it makes sense to talk of an average reduction in the lifetime. This is derived by using the sequence $f_i$ described in Section 2. However, because $f_i = 0$ for $i > \bar{l}$ for the deterministic map, the introduction of mild stochasticity would seem to make $f_i < 0$, a negative probability! This is clearly nonsense, and therefore, it is only legitimate to talk of a variation in $f_i$ for $i \leq \bar{l}$ in almost deterministic maps. Thus, for almost deterministic systems the sequence $f_i$ will be assumed to have the form

$$f_i = \begin{cases} \int_{B^i} \prod_{j=1}^{i} \left\{D\left(x_j \mid x_{j-1}\right) dx_j\right\}, & i \leq \bar{l}, \\ 0, & i > \bar{l}, \end{cases} \tag{18}$$

where the almost deterministic conditional probability distribution is given by (16). The following expansion for $f_i$ is now possible.

LEMMA 2. *For $i \leq \bar{l}$, the sequence $f_i$ can be formally expanded in $\sigma$ in the form*

$$f_i = 1 - \sigma \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx + \mathcal{O}(\sigma^2), \tag{19}$$

*where the sets $K_j$ are defined for $j \in \mathbb{N}$ by*

$$K_j = \bigcap_{k=0}^{j} H^k(B). \tag{20}$$

*Proof.* Substituting the expansion (16) in (18), and assuming that a formal expansion in $\sigma$ remains legitimate, one obtains

$$f_i = \int_{B^i} \sum_{j=1}^{i} \left\{\delta\left[x_j - H(x_{j-1})\right] - \sigma g\left[x_j - H(x_{j-1})\right] dx_j\right\} + \mathcal{O}(\sigma^2) \tag{21}$$

for $i \le \bar{l}$. Evaluating at $i = 1$,

$$f_1 = \int_B \delta[x_1 - H(x_0)]\, dx_1 - \sigma \int_B g[x_1 - H(x_0)]\, dx_1 + \mathscr{O}(\sigma^2)$$
$$= \mathscr{I}_B[H(x_0)] - \sigma \int_B g[x_1 - H(x_0)]\, dx_1 + \mathscr{O}(\sigma^2),$$

where the standard property of the delta distribution is used to obtain the indicator function of the set $B(\mathscr{I}_B)$ from the first integral. Similarly, evaluating (21) at $i = 2$,

$$f_2 = \int_{B^2} \delta[x_1 - H(x_0)]\, \delta[x_2 - H(x_1)]\, dx_1\, dx_2$$
$$- \sigma \int_B g[x_1 - H(x_0)]\, \delta[x_2 - H(x_1)]\, dx_1\, dx_2$$
$$- \sigma \int_{B^2} \delta[x_1 - H(x_0)]\, g[x_2 - H(x_1)]\, dx_1\, dx_2 + \mathscr{O}(\sigma^2)$$
$$= \int_B \delta[x_1 - H(x_0)] \mathscr{I}_B[H(x_1)]\, dx_1$$
$$- \sigma \int_B g[x_1 - H(x_0)] \mathscr{I}_B[H(x_1)]\, dx_1$$
$$- \sigma \int_B g[x_2 - H^2(x_0)] \mathscr{I}_B[H(x_0)]\, dx_2 + \mathscr{O}(\sigma^2)$$
$$= \mathscr{I}_B[H^2(x_0)] \mathscr{I}_B[H(x_0)] - \sigma \int_{B \cap H(B)} g[x - H(x_0)]\, dx$$
$$- \sigma \int_B g[x - H^2(x_0)]\, dx\, \mathscr{I}_B[H(x_0)] + \mathscr{O}(\sigma^2),$$

Analogous tedious algebra yields the expression for $f_3$:

$$f_3 = \mathscr{I}_B[H(x_0)] \mathscr{I}_B[H^2(x_0)] \mathscr{I}_B[H^3(x_0)]$$
$$- \sigma \int_{B \cap H(B) \cap H^2(B)} g[x - H(x_0)]\, dx$$

$$- \sigma \mathscr{I}_B[H(x_0)] \int_{B \cap H(B)} g[x - H^2(x_0)]\, dx$$

$$- \sigma \mathscr{I}_B[H(x_0)] \mathscr{I}_B[H^2(x_0)] \int_B g[x - H^3(x_0)]\, dx + \mathscr{O}(\sigma^2)$$

$$= \prod_{j=0}^{3} \mathscr{I}_B[H^j(x_0)] - \sigma \sum_{j=1}^{3} \left\{ \prod_{k=1}^{j} \mathscr{I}_B[H^{k-1}(x_0)] \right\}$$

$$\times \int_{\bigcap_{l=0}^{3-j} H^l(B)} g[x - H^j(x_0)]\, dx + \mathscr{O}(\sigma^2).$$

The details of this calculation are omitted for clarity, but are obtained in precisely the same fashion as for $f_2$. By defining the sets $K_j = \bigcap_{k=0}^{j} H^k(B)$, and by extrapolating the procedure used to arrive at the expression for $f_3$, the general expansion for $f_i$ in the form

$$f_i = \prod_{j=0}^{i} \mathscr{I}_B[H^j(x_0)] - \sigma \sum_{j=1}^{i} \left\{ \prod_{k=1}^{j} \mathscr{I}_B[H^{k-1}(x_0)] \right\}$$

$$\times \int_{K_{i-j}} g[x - H^j(x_0)]\, dx + \mathscr{O}(\sigma^2)$$

can be derived. Now note that, for $i \le \bar{l}$, $\prod_{j=0}^{i} \mathscr{I}_B[H^j(x_0)] = 1$, because each indicator function is unity ($H^j(x_0)$ remains in $B$ for all $j \le \bar{l}$ by the definition of $\bar{l}$). Thus, for $i \le \bar{l}$, the indicator functions can be all set to one in the derived expression, resulting in

$$f_i = 1 - \sigma \sum_{j=1}^{i} \int_{K_{i-j}} g[x - H^j(x_0)]\, dx + \mathscr{O}(\sigma^2),$$

as required. ∎

Note that in the above lemma the values of $f_i$ are only obtained for $i \le \bar{l}$; as explained earlier, nonzero $f_i$ for larger values of $i$ makes no sense in the present context. Moreover, it must be remembered that this is a formal result, because the expansion in $\sigma$ has not been justified. Even if the expansion is valid, it must be borne in mind that the result is of use only in the case of very small $\sigma$. In any case, Lemma 2 is useful in that it shows, as an expansion in the noise parameter $\sigma$, the way in which the probability of escape from the set $B$ by the $i$th iterate varies. The effect this has on the expected lifetime of the trajectories is analysed in the next section.

## 4. ILLNESS AND AGEING

It must be emphasised that the entire development of the previous section is based on a linearisation argument in $\sigma$, which has not been adequately justified. Therefore, the results achieved are only correct in a *formal* sense. That is, the results proven implicitly assume that the expansion (16) is valid, and moreover can be extended to the various functions of $\sigma$ that are of interest. In other words, the proofs are not absolutely rigorous in the sense of analytical mathematics, but should provide a first step in a subsequent rigorous analysis.

Recalling that $f_i = 1$ for the purely deterministic case if $i \leq \bar{l}$, Lemma 2 implies that the reduction in the probability that the trajectory remaining in the set $B$ up to the $i$th iterate, is given by

$$\sigma \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx + \mathscr{O}(\sigma^2).$$

This prompts the following definition.

DEFINITION 2. The *illness* at age $i$ is defined by

$$I_i := \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx,$$

for $i \leq \bar{l}$.

The illness is the leading order term in the reduction of the probability of the trajectory being alive at age $i$. Thus, if the expansion in $\sigma$ is valid, a large $I_i$ would imply that the expected chance of being alive at age $i$ is less. The special case $i = 1$ has an illness therefore defined by

$$I_1 = \int_B g\left[x - H(x_0)\right] dx.$$

Because this represents the probability of exiting the set $B$ by age 1, $I_1$ is called the *infant mortality*. It should also be noted that the illness $I_i$ is increasing in $i$, as befits the aged being more likely to become sick. This is clear if it is observed that, because $f_i \geq f_{i+1}$, then

$$1 - \sigma I_i \geq 1 - \sigma I_{i+1} + \mathscr{O}(\sigma^2),$$

and hence $I_{i+1} \geq I_i$ to leading order. Now, possibilities of using the illness as a measure of the instability of a deterministic trajectory under external noise, is discussed, under two cases.

### 4.1. *Mortal Trajectories* ($\bar{l} < \infty$)

The expansion (19) can be used to derive the reduction in the expected lifetime because of external noise. Thus, of interest is deriving an expression for the reduction in the expected lifetime from $\bar{l}$ when small stochasticity is present.

PROPOSITION 2. *If* $\bar{l} < \infty$, *then the* (*modified*) *expected lifetime can be expanded formally as*

$$\langle L \rangle = \bar{l} - \sigma \sum_{i=1}^{\bar{l}} I_i + \mathscr{O}(\sigma^2).$$

*Proof.* The proof of this fact is easily obtained via the chain of equalities

$$\langle L \rangle = \sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\bar{l}} f_i = \sum_{i=1}^{\bar{l}} [1 - \sigma I_i] + \mathscr{O}(\sigma^2) = \bar{l} - \sigma \sum_{i=1}^{\bar{l}} I_i + \mathscr{O}(\sigma^2).$$

∎

Thus, the presence of the noise, characterised by the parameter $\sigma$, reduces the expected lifetime by an amount $\sigma \sum_{i=1}^{\bar{l}} I_i$ to leading order.

DEFINITION 3. If $\bar{l} < \infty$, the *ageing* $R$ is defined by

$$R := \sum_{i=1}^{\bar{l}} I_i,$$

where $I_i$ are the illnesses.

Thus, $R$ is the coefficient of $\sigma$ in the expansion for the reduction of the lifetime due to stochasticity of (small) measure $\sigma$. Notice from Definition 3 that $R$ is the effectively accumulated illness over a lifetime. Now, because $I_i$ is given by Definition 2, $R$ has the form

$$R = \sum_{i=1}^{\bar{l}} \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx. \tag{22}$$

The double summation above is over $\frac{\bar{l}(\bar{l}+1)}{2}$ points of the form $(i, j)$ in the space $\mathbb{N}^2$. Equation (22) therefore provides a compact expression which measures the reduction in the lifespan (the ageing) of trajectories under small external perturbations. Should the value of $R$ be large, the lifetime is expected to shorten significantly. It is clear that $R$ can become large if $\bar{l}$ is large, because a large number of points will be summed over. Therefore, a longer lived deterministic trajectory is more prone to reduction in lifespan under the influence of small stochasticity, as is perhaps obvious intuitively.

The ageing $R$ could also become large by the integrals in (22) providing significant contributions. This is analysed further by assuming a particular form for $g$, which is justified as follows. Notice from the initial expansion (16) that, if the not unreasonable assumption that the conditional distribution function $D(x \mid y)$ is isotropic around $H(y)$ is made, then a possible form for $g$ is

$$g(x) = \|x\|, \tag{23}$$

where $\|\cdot\|$ is the standard Euclidian norm on $\mathbb{R}^n$. This form of $g$ will result in a distribution function not biased in any particular direction; hence the term "isotropic." Moreover, the Euclidian norm can be chosen rather than any other because all norms on $\mathbb{R}^n$ are equivalent. Under this choice of $g$, the infant mortality $I_1$ becomes

$$I_1 = \int_B \|x - H(x_0)\| \, dx.$$

This quantifies how the set $B$ (excluding sets of Lebesgue measure zero) is spread around the point $H(x_0)$; if $I_1$ is small, that means that $B$ is clustered very tightly around $H(x_0)$, or in other words, that $H(x_0)$ is positioned well into the centre of the set $B$. Naturally, this would imply that the infant mortality is small, because in this case $H(x_0)$ must be perturbed a large distance if it is to escape $B$.

All the other expressions derived so far are similarly modified, and in particular, Eq. (22) becomes

$$R = \sum_{i=1}^{\bar{l}} \sum_{j=1}^{i} \int_{K_{i-j}} \|x - H^j(x_0)\| \, dx =: \sum_{i=1}^{\bar{l}} \sum_{j=1}^{i} J_{ij},$$

where the tensor $J_{ij}$ has the structure of a triangular matrix. Each $J_{ij}$ therefore quantifies the lack of tightness of the set $\bigcap_{k=0}^{i-j} H^k(B)$ about the point $H^j(x_0)$, ignoring a set of measure zero. Thus, the ageing is small if the $J_{ij}$s are small; i.e., the set $K_{i-j}$ is closely packed about $H^j(x_0)$, or $H^j(x_0)$ is, qualitatively speaking, close to the centre of the set $K_{i-j}$. That this predicts that, for enhanced lifetime, the point $H^j(x_0)$ must be close to the centre of *a* set is not surprising, but that it should be precisely the set $K_{i-j}$ is somewhat mysterious. In any event, the expression (22) provides an immediate quantification of the lifespan reduction that can be expected of a trajectory subject to small noise, and hence a measure of the instability of the trajectory relative to the set $B$.

### 4.2. *Immortal Trajectories* ($\bar{l} = \infty$)

In the case where $\bar{l} = \infty$, it makes little sense to speak of a reduction in the lifespan, and therefore a slightly different viewpoint is in order. However, the special relevance of the case $\bar{l} = \infty$ must first be mentioned. Of interest in the theory of dynamical systems in the persistence of positively and negatively invariant sets under external stochasticity. This set may be an attractor, a set of fixed points, a periodic cycle, basins of attraction of some object, etc. For example, if the set $B$ is positively invariant under the deterministic map (14), then all iterates $H^i(x_0)$ remain within $B$, and hence $\bar{l} = \infty$.

Consider first the special case where $B$ is both positively and negatively invariant with respect to the map $H$ of (14). That is, it is assumed that $H(B) = B$. It is clear that this means that $H^j(B) = B$ for all $j \in \mathbb{N}$, and hence $K_j = \bigcap_{k=0}^{j} H^k(B) = B$ for all $j \in \mathbb{N}$. This simplifies the illness to

$$I_i = \sum_{j=1}^{i} \int_B g\left[x - H^j(x_0)\right] dx, \tag{24}$$

which is valid for all $i \in \mathbb{N}$. In this case, it is even more apparent than before that $I_i$ is increasing in $i$; the trajectory becomes more prone to exiting $B$ as time progresses. Even if $I_i$ approaches a finite limit as $i \to \infty$, the sum $\sum_{i=1}^{\infty} I_i$ diverges, rendering the expression derived in Proposition 2 indeterminate. Nevertheless, Eq. (24) can be used as a measure of the instability of the iteration process, as a function of $i$. For a given map $H$ and invariant set $B$, the illness $I_i$ is seen to be affected by either changing the initial condition $x_0$, or by choosing a slightly different form of stochastic perturbation (i.e., changing $g$). Thus, even if the precise reduction of the lifetime is not readily quantified for the case of invariant $B$, it is still possible to determine the effect of different forms of small noise on the stability of an invariant set. Should $B$ not necessarily be both positively and negatively invariant, it is still possible to establish some results.

LEMMA 3. *Suppose $B$ is a suitably small neighbourhood of an attracting fixed point* (*or attracting periodic cycle*) *of the deterministic map* (14). *Then the illness $I_i$ is bounded for $i \in \mathbb{N}$.*

*Proof.* Consider first the case of the attracting fixed point. Let $\gamma \in (0, 1)$ be the largest eigenvalue corresponding to the attracting fixed point of (14) contained in $B$. Thus, length contracts by a factor of at most $\gamma$ in any of the $n$ eigenvector directions of the fixed point, if remaining in a suitably small neighbourhood. Since $H(B) \subset B$ because of the attracting nature of the fixed point, the Lebesgue measure $\mu$ of the set $K_j$ satisfies

$$\mu(K_j) = \mu\left(\bigcap_{k=0}^{j} H^k(B)\right) = \mu\big(H^j(B)\big) \leq \mu(B)\gamma^{nj}.$$

Therefore, for any $i \in \mathbb{N}$,

$$\begin{aligned} |I_i| &= \left| \sum_{j=1}^{i} \int_{K_{i-j}} g\big[x - H^j(x_0)\big]\, dx \right| \\ &\leq \left\{\sup_B |g|\right\} \sum_{j=1}^{i} |\mu(K_{i-j})| \\ &\leq a \sum_{j=1}^{i} (\gamma^n)^{i-j} \\ &= a\frac{1-\gamma^{ni}}{1-\gamma^n}, \end{aligned}$$

where $a$ is a constant for suitably small $B$. Thus,

$$I_i \leq \frac{a}{1-\gamma^n},$$

and the illness remains bounded. Now, if $B$ is a small neighbourhood of an attracting periodic cycle of period $m$, a similar argument can be made by considering the map $H^m$. ∎

Because $I_i$ is in general nondecreasing, the above lemma shows that, for this particular form of $B$, the sequence $I_i$ approaches a finite limit; the illness does not increase without bound as the trajectory grows older. Notice also that, for a set $B$ with the strongly attracting properties given in Lemma 3, it can be expected that any trajectory beginning in $B$ would be impervious to small external noise. Hence, if the illness is bounded, one would expect the set $B$ to be extremely stable toward small stochasticity. This is an important criterion that can be used to characterise stability of a set under small noise. Another instructive case is presented in the following lemma.

LEMMA 4. *If the deterministic map* (14) *is Hamiltonian,* $\bar{l} = \infty$, *and the set* $B$ *is finite, then the illness* $I_i$ *grows at most linearly in* $i$.

*Proof.* A Hamiltonian dynamical system conserves phase space area, and hence, if $B$ is any subset of its phase space, $\mu(H(B)) = \mu(B)$ [4]. That is, the Lebesgue measure $\mu$ of the iterated sets of $B$ is conserved, and hence

$$\mu(K_j) = \mu\left( \bigcap_{k=0}^{j} H^k(B) \right) \leq \mu(B),$$

for all $j \in \mathbb{N}$. Furthermore, $K_j \subset B$, and therefore

$$\begin{aligned} I_i &= \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx \\ &\leq \sum_{j=1}^{i} \sup_{K_{i-j}} |g|\, \mu(K_{i-j}) \\ &\leq \sum_{j=1}^{i} \sup_{B} |g|\, \mu(B) \\ &= ai, \end{aligned}$$

where $a$ is some positive constant, and the result is proven. ∎

It is possible to quantify in some sense an ageing for the case where $\bar{l} = \infty$, but it is necessary to index this quantity. Define

$$R_k := \sum_{i=1}^{k} \sum_{j=1}^{i} \int_{K_{i-j}} g\left[x - H^j(x_0)\right] dx = \sum_{i=1}^{k} \sum_{j=1}^{i} J_{ij}. \tag{25}$$

The proof of Proposition 2 indicates that, for the case where $\bar{l} = \infty$, an informal expression for the expected lifetime after the introduction of noise is

$$\langle L \rangle \approx \lim_{k \to \infty} \left[k - \sigma R_k\right] + \mathcal{O}(\sigma^2).$$

It is clear that, if $R_k$ grows greater than linearly in $k$ for large enough $k$, the so-called perturbative term above would dominate: the lifetime computed from the limit may well be finite. Hence, the speed of divergence of $R_k$ is a relevant measure of the suppression of the lifespan of a trajectory with the introduction of noise. For the conditions of Lemma 3,

$$R_k = \sum_{i=1}^{k} I_i \leq \sum_{i=1}^{k} C = Ck,$$

where $C$ is the constant which bounds the illnesses. Therefore, the lifetime expectancy is expected to satisfy

$$\langle L \rangle \approx \lim_{k \to \infty} [k - \sigma C k] + \mathscr{O}(\sigma^2) \to \infty,$$

for small enough $\sigma$. Because the expected lifetime remains infinite, this implies extreme stability of the set $B$. Moreover, $R_k$ grows at most linearly in $k$. However, because the $I_i$s are nondecreasing in $i$, it is clear that the lowest rate of increase of $R_k$ with $k$ that is possible is linear, and therefore the conditions of Lemma 3 represent the most stable case possible. A reasonable quantity to measure the stability in this sense, for the case where $\bar{l} = \infty$, is therefore given in the following definition.

DEFINITION 4. If $\bar{l} = \infty$, then the *ageing exponent* $r$ is defined by

$$r := \inf\left\{t \geq 1 \colon \exists K\ \forall k > K\ \exists C \colon R_k \leq C k^t\right\}.$$

The ageing exponent is therefore the smallest possible choice for $r$ which ensures that $R_k / k^r$ remains bounded for large enough $k$. In other words, it is the order of the rate of increase of $R_k$ with $k$, and it is readily seen that the rate of increase of the illness $I_k$ with $k$ then is of the order $r - 1$. As has been discussed, $r = 1$ implies that the trajectory is very stable under small stochastic perturbations (in the sense of remaining within $B$). A large $r$, on the other hand, means that the lifetime reduction is very large—instability. The ageing exponent $r$ therefore provides a good theoretical object which precisely quantifies stability in the sense that is of interest: for small noise imposed on a (initially immortal) trajectory. Notice that, by these arguments, the following proposition has already been established.

PROPOSITION 3. *If $B$ is a suitably small neighbourhood enclosing an attracting periodic cycle of the deterministic map $H$, then, for a trajectory with initial condition $x_0 \in B$ for the almost deterministic map, the ageing exponent $r = 1$.*

If the map $H$ is Hamiltonian, this imposes additional constraints on the problem, and actually contributes to the stability of the finite set $B$ in the sense that has been described. In fact, as is apparent from the following lemma, the ageing exponent for this case cannot be more than 2, and hence, if $H$ is Hamiltonian, there is very little reduction in the lifespan because of small noise.

PROPOSITION 4. *If $\bar{l} = \infty$, the map $H$ is Hamiltonian, and the set $B$ is finite, then the ageing exponent $r \leq 2$.*

*Proof.* It is shown in Lemma 4 that $I_i \leq ai$ for some constant $i$, for large enough $i$. Thus,

$$R_k = \sum_{i=1}^{k} I_i \leq a \sum_{i=1}^{k} i = a \frac{k(k+1)}{2},$$

which goes as $k^2$ to infinity for large $k$. Hence, there exists a constant $C$ such that for large enough $k$, $R_k \leq Ck^2$, and therefore the ageing exponent $r \leq 2$. ∎

The fact that a Hamiltonian system possesses an ageing exponent which is less than 2 is an indication that such systems are stable in some sense toward small noise. This is not surprising because area preservation ensures that, if a certain direction is unstable, a corresponding stable direction exists. For general maps, however, the quantitative reduction of immortality must be determined through the evaluation of the ageing exponent $r$ via a calculation of $R_k$ from (25).

## 5. CONCLUSIONS AND DISCUSSION

The stability of sets in stochastic maps has been examined, using a completely new approach. The stability in this sense has been quantified via the expected lifetime of trajectories within the set; i.e., the average number of iterates that stay within the set of interest. If this is large, one would expect the set to be stable in the sense that computed, or observed, trajectories would appear to remain within the set. Because deterministic models are only approximately correct for real life systems, observed stable sets from such models would only give appropriate information if the set were stable under small noise.

Thus, this paper has focused primarily on almost deterministic maps; that is, maps in which the stochasticity can be represented in a perturbative form. The result of this stochasticity upon the previously deterministic lifetime expectancy is calculated, and formulae are developed. These formulae are in terms of the deterministic map, and the form of the perturbation only, and do not depend on precise knowledge of the perturbed (stochastic) trajectory. In general, mild stochasticity causes the lifetime expectancy to decrease; the leading order decrease is quantified by the *ageing*. The ageing is the result of accumulated *illness*, which measures the probability that the iterate escapes the set at each step. The mathematical definition of these terms helps in comprehending how the stochasticity affects the trajectory to leading order.

The development in this paper is based on fundamental ideas, and is a new approach to the problem of quantifying stability of sets under small noise. Much of the formulae derived depend on formal perturbative analysis, and as such lack absolute mathematical rigour. Nevertheless, it is expected that these formulae enhance our understanding of the phenomena associated with the stability of sets. In particular, they provide guidelines on using evidence from deterministic models to, say, predict the presence of a basin of attraction for a strange attractor in the *real* problem, in which factors ignored from the deterministic model actually must have a small effect.

There are may possible extensions or improvements that arise out of this paper. Whether the formal perturbative arguments presented here could be rigorously proven is one issue of interest. Extensive analysis of particular examples of almost deterministic systems using the theory developed here is also suggested; it would be instructive to compare the formulae developed with numerical investigations of the stability of sets. A more ambitious project would be to analogously determine the effect of small stochasticity upon continuous dynamical systems; that is, to analyse almost deterministic differential equations with the same viewpoint.

## ACKNOWLEDGMENT

The author thanks Anuradha Ratnaweera and Keerthi Tennekone for initial ideas concerning the lifetime approach to stability.

## REFERENCES

1. S. Karlin and H. M. Taylor, ‘‘A First Course in Stochastic Processes,’’ Academic Press, San Diego, 1975.
2. L. Breiman, ‘‘Probability,’’ SIAM, Philadelphia, 1992.
3. R. G. Bartle, ‘‘The Elements of Integration,’’ Wiley, New York, 1966.
4. J. Guckenheimer and P. Holmes, ‘‘Non-linear Oscillations, Dynamical Systems and Bifurcation of Vector Fields,’’ Springer-Verlag, New York, 1983.