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The general caloron correspondence

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ABSTRACT

We outline in detail the general caloron correspondence for the group of automorphisms of an arbitrary principal G-bundle Q over a manifold X, including the case of the gauge group of Q. These results are used to define characteristic classes of gauge group bundles. Explicit but complicated differential form representatives are computed in terms of a connection and Higgs field.

1. Introduction

The caloron correspondence arose originally [1] as a correspondence between calorons (instantons on $\mathbb{R}^3 \times S^1$) with structure group G and Bogomolny monopoles on $\mathbb{R}^3$ with structure group the loop group of G. Later it was realised that the self-duality and Bogomolny equations can be disregarded and interesting results can be obtained by regarding the caloron correspondence as a correspondence between G-bundles with connection on $M \times S^1$, for some manifold M and loop group bundles with connection and Higgs field on M. In particular in [2], the caloron correspondence was used to calculate the string class of an LG-bundle, and generalised in [3,4], where it was used to define characteristic classes for ΩG-bundles and $LG \rtimes S^1$-bundles. See also [5–7] for related applications of the caloron correspondence.

In the current work, we generalise these constructions by replacing the circle S^1 by an arbitrary compact, connected manifold X. We restrict X to be compact in order to make the various spaces of maps and sections associated to X Fréchet spaces. In summary, if $Y \to M$ is a fibration with fibre X and $\widetilde{P} \to Y$ is a G-bundle, which over a fibre of $Y \to M$ is isomorphic to some G-bundle $Q \to X$, we show that $\widetilde{P} \to Y$ is equivalent to an infinite-dimensional principal bundle $P \to M$ whose structure group is the group $\mathrm{Aut}(Q)$ of automorphisms of the G-bundle $Q \to X$ or a subgroup thereof. Which subgroup occurs depends on whether the fibration $Y \to M$ is a product and whether the bundles are framed. The resulting four possible cases are detailed in Section 2 without proof.

In Section 3, we give the detailed proofs of the correspondence in the most general cases, leaving some of the specialisations for the reader. In the following section, we show how the correspondence works when $\widetilde{P}$ has a connection. In this case, P has a connection and some extra geometric data which we call a Higgs field in analogy with the case when $X = S^1$.

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The main results of the paper are in these two sections and summarised in Theorems 4.8 and 4.9 for the case of a general fibration and Theorems 4.11 and 4.12 for the case when the fibration is a product. In Section 5 we use these constructions to define characteristic classes of gauge group bundles and define differential form representatives for them in terms of connections and Higgs fields. Finally in Section 6 we consider the group of based gauge transformations and provide explicit formulae for its universal characteristic classes.

2. Description of the caloron correspondences

There are four basic caloron correspondences. To introduce notation we will discuss here what they are correspondences between and leave the definition of the actual correspondences until later.

Let us fix $Q \to X$ a (principal) G-bundle. Define $\mathrm{Aut}(Q)$ to be all bundle automorphisms of Q. A bundle automorphism defines a diffeomorphism of X and thus there is a homomorphism $\mathrm{Aut}(Q) \to \mathrm{Diff}(X)$ whose kernel, the gauge group of Q, we denote by $\mathcal{G}$ and whose image we denote by $\mathrm{Diff}^Q(X)$. We therefore have a short exact sequence

$$1 \to \mathcal{G} \to \mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X) \to 1. \tag{2.1}$$

Note that, unlike the case where $X = S^1$, we may not have $\mathrm{Diff}^Q(X)$ equal to $\mathrm{Diff}(X)$. To understand how this can happen note that a diffeomorphism $\psi \in \mathrm{Diff}(X)$ is in the image of the map $\mathrm{Aut}(Q) \to \mathrm{Diff}(X)$ if and only if $\psi^*(Q) \simeq Q$. For example, if $Q \to S^2$ is the standard Hopf bundle of Chern class 1 and ψ is the antipodal map then $\psi^*(Q)$ has Chern class -1, so it cannot be isomorphic to Q. It is a standard fact that if ψ and χ are homotopic then $\psi^*(Q) \simeq \chi^*(Q)$, which shows that the connected component of the identity $\mathrm{Diff}(X)_0$ is a subgroup of $\mathrm{Diff}^Q(X)$.

Notice that if $R \to M$ is an $\mathrm{Aut}(Q)$-bundle it induces an associated bundle with fibre X defined by $R \times_{\mathrm{Aut}(Q)} X$, where $\mathrm{Aut}(Q)$ acts on X via the homomorphism to $\mathrm{Diff}^Q(X)$ above. If $Y \to M$ is a locally trivial fibre bundle with fibre X and structure group $\mathrm{Diff}^Q(X)$ denote by $F(Y) \to M$ its $\mathrm{Diff}^Q(X)$ frame bundle. If $m \in M$ and $f \in F_m(Y)$ then, by definition, $f : X \to Y$ is a diffeomorphism onto the fibre Y_m which we call a *frame* at m. Notice that not all diffeomorphisms $X \simeq Y_m$ are frames unless $\mathrm{Diff}^Q(X) = \mathrm{Diff}(X)$.

Definition 2.1. Let $Y \to M$ be a locally trivial fibre bundle with fibre X and structure group $\mathrm{Diff}^Q(X)$ and $\widetilde{P} \to Y$ be a principal G-bundle. We say that $\widetilde{P}$ has *type* $Q \to X$ if for all $m \in M$ and for all $f \in F(Y)$ we have $f^*(\widetilde{P}) \simeq Q$.

Remark 2.1. Notice that if $f, \tilde{f} \in F(Y)_m$, then $f = \tilde{f} \circ \chi$ for $\chi \in \mathrm{Diff}^Q(X)$ so that $f^*(Q) \simeq \tilde{f}^*(Q)$. Moreover if M is connected and $m' \in M$, we can join m and m' by a path which can be lifted to $F(Y)$ as a path joining $f \in F(Y)_m$ and some $f' \in F(Y)_{m'}$ so that $f^*(Q) \simeq (f')^*(Q)$. Hence for a connected manifold M it suffices to check the type condition at a single point $m \in M$ for a single framing of Y_m.

We have the following correspondences:

1. *The unframed caloron correspondence for fibrations.*
   There is a bijection between isomorphism classes as follows.
   - G-bundles $\widetilde{P} \to Y$ of type $Q \to X$ and;
   - $\mathrm{Aut}(Q)$-bundles $P \to M$ with an isomorphism of spaces over M from $P \times_{\mathrm{Aut}(Q)} X \to M$ to $Y \to M$.
2. *The unframed caloron correspondence for products.*
   If the fibration is a product $Y = M \times X$ then this becomes a bijection between isomorphism classes as follows.
   - G-bundles $\widetilde{P} \to M \times X$ of type $Q \to X$ and;
   - $\mathcal{G}$-bundles $P \to M$.

We will call these two correspondences the *unframed case* to distinguish them from the next two cases. First we make a general definition.

Definition 2.2. We say a fibre bundle $W \to Z$ is *framed* (over a submanifold $Z_0 \subset Z$) if we have chosen a section $s : Z_0 \to W$. We call s a framing (over Z_0).

We need a number of cases of this definition. Firstly for the G-bundle $Q \to X$ we can pick $x_0 \in X$ and $q_0 \in Q_{x_0}$. This amounts to a framing s of Q over the point $Z_0 = \{x_0\}$ which is determined by the image $q_0 = s(x_0)$. In this case we call x_0 and q_0 basepoints for X and Q. Secondly for the fibration $Y \to M$ a framing over M is simply called a framing. Thirdly if $s : M \to Y$ is a framing, we will be interested in G-bundles framed over $s(M) \subset Y$. Again we will call these just framed.

In each case there is a natural notion of morphism that preserves the framing. Consider then $x \in X$ and $q \in Q_x$. By restricting the short exact sequence (2.1) to framed isomorphisms we have a short exact sequence

$$1 \to \mathcal{G}_0 \to \mathrm{Aut}_0(Q) \to \mathrm{Diff}_0^Q(X) \to 1. \tag{2.2}$$

We will be interested in the case where X and Q have basepoints, that is, Q is framed over a point. In this case $\mathcal{G}_0$ and $\mathrm{Aut}_0(Q)$ are the subgroups of $\mathcal{G}$ and $\mathrm{Aut}(Q)$, respectively, which fix the basepoint in Q, and $\mathrm{Diff}_0^Q(X)$ is the subgroup of $\mathrm{Diff}^Q(X)$ which fixes the basepoint of X.

Note that, because $Y \to M$ is not a principal bundle, a global section does not necessarily make it trivial. However, in the case that it *is* trivial we take the framing of $M \times X \to M$ to be $m \mapsto (m, x)$ where x is the basepoint for X. Again we have a notion of the *framed type* of a framed bundle by restricting all morphisms in Definition 2.1 to be framed. Notice that if $R \to M$ is an $\mathrm{Aut}_0(Q)$-bundle it induces an associated framed bundle with fibre X defined by $R \times_{\mathrm{Aut}_0(Q)} X$, where as above $\mathrm{Aut}_0(Q)$ acts on X via the homomorphism to $\mathrm{Diff}_0^Q(X)$.

With these definitions we have the following correspondences.

3. *The framed caloron correspondence for fibrations.*
   There is a bijection between isomorphism classes as follows.
   - Framed G-bundles $\widetilde{P} \to Y$ over a framed fibration $Y \to M$ of framed type $Q \to X$ and;
   - $\mathrm{Aut}_0(Q)$-bundles $P \to M$ with a framed isomorphism of framed spaces over M from $P \times_{\mathrm{Aut}_0(Q)} X \to M$ to $Y \to M$.
4. *The framed caloron correspondence for products.*
   If the fibration is a product $Y = M \times X$ then this becomes a bijection between isomorphism classes as follows.
   - Framed G-bundles $\widetilde{P} \to M \times X$ of type $Q \to X$ and;
   - $\mathcal{G}_0$-bundles $P \to M$.

3. Construction of the caloron correspondences

We note first from [8] that the groups in Eq. (2.1) are Fréchet Lie groups with $\mathrm{Aut}(Q)$ a Fréchet Lie subgroup of $\mathrm{Diff}(Q)$ and $\mathrm{Diff}^Q(X)$ an open subgroup of $\mathrm{Diff}(X)$. We will assume throughout that all infinite-dimensional spaces are Fréchet manifolds. For details see for example [9].

To establish the caloron correspondence we need the following useful fact.

3.1. Principal bundles and extensions

Let

$$1 \to L \xrightarrow{\alpha} H \xrightarrow{\beta} K \to 1$$

be an extension of Fréchet Lie groups, that is a short exact sequence of groups such that α is an immersion and β a submersion in the Fréchet sense. In particular β admits local sections and thus $H \to K$ is a locally trivial L-bundle. Indeed if $s\colon U \to H$ is a local section of $H \to K$ for $U \subset K$, then we can define $U \times L \to H$ by $(k, l) \mapsto s(k)l$ with the inverse $h \mapsto (\beta(h), s(\beta(h))^{-1}h)$.

We are interested in the relationships between the principal bundles in the following diagrams

(3.1)

If $S \to M$ is an H-bundle, then L acts on S on the right and the set of orbits S/L is a principal $H/L = K$-bundle over M. Moreover $S \to S/L$ is a principal L-bundle. Notice that H acts on S/L on the right and the action on S covers this.

As L is normal in H the adjoint action of H on itself fixes L. Given a principal L-bundle $T \to R$ with H acting on R, we say that T is an *H-equivariant bundle* if the L-action on T can be extended to an H-action on T which moreover covers the H-action on R. In the case above we clearly have that $S \to S/L$ is an H-equivariant bundle. We have

Proposition 3.1. *Fix a principal K-bundle $R \to M$. We have a bijective correspondence between isomorphism classes of the following objects:*

(1) *Principal H-bundles $S \to M$ with $S/L \to M$ isomorphic to $R \to M$ as K-bundles; and*
(2) *principal L-bundles $T \to R$ which are H-equivariant for the H-action on R induced by the K-action on R using the homomorphism β.*

Proof. The correspondence is as described above except that we need to check local triviality. In the forward direction if $S \to M$ is locally trivial we can cover M by open sets U so that $S_U \simeq U \times H$ and thus $(S/L)_U \simeq U \times H/L$ so that $S_U \to R_U$ is isomorphic as a principal L-bundle to $U \times H \to U \times H/L$. As the extension $H \to H/L$ is a locally trivial L-bundle the result follows.

In the backwards direction let $m \in M$ and we want to show that there is an open set U containing m such that $T_U \to U$ is a locally trivial H-bundle, namely $T_U \simeq U \times H$. First choose U so that $R \to M$ is trivial, that is $R_U \simeq U \times K$. As a consequence we have a section $U \to R$ and the pullback of $T \to R$ by that section is locally trivial and thus admits a section in some open neighbourhood of m. So without loss of generality we can assume that open neighbourhood is U and we have a section $\sigma: U \times \{1\} \to T$ of T restricted to $U \times \{1\}$. We define

$$\psi : U \times H \to T_U$$
$$(x, h) \mapsto \sigma(x)h.$$

This is a smooth bijection of right H spaces. It suffices to show that its inverse is also smooth. We do this by covering T_U with open sets on which the inverse is manifestly smooth. Let $p_0 \in P_U$ with $\pi(p_0) = (m_0, k_0) \in U \times K$. Choose an open neighbourhood V of k_0 in K with a local section $s: V \to H$ of β. Because $T \to R$ is a locally trivial L-bundle, the map $\tau : T \times_R T \to L$ defined by $t_1\tau(t_1, t_2) = t_2$ is smooth. The restriction ψ^{-1} to V is the smooth map $V \to U \times \beta^{-1}(V)$ given by $p \mapsto (\pi_U(p), s(\pi_K(p))\tau(\sigma(\pi_K(p))s(\pi_K(p)), p))$, where π_U and π_K are the natural projections onto U and K respectively. We have established the desired local triviality. □

3.2. *The unframed caloron correspondence for fibrations*

We wish to define the correspondences which establish the bijection between isomorphism classes of the following objects.

- G-bundles $\widetilde{P} \to Y$ of type $Q \to X$ and;
- $\mathrm{Aut}(Q)$-bundles $P \to M$ with an isomorphism of spaces over M from $P \times_{\mathrm{Aut}(Q)} X \to M$ to $Y \to M$.

We start with $\widetilde{P} \to Y$ a G-bundle of type $Q \to X$. If Z is a space on which G acts on the right (possibly trivially) denote by $\mathrm{Eq}(Q, Z)$ the space of all G-equivariant maps. If G acts trivially on Z then $\mathrm{Eq}(Q, Z) = \mathrm{Map}(X, Z)$. Thinking of $\mathrm{Eq}(Q, \ )$ as a functor apply it to $\widetilde{P} \to Y$. Denote by $\mathrm{Map}^Q(X, Y) \subset \mathrm{Map}(X, Y)$ the image of $\mathrm{Eq}(Q, \widetilde{P})$ under the map $\mathrm{Eq}(Q, \widetilde{P}) \to \mathrm{Map}(X, Y)$. We claim that $\mathrm{Eq}(Q, \widetilde{P}) \to \mathrm{Map}^Q(X, Y)$ is a $\mathcal{G}$-bundle, noting that $\mathrm{Eq}(Q, G) = \mathcal{G}$. To establish local triviality, pick $m \in M$ and choose a contractible open neighbourhood U which can be contracted to m. Because U is contractible it follows that $Y_U = \pi^{-1}(U) \simeq U \times X$. Moreover we have that the restriction of $\widetilde{P}$ to $Y_m \simeq \{m\} \times X$ is isomorphic to Q and so by contractibility of U it follows that $\widetilde{P}_U$, the restriction of $\widetilde{P}$ to Y_U, is diffeomorphic to $U \times Q$. Then $\mathrm{Eq}(Q, \widetilde{P}_U) \simeq \mathrm{Map}(X, U) \times \mathrm{Aut}(Q)$ and $\mathrm{Map}^Q(X, Y_U) \simeq \mathrm{Map}(X, U) \times \mathrm{Diff}^Q(X)$ and the projection is the product of the identity on $\mathrm{Map}(X, U)$ with the projection on $\mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X)$. Local triviality then follows from the fact that $\mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X)$ is a principal $\mathcal{G}$-bundle [8].

Define $\eta: F(Y) \to \mathrm{Map}(X, Y)$ by recalling that elements of $F_m(Y)$ are maps $X \to Y_m \subset Y$. Then $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \to F(Y)$ is a $\mathcal{G}$-bundle and $F(Y) \to M$ is a $\mathrm{Diff}^Q(X)$-bundle. It follows that there is a projection $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \to M$ which we describe as follows. An element of $\eta^*(\mathrm{Eq}(Q, \widetilde{P}))$ is a G-bundle map $\widehat{f}: Q \to \widetilde{P}$ covering some frame $f : X \to Y$. In fact if $\widetilde{P}_m$ denotes the restriction of $\widetilde{P}$ to Y_m, then the fibre of $\eta^*(\mathrm{Eq}(Q, \widetilde{P}))$ above $m \in M$ consists of all G-bundle isomorphisms from $Q \to X$ to $\widetilde{P} \to Y_m$ which cover a frame $X \to Y_m$. That is

$$(\eta^*(\mathrm{Eq}(Q, \widetilde{P})))_m = \mathrm{Eq}(Q, \widetilde{P}_m).$$

We can apply Proposition 3.1 to the case of the exact sequence (2.1) if we can show that the $\mathcal{G}$-action on $\eta^*(\mathrm{Eq}(Q, \widetilde{P}))$ extends to an $\mathrm{Aut}(Q)$-action. If $\hat{\rho} \in \mathrm{Aut}(Q)$ is a lift to $\rho \in \mathrm{Diff}^Q(X)$, then we can make $\hat{\rho}$ act on $\hat{f}$ by pre-composition and this extends the $\mathcal{G}$-action. We denote

$$\mathcal{C}(\widetilde{P}) = \eta^*(\mathrm{Eq}(Q, \widetilde{P}))$$

thought of as an $\mathrm{Aut}(Q)$-bundle.

To see that this is a locally trivial $\mathrm{Aut}(Q)$-bundle, let us choose again $U \subset M$ contractible so that $Y_U \simeq U \times X$ and $\widetilde{P}$ restricted to Y_U is diffeomorphic to $U \times Q$. If we consider now the construction just defined, we will see that there is a natural isomorphism from $\mathcal{C}(\widetilde{P})$ restricted to U to $U \times \mathrm{Aut}(Q)$ which gives a local trivialisation.

To complete the correspondence as described in the previous section, we need to describe the isomorphism

$$\mathcal{C}(\widetilde{P}) \times_{\mathrm{Aut}(Q)} X \to Y$$

of fibre bundles over M. An element of the first space over $m \in M$ has the form $[(\hat{f}, f), x]$ where $f : X \to Y_m$ is a frame. We map this to $f(x) \in Y_m$. The action of $(\hat{\rho}, \rho)$ is given by $((\hat{f} \circ \hat{\rho}, f \circ \rho), \rho^{-1}(x))$ which maps to $f(\rho(\rho^{-1}(x))) = f(x)$.

Consider now the reverse direction. We start with an $\mathrm{Aut}(Q)$-bundle $P \to M$ with an isomorphism of spaces over M from $P \times_{\mathrm{Aut}(Q)} X \to M$ to $Y \to M$. Consider the map

$$P \times_{\mathrm{Aut}(Q)} Q \to P \times_{\mathrm{Aut}(Q)} X.$$

Because $\mathrm{Aut}(Q)$ acts by bundle automorphisms we have a natural action of G on $P \times_{\mathrm{Aut}(Q)} Q$ by $[p, q]g = [p, qg]$. It is straightforward to check that this makes

$$\mathcal{C}^{-1}(P) = P \times_{\mathrm{Aut}(Q)} Q \to P \times_{\mathrm{Aut}(Q)} X = Y$$

a G-bundle.

In this case we can assume that locally we have $P \to M$ of the form $U \times \mathrm{Aut}(Q) \to U$ so that the bundle

$$\mathcal{C}^{-1}(P) = P \times_{\mathrm{Aut}(Q)} Q \to P \times_{\mathrm{Aut}(Q)} X$$

locally looks like

$$U \times \mathrm{Aut}(Q) \times_{\mathrm{Aut}(Q)} Q \to U \times \mathrm{Aut}(Q) \times_{\mathrm{Aut}(Q)} X$$

or

$$U \times Q \to U \times X.$$

We can now use the local triviality of $Q \to X$ as a G-bundle to establish that $\mathcal{C}^{-1}(P) \to Y$ is locally trivial.

The correspondences $\mathcal{C}$ and $\mathcal{C}^{-1}$ are best understood as in [3] as functors between the obvious categories of objects and morphisms although we will not pursue that perspective in the present discussion. They are not inverses in the set-theoretic sense but only in the categorical sense. That is $\mathcal{C}^{-1} \circ \mathcal{C}(\widetilde{P}) \simeq \widetilde{P}$ and $\mathcal{C} \circ \mathcal{C}^{-1}(P) \simeq P$. We construct these isomorphisms next.

We start with $\widetilde{P} \to Y$ a G-bundle of type $Q \to X$ and recall that

$$\mathcal{C}(\widetilde{P})_m = \mathrm{Eq}(Q, \widetilde{P}_m)$$

applying $\mathcal{C}^{-1}$ we have

$$\mathcal{C}^{-1} \circ \mathcal{C}(\widetilde{P}) = \mathrm{Eq}(Q, \widetilde{P}_m) \times_{\mathrm{Aut}(Q)} Q.$$

There is a natural map from this space to $\widetilde{P}_m$ given by $[\widehat{\psi}, q] \mapsto \widehat{\psi}(q)$ which defines a smooth isomorphism of G-bundles which we denote

$$\tau_{\widetilde{P}} : \mathcal{C}^{-1} \circ \mathcal{C}(\widetilde{P}) \to \widetilde{P}. \tag{3.2}$$

In the other direction let $P \to M$ be an $\mathrm{Aut}(Q)$-bundle. Then

$$\mathcal{C}^{-1}(P) = P \times_{\mathrm{Aut}(Q)} Q$$

so that

$$\mathcal{C} \circ \mathcal{C}^{-1}(P)_m = \mathrm{Eq}(Q, P_m \times_{\mathrm{Aut}(Q)} Q).$$

There is a natural map

$$P_m \to \mathrm{Eq}(Q, P_m \times_{\mathrm{Aut}(Q)} Q)$$

defined by $p \mapsto (q \mapsto [p, q])$ which extends to an isomorphism of $\mathrm{Aut}(Q)$-bundles

$$\tau_P : \mathcal{C} \circ \mathcal{C}^{-1}(P) \to P. \tag{3.3}$$

From the categorical viewpoint both these isomorphisms are natural transformations.

3.3. *The unframed caloron correspondence for products*

In the case that the fibration is a product $Y = M \times X$ the caloron correspondence becomes a bijection between isomorphism classes as follows.

- G-bundles $\widetilde{P} \to M \times X$ of type $Q \to X$ and;
- $\mathcal{G}$-bundles $P \to M$.

This can be deduced from the general case as follows. Firstly as $\mathcal{G} \subset \mathrm{Aut}(Q)$, we can apply the construction as before to P to obtain $\widetilde{P} = \mathcal{C}^{-1}(P)$. But in this case we have

$$\mathcal{C}^{-1}(P) = (P \times Q)/\mathcal{G} \to (P \times X)/\mathcal{G} = M \times X$$

as $\mathcal{G}$ acts trivially on X.

Secondly, in the other direction, consider the fibre of $\mathcal{C}(P)_m$ which is

$$\mathrm{Eq}(Q, \widetilde{P}_m).$$

Because $Y = X \times M$ we can pick out in here the subset of isomorphisms $Q \to P_m$ which cover the obvious inclusion $X \to X \times M, x \mapsto (x, m)$. This is naturally acted on by $\mathcal{G}$ and the result is a reduction of the $\mathrm{Aut}(Q)$-bundle to $\mathcal{G}$ which we denote by $\mathcal{C}(\widetilde{P})$. Alternatively note that with the previous construction we obtained a $\mathcal{G}$-bundle

$$\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \to F(Y) = X \times \mathrm{Diff}^Q(X)$$

because $Y = X \times M$. Then we can pull back this $\mathcal{G}$-bundle with the obvious section of $X \times \mathrm{Diff}^Q(X)$. Finally using the map $\bar{\eta} : M \to \mathrm{Eq}(X, M \times X)$ given by $m \mapsto (x \mapsto (m, x))$, we get $\mathcal{C}(\widetilde{P}) = \bar{\eta}^*(\mathrm{Eq}(Q, \widetilde{P}))$.

3.4. The framed caloron correspondence for fibrations

We sketch briefly the correspondences in the framed case. We want to show that there is a bijection between isomorphism classes as follows.

- Framed G-bundles $\widetilde{P} \to Y$ over a framed fibration $Y \to M$ of framed type $Q \to X$ and;
- $\mathrm{Aut}_0(Q)$-bundles $P \to M$ with a framed isomorphism of framed spaces over M from $P \times_{\mathrm{Aut}_0(Q)} X \to M$ to $Y \to M$.

It suffices to show how to define the framings on the transformed objects. Consider $\mathcal{C}(\widetilde{P})$ whose fibre at $m \in M$ is

$$\mathcal{C}(\widetilde{P})_m = \mathrm{Eq}(Q, \widetilde{P}_m).$$

But now both sides have basepoints so we can restrict to those isomorphisms preserving the basepoints and call this

$$\mathrm{Eq}_0(Q, \widetilde{P}_m) \subset \mathrm{Eq}(Q, \widetilde{P}_m),$$

which defines a reduction to $\mathrm{Aut}_0(Q)$. Consider the isomorphism $P \times_{\mathrm{Aut}_0(Q)} X \to M$ to $Y \to M$ which we have defined above to be

$$[(\hat{\psi}, \psi), x] \mapsto \psi(x).$$

The framing of the first space is given by the basepoint $[(\hat{\psi}, \psi), x_0]$ and ψ preserves basepoints so that $[(\hat{\psi}, \psi), x_0]$ maps to $\psi(x_0)$ which must be the basepoint.

In the other direction the construction of $\mathcal{C}^{-1}(P)$ is

$$\mathcal{C}^{-1}(P) = P \times_{\mathrm{Aut}_0(Q)} Q \to P \times_{\mathrm{Aut}_0(Q)} X \simeq Y$$

and both $P \times_{\mathrm{Aut}_0(Q)} Q$ and $P \times_{\mathrm{Aut}_0(Q)} X$ have natural framings coming from the basepoints of Q and P and the fact that these are preserved by $\mathrm{Aut}_0(Q)$.

We leave it as an exercise to show that the isomorphisms $\mathcal{C} \circ \mathcal{C}^{-1}(P) \simeq P$ and $\mathcal{C}^{-1} \circ \mathcal{C}(\widetilde{P}) \simeq \widetilde{P}$ preserve the framings just defined.

3.5. The framed caloron correspondence for products

If the fibration is a product $Y = M \times X$ then this becomes a bijection between isomorphism classes as follows.

- Framed G-bundles $\widetilde{P} \to M \times X$ of type $Q \to X$ and;
- $\mathcal{G}_0$-bundles $P \to M$.

We leave this case also as an exercise for the reader.

4. The caloron correspondence with connections and Higgs fields

In this section we will extend the various caloron correspondences from the previous sections to include the data of connections on the bundles involved.

4.1. Introduction

Before considering how to extend the caloron correspondence to a correspondence for bundles with connections, we need to recall some facts about principal bundles and to introduce some notation. If $\pi : P \to M$ is a principal L-bundle we denote the right action of $l \in L$ on P by $R_l : P \to P$ and the induced action on forms and tangent vectors R_l^* and $(R_l)_*$ respectively. If $\lambda \in \mathfrak{l} = T_e L$ the Lie algebra of L, then λ defines a so-called fundamental vector field at any $p \in P$ which we denote by $\iota_p(\lambda)$. Writing $t_0(t \mapsto \gamma(t))$ for the tangent to the map γ at $t = 0$, we have

$$\iota_p(\lambda) = t_0(t \mapsto p \exp(t\lambda)) \in T_p P.$$

If $l \in L$ then $(R_l)_*(\iota_p(\lambda)) = t_0(t \mapsto p \exp(t\lambda) l) = t_0(t \mapsto pl(l^{-1}) \exp(t\lambda) l) = \iota_{pl}(\mathrm{ad}(l^{-1})(\lambda))$. A connection one-form ω on P is an $\mathfrak{l}$-valued one-form on P satisfying $\omega_p(\iota_p(\lambda)) = \lambda$ and $R_l^*(\omega) = \mathrm{ad}(l^{-1})\omega$.

If L is a subgroup of H which also acts freely on the right of P, extending the action of L, the connection ω is called H-invariant if it is fixed by H. A straightforward calculation shows the following lemma.

Lemma 4.1. *The connection ω is H-invariant if and only if*

$$R_h^*(\omega) = \mathrm{ad}(h^{-1})\omega.$$

Consider a fibration $\pi : Y \to M$ with fibre X and $F(Y)$ its associated frame bundle which is a principal $\mathrm{Diff}^Q(X)$-bundle. We denote by $T_y^v Y$ the vertical tangent vectors or the tangents to the fibres of π at $y \in Y$. A connection a on Y is a complementary subspace to T_y^v at every y or, equivalently a projection $v_a\colon T_yY \to T_y^vY$. A choice of connection on $Y \to M$ defines a connection on $F(Y)$ as follows. The tangent space to $f \in F(Y)$ is the subspace of $\Gamma(X, f^*(TY))$ of vectors whose projection to $T_{\pi(f)}M$ is constant. The Lie algebra of $\mathrm{Diff}^Q(X)$ is $\Gamma(X, TX)$. If $\xi \in \Gamma(X, f^*(TY))$ is a tangent vector then $v_a(\xi(x))$ is a vertical vector at $f(x)$. We have $f\colon X \to Y_{\pi(f(x))}$ a diffeomorphism so we can define $a_f(\xi) \in \Gamma(X, TX)$ by

$$a_f(\xi)(x) = (f^{-1})_*(v_a(\xi(x))).$$

Equivalently ξ is horizontal if and only if $\xi(x)$ is horizontal for all $x \in X$.

4.2. *Principal bundles with connections and extensions*

First we need to reconsider the discussion from Section 3.1 taking into account connections on the bundles in question. We have the same structure as before. An exact sequence of groups

$$1 \to L \xrightarrow{\alpha} H \xrightarrow{\beta} K \to 1$$

and a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\;L\;} & R \\ {\scriptstyle H}\Big\downarrow & \swarrow{\scriptstyle K} & \\ M & & \end{array} \tag{4.1}$$

If we fix a point $s \in S_m$, the fibre of S above $m \in M$, there is an H-equivariant isomorphism

$$\begin{array}{ccc} H & \longrightarrow & S_m \\ \Big\downarrow & & \Big\downarrow \\ K & \longrightarrow & R_m \end{array} \qquad\qquad \begin{array}{ccc} h & \longmapsto & sh \\ \Big\downarrow & & \Big\downarrow \\ k & \longmapsto & \pi(s)\beta(h) \end{array} \tag{4.2}$$

determined by the choice of s. Here π is the projection $S \to R$. The L-bundle $H \to K$ has two H-actions given by left and right multiplication. We are interested in the right action which does not commute with the L-action but extends it. An H-invariant connection for this action is determined by its value at the identity in H which is a splitting of the exact sequence of Lie algebras

$$0 \to \mathfrak{l} \xrightarrow{\alpha} \mathfrak{h} \xrightarrow{\beta} \mathfrak{k} \to 0. \tag{4.3}$$

Let $\mathrm{Split}(\mathfrak{h}, \mathfrak{k})$ denote the affine space of all right splittings of (4.3). We define a left action of H on $\sigma \in \mathrm{Split}(\mathfrak{h}, \mathfrak{k})$ by $h\sigma = \mathrm{ad}(h)\sigma\,\mathrm{ad}(\beta(h)^{-1})$. To extend Proposition 3.1 we first need the following Definition.

Definition 4.2. If $S \to M$ is an H-bundle, a *Higgs field* is a section Φ of the associated bundle $S \times_H \mathrm{Split}(\mathfrak{h}, \mathfrak{k})$ or equivalently a function $\Phi\colon S \to \mathrm{Split}(\mathfrak{h}, \mathfrak{k})$ satisfying

$$\Phi(sh) = \mathrm{ad}(h^{-1})\Phi(s)\,\mathrm{ad}(\beta(h)).$$

Recall that a right splitting of the exact sequence (4.3) gives rise to a *left splitting* and vice-versa. Occasionally we will need to distinguish between these and we will use the notation Φ^r and Φ^l for right and left splittings respectively. They are related by

$$\Phi^r\beta + \alpha\Phi^l = \mathrm{id}_{\mathfrak{h}}\,. \tag{4.4}$$

For the moment it is most natural to use right splittings for the value of the Higgs field but we warn the reader that from Proposition 4.6 onwards we will be assuming the Higgs field is a left splitting. We will also adopt the convention that α is an inclusion and hence not explicitly referred to.

The reason for introducing the Higgs field is the following. If we start with a connection ω^H on $S \to M$ then it is an $\mathfrak{h}$-valued one-form on S. It defines, in standard fashion, a connection ω^K on the associated bundle $R \to M$ which is a $\mathfrak{k}$ valued one-form on R. If we introduce an H-invariant connection ω^L on $S \to R$ then this is an $\mathfrak{l}$-valued one-form on S and these objects are related by the equation

$$\omega^H = \omega^L + \Phi(\pi^*\omega^K) \tag{4.5}$$

where we have denoted the projection $S \to R$ by π and have used the fact that $\Phi(s)\colon \mathfrak{k} \to \mathfrak{h}$ for all $s \in S$. This equation gives a correspondence between connections on the one hand and connections and Higgs fields on the other hand, which we establish in detail in the next proposition.

Proposition 4.3. *Fix a principal K-bundle $R \to M$ with connection ω^K. We have a bijective correspondence between isomorphism classes of the following objects:*

(1) *Principal H-bundles $S \to M$, with $S/L \to M$ isomorphic to $R \to M$ as K-bundles with connection ω^H, which projects to ω^K under the isomorphism, and Higgs field Φ; and*
(2) *Principal L-bundles $T \to R$ which are H-equivariant for the H-action on R induced by the K-action on R using the homomorphism β, with connection ω^L which is H-invariant.*

Proof. We assume the isomorphisms from Proposition 3.1 are in place so it is just a question of constructing the connections. As noted above for convenience we regard $\mathfrak{l}$ as included in $\mathfrak{h}$ and suppress the function α and we expect the various connections and the Higgs field to be related by Eq. (4.5). We have from Lemma 4.1 that H-invariance of ω^L is equivalent to $R_h^*\omega^L = \mathrm{ad}(h^{-1})\omega^L$. Notice that this make sense because $\mathrm{ad}(h^{-1})(\mathfrak{l}) = \mathfrak{l}$ and that if $h \in L$ this is just part of the condition satisfied by any connection.

First we do the forwards direction. Assuming that ω^H and Φ are given, we show that the one-form ω^L defined by Eq. (4.5) is an H-invariant connection on $S \to R$. Let $\lambda \in \mathfrak{l}$ and $s \in S$, then

$$\begin{aligned}\omega^L_s\big(\iota^L_s(\lambda)\big) &= \omega^H\big(\iota^L_s(\lambda)\big) - \Phi(s)\big(\omega^K_{\beta(s)}(\pi_*(\iota^L_s(\lambda)))\big)\\ &= \omega^H\big(\iota^H_s(\alpha(\lambda))\big) - \Phi(s)\big(\omega^K_{\beta(s)}(0)\big)\\ &= \lambda\end{aligned}$$

as required. Letting $h \in H$ and $\xi \in T_sS$ we have

$$\begin{aligned}\alpha\big((R_h^*\omega^L_{sh})(\xi)\big) &= \alpha\big(\omega^L_{sh}((R_h)_*(\xi))\big)\\ &= \omega^H_{sh}\big((R_h)_*(\xi)\big) - \Phi(sh)\big(\omega^K_{\pi(s)\beta(h)}((R_{\beta(h)})_*(\pi_*(\xi)))\big)\\ &= \mathrm{ad}(h^{-1})\omega^H_s(\xi) - \mathrm{ad}(h^{-1})\Phi(s)\,\mathrm{ad}(\beta(h))\,\mathrm{ad}(\beta(h)^{-1})\omega^K_{\pi(s)}(\pi_*(\xi))\\ &= \mathrm{ad}(h^{-1})\big\{\omega^H_s(\xi) - \Phi(s)(\pi^*(\omega^K_s(\xi)))\big\}\\ &= \mathrm{ad}(h^{-1})(\omega^L_s)(\xi).\end{aligned}$$

Hence $R_h^*\omega^L = \mathrm{ad}(h^{-1})\omega^L$ and we conclude that ω^L is an H-invariant connection on $S \to R$.

Consider the reverse direction. We are given ω^L which is H-invariant and we wish to manufacture Φ and define ω^H using Eq. (4.5). We define Φ as follows. Let $s \in S$ and $\kappa \in \mathfrak{k}$. Consider $\iota^K_{\pi(s)}(\kappa) \in T_{\pi(s)}R$ and lift it to a horizontal vector $\widehat{\iota^K_{\pi(s)}(\kappa)}$ at $s \in S$ using ω^L. The projection of this vector to M is the projection of $\iota^K_{\pi(s)}(\kappa)$ to M which is zero, so it must be vertical for H. Hence we can define $\Phi(s)\colon \mathfrak{k} \to \mathfrak{h}$ by

$$\iota^H_s(\Phi(s)(\kappa)) = \widehat{\iota^K_{\pi(s)}(\kappa)}.$$

We need to check that this is a Higgs field. First we check it splits. We have

$$\begin{aligned}\iota^K_{\pi(s)}(\kappa) &= \pi_*\big(\widehat{\iota^K_{\pi(s)}(\kappa)}\big)\\ &= \pi_*\big(\iota^H_s(\Phi(s)(\kappa))\big)\\ &= \iota^K_{\pi(s)}\big(\beta(\Phi(s)(\kappa))\big),\end{aligned}$$

where the last line follows from the fact that the H-action on S covers the H-action on R induced by $\beta\colon H \to K$. Thus we have

$$\kappa = \beta\big(\Phi(s)(\kappa)\big)$$

as required. Next we check that it transforms correctly:

$$\begin{aligned}\iota^H_{sh}\big(\mathrm{ad}(h^{-1})\Phi(s)(\kappa)\big) &= (R_h)_*\big(\iota^H_s(\Phi(s)(\kappa))\big)\\ &= (R_h)_*\big(\widehat{\iota^K_{\pi(s)}(\kappa)}\big)\\ &= \widehat{(R_{\beta(h)})_*\big(\iota^K_{\pi(s)}(\kappa)\big)}\\ &= \iota^K_{\pi(sh)}\big(\widehat{\mathrm{ad}(\beta(h)^{-1})(\kappa)}\big)\\ &= \iota^H_{sh}\Big(\Phi(sh)\big((\mathrm{ad}(\beta(h)^{-1})(\kappa))\big)\Big)\end{aligned}$$

hence

$$\Phi(sh) = \mathrm{ad}(h^{-1})\Phi(s)\,\mathrm{ad}(\beta(h)).$$

It remains to show that ω^H defined by Eq. (4.5) is a connection. First we show that $\omega^H_s(\iota^H_s(\eta)) = \eta$ for all $\eta \in \mathfrak{h}$. For such an η we have

$$\eta = \lambda + \Phi(s)(\beta(\eta))$$

for some $\lambda \in \mathfrak{l}$. Again we suppress $\alpha : \mathfrak{l} \to \mathfrak{h}$. It follows that

$$\begin{aligned}\iota^H_s(\eta) &= \iota^H_s(\lambda) + \iota^H_s(\Phi(s)(\beta(\eta)))\\ &= \iota^L_s(\lambda) + \iota^K_{\pi(s)}\widehat{(\beta(\eta))}\end{aligned}$$

and thus we have

$$\begin{aligned}\omega^H_s(\iota^H_s(\eta)) &= \omega^L_s(\iota^H_s(\eta)) + \Phi(s)(\omega^K_{\pi(s)}\pi_*(\iota^H_s(\eta)))\\ &= \omega^L_s(\iota^L_s(\lambda)) + \omega^L_s(\iota^K_{\pi(s)}\widehat{(\beta(\eta))}) + \Phi(s)(\omega^K_{\pi(s)}\pi_*(\iota^L_s(\lambda))) + \Phi(s)(\omega^K_{\pi(s)}\pi_*(\iota^K_{\pi(s)}\widehat{(\beta(\eta))}))\\ &= \omega^L_s(\iota^L_s(\lambda)) + 0 + 0 + \Phi(s)(\omega^K_{\pi(s)}(\iota^K_{\pi(s)}(\beta(\eta))))\\ &= \lambda + \Phi(s)(\beta(\eta))\\ &= \eta.\end{aligned}$$

Next we establish the right equivariance of the connection form, $R_h^*\omega^H_{sh} = \mathrm{ad}(h^{-1})\omega^H_s$. Let $\xi \in T_sS$ and write

$$\xi = \hat{\xi} + \iota_s(\lambda)$$

where $\hat{\xi}$ is horizontal for ω^L and $\lambda \in \mathfrak{l}$. Then

$$(R_h)_*(\xi) = (R_h)_*(\hat{\xi}) + \iota_{sh}(\mathrm{ad}(h^{-1})(\lambda))$$

and $(R_h)_*(\hat{\xi})$ is horizontal as ω^L is H-invariant. We have

$$\begin{aligned}(R_h^*\omega^H_{sh})(\xi) &= \omega^H_{sh}\,((R_h)_*(\xi))\\ &= \omega^L_{sh}\,((R_h)_*(\xi)) + \Phi(sh)\big(\omega^K_{\pi(s)\beta(h)}(\pi_*((R_h)_*(\xi)))\big)\\ &= \omega^L_{sh}\big((R_h)_*(\hat{\xi})\big) + \omega^L_{sh}\big(\iota_{sh}(\mathrm{ad}(h^{-1})(\lambda))\big) + \Phi(sh)\big(\omega^K_{\pi(s)\beta(h)}(\pi_*((R_h)_*(\xi)))\big)\\ &= 0 + \omega^L_{sh}((R_h)_*\iota_s(\lambda)) + \Phi(sh)\big(\omega^K_{\pi(s)\beta(h)}((R_{\beta(h)})_*\pi_*(\xi))\big)\\ &= \mathrm{ad}(h^{-1})\omega^L_s(\lambda) + \mathrm{ad}(h^{-1})\Phi(s)\big(\mathrm{ad}(\beta(h))\,\mathrm{ad}(\beta(h)^{-1})\omega^K_{\pi(s)}(\pi_*(\xi))\big)\\ &= \mathrm{ad}(h^{-1})\omega^L_s(\xi) + \mathrm{ad}(h^{-1})\Phi(s)\big(\omega^K_{\pi(s)}(\pi_*(\xi))\big)\\ &= \mathrm{ad}(h^{-1})\omega^H_s(\xi)\end{aligned}$$

as required.

We leave it as an exercise for the reader to show that when we apply the bijection twice the isomorphisms introduced in Proposition 3.1 preserve the connections defined here. □

For the caloron correspondence we need the following slightly more complicated version of Proposition 4.3. Choose an affine subspace $\mathrm{Split}(\mathfrak{h}, \mathfrak{k})_0 \subset \mathrm{Split}(\mathfrak{h}, \mathfrak{k})$ which is invariant under the action of H. Using the same notation as before we say a connection ω^L is a $\mathrm{Split}(\mathfrak{h}, \mathfrak{k})_0$-connection if the corresponding Higgs field takes its values in $\mathrm{Split}(\mathfrak{h}, \mathfrak{k})_0$. It is then obvious that we have

Corollary 4.4. *Fix a principal K-bundle $R \to M$ with connection ω^K. We have a bijective correspondence between isomorphism classes of the following objects:*

(1) *Principal H-bundles $S \to M$, with $S/L \to M$ isomorphic to $R \to M$ as K-bundles with connection ω^H, which projects to ω^K under the isomorphism, and Higgs field $\Phi : S \to \mathrm{Split}(\mathfrak{h}, \mathfrak{k})_0$; and*

(2) *principal L-bundles $T \to R$ which are H-equivariant for the H-action on R induced by the K-action on R using the homomorphism β, with a $\mathrm{Split}(\mathfrak{h}, \mathfrak{k})_0$-connection ω^L which is H-invariant.*

4.3. The unframed caloron correspondence for fibrations with connections and Higgs fields

Let a be a connection on $F(Y) \to M$, where $F(Y)$ is the principal $\mathrm{Diff}^Q(X)$-bundle associated to Y. This induces a connection also on Y. We wish to show that there is a bijection between isomorphism classes of

- G-bundles $\widetilde{P} \to Y$ of type $Q \to X$ with connection $\widetilde{A}$ and connection a for $F(Y) \to M$ and;
- $\mathrm{Aut}(Q)$-bundles $P \to M$ with Higgs field Φ and connection A and isomorphism of X fibrations from $P \times_{\mathrm{Aut}(Q)} X$ to Y which sends the connection A to a.

We will prove this result using the constructions from Section 3.2 and Corollary 4.4. Recall that in that case the extension of groups is given by (2.1)

$$1 \to \mathcal{G} \to \mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X) \to 1.$$

The corresponding sequence of Lie algebras is

$$0 \to \Gamma(X, \mathrm{ad}(Q)) \to \Gamma(X, TQ/G) \to \Gamma(X, TX) \to 0, \tag{4.6}$$

which is the functor $\Gamma(X,\ )$ applied to the Atiyah sequence [10] of $Q \to X$ given by

$$0 \to \mathrm{ad}(Q) \to TQ/G \to TX \to 0. \tag{4.7}$$

Remark 4.1. We note for later use that we can identify $\Gamma(X, \mathrm{ad}(Q)) = \Gamma_G(Q, \mathfrak{g})$, the space of G-equivariant maps from Q into $\mathfrak{g}$ and $\Gamma(X, TQ/G) = \Gamma_G(Q, TQ)$, the space of G-equivariant vector fields on Q. The inclusion map of the former into the latter then maps a function $\mu : Q \to \mathfrak{g}$ into the vector field $q \mapsto \iota_q(\mu(q))$.

Recall that a connection on $Q \to X$ is a splitting of the Atiyah sequence (4.7) with a right splitting corresponding to a horizontal distribution and a left splitting to a connection one-form on Q. A splitting of (4.7) also induces a splitting of (4.6), which in turn induces an $\mathrm{Aut}(Q)$-invariant connection on $\mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X)$. We take as $\mathrm{Split}(\Gamma(X, TQ/G), \Gamma(X, TX))_0$ only those splittings arising in this way and we denote them by $\mathcal{A}$. It is straightforward to check that $\mathcal{A}$ is an affine subspace invariant under $\mathrm{Aut}(Q)$ and we are in the setting of Corollary 4.4. Note also that in this situation a Higgs field is an $\mathrm{Aut}(Q)$-equivariant map from P into splittings of the sequence (4.6). In fact, as we shall see below the construction of the Higgs field from the proof of Proposition 4.3 adapted to this context naturally takes values in $\mathrm{Split}(\Gamma(X, TQ/G), \Gamma(X, TX))_0 = \mathcal{A}$. Thus, we can view the Higgs field as an $\mathrm{Aut}(Q)$-equivariant map $P \to \mathcal{A}$, or equivalently a section of $P \times_{\mathrm{Aut}(Q)} \mathcal{A}$.

Proposition 4.5. *Let $f \in \mathrm{Eq}(Q, \widetilde{P})$ and Φ be the Higgs field as constructed in Proposition 4.3. The value of $\Phi(f)$ is the connection $f^*(\widetilde{A})$.*

Proof. We have $f : Q \to \widetilde{P}_m$ covering $\bar{f} : X \to Y_m$. Following the definition of the Higgs field above we take $\mu \in \Gamma(X, TX)$ and choose a one-parameter family of diffeomorphisms in $\mathrm{Diff}^Q(X)$, $\chi_t : X \to X$ such that $\chi_0 = \mathrm{id}_X$ and $\chi_t'(0) = \mu$. We have

$$\iota_f(\eta) = t_0(t \mapsto f \circ \chi_t) \in T_f(F(Y)).$$

The construction of $\Phi(f)$ implies that there is a one-parameter family of bundle automorphisms $\hat{\chi}_t : Q \to Q$ such that $\hat{\chi}_0 = \mathrm{id}_Q$, $\hat{\chi}_t$ covers χ_t and $\mathrm{Eq}(Q, \widetilde{A})_f((f \circ \hat{\chi}_t)'(0)) = 0$ which is equivalent to $\widetilde{A}_{f(q)}((f \circ \hat{\chi}_t(q))'(0)) = 0$ for all $q \in Q$. The splitting $\Phi(f)$ of (4.6) is therefore the map $\mu \mapsto \hat{\chi}_t'(0)$. But because $\widetilde{A}_{f(q)}((f \circ \hat{\chi}_t(q))'(0)) = 0$ for all $q \in Q$, we see that this must be a splitting in $\mathcal{A}$ and one which is associated to the connection one-form $f^*(\widetilde{A})$. □

We also note that in this situation the equivariance condition on the Higgs field is particularly nice.

Proposition 4.6. *Let Φ be a Higgs field for an $\mathrm{Aut}(Q)$-bundle P, viewed as a map $P \to \mathcal{A}$. Then the condition from Definition 4.2 is equivalent to*

$$\Phi(p\psi) = \psi^*\Phi(p),$$

where $\psi \in \mathrm{Aut}(Q)$ and $\beta : \mathrm{Aut}(Q) \to \mathrm{Diff}^Q(X)$.

Proof. First note that if we view Φ as a connection one-form for each $p \in P$, then what we called the Higgs field in Definition 4.2 is really the associated splitting $\Gamma(X, TX) \to \Gamma(X, TQ/G)$. That is, the Higgs field is the assignment of horizontal subspaces of TQ at each point in Q. To avoid confusion for this proof we denote this splitting Φ^r and the connection one-form by Φ^l. They are related by (4.4). Then by definition the image of Φ^r is the kernel of the one-form Φ^l.

We wish to show that the connection one-form corresponding to the splitting $\mathrm{ad}(\psi^{-1})\Phi^r\,\mathrm{ad}(\beta(\psi))$ is $\psi^*\Phi^l$. It is straightforward to check that $\psi^*\Phi^l$ is a connection one-form so we just need to show that $\psi^*\Phi^l$ annihilates $\big(\mathrm{ad}(\psi^{-1})\Phi^r\,\mathrm{ad}(\beta(\psi))\big)(\xi)$ for ξ a vector field on X. The adjoint action of $\mathrm{Aut}(Q)$ on an element μ in its Lie algebra, $\mathrm{Lie}(\mathrm{Aut}(Q)) = \Gamma(X, TQ/G)$ is given by $\mathrm{ad}(\psi)(\mu)_q = \psi_*\mu_{\psi^{-1}(q)}$. Similarly, if $\xi \in \mathrm{Lie}(\mathrm{Diff}^Q(X))$ then $\mathrm{ad}(\beta(\psi))(\xi)_x =$

$\beta(\psi)_*\xi_{\beta(\psi)^{-1}(x)}$. Notice that this implies that if ω is a one-form on Q and ζ is a vector field on Q then $(\psi^*\omega)_q(\mathrm{ad}(\psi^{-1})\zeta_q) = \omega_{\psi(q)}(\psi_*(\psi_*^{-1}\zeta)_{\psi(q)}) = \omega_{\psi(q)}(\zeta_{\psi(q)})$. Letting $\zeta = \Phi^H\,\mathrm{ad}(\beta(\psi))(\xi)$ we note that $\Phi(\zeta) = 0$ and then we have

$$(\psi^*\Phi^l)\big(\mathrm{ad}(\psi^{-1})\Phi^r\,\mathrm{ad}(\beta(\psi))\big)(\xi) = \Phi^l(\zeta) = 0$$

as required. □

We note that from now on the default choice for the value of a Higgs field Φ will be that it is a connection one-form, that is a left-splitting. When this is not the case we will write Φ^r.

Now we are in a position to extend the caloron correspondence to include connections and Higgs fields. We start with a G-bundle $\widetilde{P} \to Y$ of type $Q \to X$. We have seen above that $\mathrm{Eq}(Q, \widetilde{P}) \to \mathrm{Map}^Q(X, Y)$ is a $\mathcal{G}$-bundle. We can apply the functor $\mathrm{Eq}(Q,\ )$ to the connection $\widetilde{A}$ to obtain a connection $\mathrm{Eq}(Q, \widetilde{A})$. Indeed if $\rho \in T_f(\mathrm{Eq}(Q, \widetilde{P}))$ then $\rho \in \Gamma_G(Q, f^*(T\widetilde{P}))$, so for any $q \in Q$ we have $\rho_q \in T_{f(q)}\widetilde{P}$ and we define

$$\mathrm{Eq}(Q, \widetilde{A})_f(\rho) = (q \mapsto \widetilde{A}_{f(q)}(\rho_q)) \in \Gamma(X, \mathrm{ad}(Q)).$$

Proposition 4.7. *In the situation just described* $\mathrm{Eq}(Q, \widetilde{A})$ *is an* $\mathcal{A}$*-connection which is* $\mathrm{Aut}(Q)$*-invariant on the* $\mathcal{G}$*-bundle* $\mathrm{Eq}(Q, \widetilde{P}) \to \mathrm{Map}^Q(X, Y)$.

Proof. Consider first what happens when we apply $\mathrm{Eq}(Q, \widetilde{A})$ to a vertical vector in the tangent space to $\mathrm{Eq}(Q, \widetilde{P})$ at f. The tangent space to f is $\Gamma_G(Q, f^*(TQ))$ and a vertical vector is generated by an element of the Lie algebra of $\mathcal{G}$ which is given by an equivariant map $\mu\colon Q \to \mathfrak{g}$. It is a straightforward exercise to check that

$$\iota_f(\mu)(q) = \iota_{f(q)}\mu(q)$$

and then

$$\mathrm{Eq}(Q, \widetilde{A})_f(\iota_f(\mu)) = q \mapsto \widetilde{A}_{f(q)}(\iota_{f(q)}\mu(q)) = \mu(q)$$

or $\mathrm{Eq}(Q, \widetilde{A})_f(\iota_f(\mu)) = \mu$.

To check invariance under $\mathrm{Aut}(Q)$ we first need to understand the adjoint action of $\mathrm{Aut}(Q)$ on $\mathcal{G}$ and its Lie algebra. If $g \in \mathcal{G}$ then $g\colon Q \to G$ and it acts on Q by $q \mapsto qg(q)$. If $\psi \in \mathrm{Aut}(Q)$ then $\psi^{-1}g\psi$ acts on Q by sending $q \in Q$ to $\psi^{-1}(\psi(q)g(\psi(q))) = qg(\psi(q))$ or $\mathrm{ad}(\psi^{-1})(g) = g \circ \psi$. Hence if $\mu\colon Q \to \mathfrak{g}$ is in the Lie algebra of $\mathcal{G}$ we have $\mathrm{ad}(\psi^{-1})(\mu) = \mu \circ \psi$. Let $f_t \in \mathrm{Eq}(Q, \widetilde{P})$ with $f_0 = f$ and $\rho = t_0(t \mapsto f_t) \in T_f(\mathrm{Eq}(Q, \widetilde{P}))$. We have

$$\begin{aligned} R_\psi^*(\mathrm{Eq}(Q, \widetilde{A}))_f(\rho)(q) &= \mathrm{Eq}(Q, \widetilde{A})_{f\circ\psi}(t_0(t \mapsto f_t \circ \psi))(q) \\ &= \widetilde{A}_{f(\psi(q))}\big(t_0(t \mapsto f_t(\psi(q)))\big) \\ &= \mathrm{Eq}(Q, \widetilde{A})_f(\rho)(\psi(q)) \\ &= \mathrm{ad}(\psi^{-1}(q))\,\mathrm{Eq}(Q, \widetilde{A})_f(\rho) \end{aligned}$$

as required. Proposition 4.5 implies that the associated Higgs field is valued in $\mathcal{A}$ and so this is an $\mathcal{A}$-connection. □

The pullback of the connection $\mathrm{Eq}(Q, \widetilde{A})$ using $\eta\colon F(Y) \to \mathrm{Map}(X, Y)$ is therefore an $\mathrm{Aut}(Q)$-invariant connection on the $\mathcal{G}$-bundle $\eta^*(\mathrm{Eq}(Q, P)) \to F(Y)$.

It follows from Corollary 4.4 that we have a connection A on $P \to M$ defined by

$$A = \eta^*(\mathrm{Eq}(Q, \widetilde{A})) + \Phi^r(\pi^* a) \tag{4.8}$$

where $\pi : P \to F(Y)$.

Putting this all together with the results from Section 4.1 we have the following theorem.

Theorem 4.8. *Let* $\widetilde{P} \to Y$ *be a* G*-bundle of type* $Q \to X$ *with connection* $\widetilde{A}$ *and let* a *be a connection for* $Y \to M$*. Then the* $\mathrm{Aut}(Q)$*-bundle* $P \to M$ *has connection* A *and Higgs field* Φ *defined as follows.*

- *Let* $f \in P$ *so that* $f : Q \to \widetilde{P}_m$ *for some* $m \in M$*, then* $\Phi(f) = f^*(\widetilde{A})$.
- *If* $\rho \in \Gamma(Q, f^*(T\widetilde{P}))$*, let* $\bar{f} : X \to Y_m$ *and* $\bar{\rho} \in \Gamma(X, \bar{f}^*(TY))$ *be the projections of* f *and* ρ *respectively. Then* $A_f(\rho) \in \Gamma_G(Q, TQ)$ *is given by*

$$A_f(\rho)(q) = \widetilde{A}_{f(q)}(\rho(q)) + \Phi^r(f)\big(x \mapsto (\bar{f}^{-1})_*(v_a(\bar{\rho}(x)))\big)(q)$$

where, as before, $\Phi^r(f)\colon \Gamma(X, TX) \to \Gamma_G(Q, TQ)$ *is the right splitting induced by the connection* $\Phi(f) = f^*(\widetilde{A})$.

Consider the caloron transform in the other direction. Given an Aut(Q)-bundle $P \to M$ we define $Y = P \times_{\mathrm{Aut}(Q)} X$ and a G-bundle $\widetilde{P} \to Y$ by $\widetilde{P} = P \times_{\mathrm{Aut}(Q)} Q$. Recall that the Lie algebra of Aut(Q) is $\Gamma_G(Q, TQ)$, the Lie algebra of G-equivariant vector fields on Q. The tangent space at $(p, q) \in P \times Q$ is

$$T_{(p,q)}(P \times Q) = T_pP \times T_qQ$$

and the vertical vector induced by $\mu \in \mathrm{Lie}(\mathrm{Aut}(Q))$ is given by $(\iota_p(\mu), -\mu(q))$.

Let A be a connection one-form on P. That is, A is an Lie(Aut(Q))-valued one-form on P so that if $\xi \in T_pP$ then $A_p(\xi) \in \mathrm{Lie}(\mathrm{Aut}(Q)) = \Gamma_G(Q, TQ)$ and thus $A_p(\xi)(q) \in T_qQ$. As above a Higgs field for $P \to M$ is an equivariant map $\Phi : P \to \mathcal{A}$, so we have

$$\Phi(p)_q \left(A_p(\xi)(q)\right) \in \mathfrak{g}.$$

We define

$$\widetilde{\omega}_{(p,q)}(\xi, \zeta) = \Phi(p)_q \left(A_p(\xi)(q) + \zeta\right) \tag{4.9}$$

on the product $P \times Q$ and we have the following theorem.

Theorem 4.9. *Let $P \to M$ be an* Aut(Q)*-bundle with connection A and Higgs field Φ. The one-form $\widetilde{\omega}$ defined in (4.9) on $P \times Q$ descends to a connection one-form $\widetilde{A}$ on the G-bundle $\widetilde{P} \to Y$.*

Proof. We have to show that the one-form

$$\widetilde{\omega}_{(p,q)}(\xi, \zeta) = \Phi(p)_q \left(A_p(\xi)(q) + \zeta\right)$$

descends to a connection one-form $\widetilde{A}$ on the G-bundle $\widetilde{P} \to Y$.

First we show that ω annihilates vectors generated by the action of Aut(Q) on $P \times Q$. In the tangent space at (p, q) such vectors have the form $(\iota_p(\mu), -\mu(q))$ for some $\mu \in \Gamma_G(Q, TQ)$ and we have

$$\begin{aligned}\widetilde{\omega}_{(p,q)}(\iota_p(\mu), -\mu(q)) &= \Phi(p)_q(A_p(\iota_p(\mu))(q) - \mu(q))\\ &= \Phi(p)_q(\mu(q) - \mu(q))\\ &= 0.\end{aligned}$$

Next we show that $\widetilde{\omega}$ is invariant under the Aut(Q)-action. Let $\psi \in \mathrm{Aut}(Q)$ then

$$\begin{aligned}(R_\psi^*\widetilde{\omega})_{(p,q)}(\xi, \zeta) &= \widetilde{\omega}_{(p\psi, \psi^{-1}(q))}\left((R_\psi)_*(\xi), (\psi_*^{-1})_q(\zeta)\right)\\ &= \Phi(p\psi)_{\psi^{-1}(q)}\left((A_p(R_\psi(\xi)))(\psi^{-1}(q)) + (\psi_*^{-1})_q(\zeta)\right)\\ &= \Phi(p\psi)_{\psi^{-1}(q)}\left(\mathrm{ad}(\psi^{-1})(A_p(\xi))(\psi^{-1}(q)) + (\psi_*^{-1})_q(\zeta)\right)\\ &= (\psi^*\Phi(p))_{\psi^{-1}(q)}\left((\psi_*^{-1})_q(A_p(\xi)(q) + \zeta)\right)\\ &= \Phi(p)_q\left((\psi_*)_q(\psi_*^{-1})_q(A_p(\xi)(q) + \zeta)\right)\\ &= \Phi(p)_q(A_p(\xi)(q) + \zeta)\\ &= \widetilde{\omega}_{(p,q)}(\xi, \zeta)\end{aligned}$$

as required.

It follows that $\widetilde{\omega}$ descends to a $\mathfrak{g}$-valued one-form on $\widetilde{P}$ which we denote by $\widetilde{A}$. We need to check that this is a connection one-form. Notice that G acts on $P \times Q$ by acting on Q and that this commutes with the action of Aut(Q) covering the action on $\widetilde{P}$. If $\iota_{[p,q]}(\chi)$ is a vertical vector in $\widetilde{P}$ it lifts to $(0, \iota_q(\chi))$ at (p, q). Applying $\widetilde{\omega}_{(p,q)}$ we have

$$\widetilde{\omega}_{(p,q)}(0, \iota_q(\chi)) = \Phi(p)_q(0 + \iota_q(\chi)) = \chi.$$

Hence $\widetilde{A}(\iota(\chi)) = \chi$. Let $g \in G$, then we have

$$\begin{aligned}(R_g^*\widetilde{\omega})(p, q)(\xi, \zeta) &= \widetilde{\omega}_{(p,qg)}(\xi, (R_g)_*(\zeta))\\ &= \Phi(p)_{qg}\left(A_p(\xi)(qg) + (R_g)_*(\zeta)\right)\\ &= \Phi(p)_{qg}\left((R_g)_*(A_p(\xi)(q) + \zeta)\right)\\ &= \mathrm{ad}(g^{-1})\Phi(p)_q(A_p(\xi)(q) + \zeta)\\ &= \mathrm{ad}(g^{-1})\widetilde{\omega}_{(p,q)}(\xi, \zeta)\end{aligned}$$

so that $R_g^*\widetilde{A} = \mathrm{ad}(g^{-1})\widetilde{A}$. Here we use the fact that $A_p(\xi) \in \Gamma_G(Q, TQ)$, so it satisfies $A_p(\xi)(qg) = (R_g)_*(A_p(\xi)(q))$. □

The final thing we need to do is to show that when we apply the caloron transform twice in either direction the connections map to each other under the isomorphisms introduced in Section 3.2.

Proposition 4.10. *Let $\widetilde{P} \to Y$ be a G-bundle of type $Q \to X$ with connection $\widetilde{A}$ and let a be a connection for $Y \to M$. Let $P \to M$ be an* Aut(Q)*-bundle with connection A and Higgs field Φ. Let the caloron transforms be as in Theorems 4.9 and 4.8. Then the isomorphism (3.2)*

$$\tau_{\widetilde{P}} : \mathcal{C}^{-1} \circ \mathcal{C}(\widetilde{P}) \to \widetilde{P}$$

preserves connections and the isomorphism (3.3)

$$\tau_P : \mathcal{C} \circ \mathcal{C}^{-1}(P) \to P$$

preserves connections and Higgs fields.

Proof. Consider first the case that we start with a G-bundle $\widetilde{P} \to Y$ with a connection $\widetilde{A}$. The isomorphism in question goes from $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times_{\mathrm{Aut}(Q)} Q$ to $\widetilde{P}$ and is given by $(f, q) \mapsto f(q)$. Under this isomorphism $\widetilde{A}$ pulls back to a connection on $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times_{\mathrm{Aut}(Q)} Q$ which we further pullback to $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times Q$. We note that

$$T_{(f,q)}\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times Q = \Gamma_G(Q, f^*(\widetilde{P})) \times T_q Q$$

and the pullback of $\widetilde{A}$ applied to a pair (ρ, ζ) is

$$\widetilde{A}_{f(q)}(\rho(q) + f_*(\zeta)) = \widetilde{A}_{f(q)}(\rho(q)) + \Phi(f)_q(\zeta).$$

Consider now the connection constructed on $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times Q$ by applying the constructions above twice. First the connection $\widetilde{A}$ gives rise to a connection

$$A = \eta^* \,\mathrm{Eq}(Q, \widetilde{A}) + \Phi^r(\pi^* a)$$

and the Higgs field identified in Proposition 4.5.

The pair (A, Φ) gives rise to a $\mathfrak{g}$-valued one-form on $\eta^*(\mathrm{Eq}(Q, \widetilde{P})) \times Q$ given by

$$\begin{aligned}\widetilde{\omega}_{(f,q)}(\rho, \zeta) &= \Phi(f)_q\big(A_f(\rho)(q) + \zeta\big)\\ &= \Phi(f)_q\big(\eta^* \,\mathrm{Eq}(Q, \widetilde{A})_f(\rho)(q) + \Phi(a(\pi_*(\rho))) + \zeta\big)\\ &= \Phi(f)_q\big(\widetilde{A}_{f(q)}(\rho(q))\big) + \Phi(f)_q\big(\Phi(a(\pi_*(\rho)))\big) + \Phi(f)_q(\zeta).\end{aligned}$$

Notice that under the identifications we are using (see Remark 4.1) $\widetilde{A}_{f(q)}(\xi(q))$ is a vertical vector in Q and already identified with a vector in $\mathfrak{g}$, so there is no need to apply $\Phi(f)_q$. Further the vector $\Phi(a(\pi_*(\rho)))$ is horizontal for $\widetilde{A}$, so applying $\Phi(f) = f^*(\widetilde{A})$ gives zero. Hence we have

$$\widetilde{\omega}_{(f,q)}(\rho, \zeta) = \widetilde{A}_{f(q)}(\rho(q)) + \Phi(f)_q(\zeta)$$

as required.

Consider the second case where we start with an Aut(Q)-bundle $P \to M$ with connection A and Higgs field Φ. From these we construct a G-bundle

$$P \times_{\mathrm{Aut}(Q)} Q \to P \times_{\mathrm{Aut}(Q)} X$$

with connection $\widetilde{A}$ whose pullback to $P \times Q$ we have called $\widetilde{\omega}$ defined by (4.9):

$$\widetilde{\omega}_{(p,q)}(\xi, \zeta) = \Phi(p)_q\left(A_p(\xi)(q) + \zeta\right).$$

From this we construct the Aut(Q)-bundle $\eta^*(\mathrm{Eq}(Q, P \times_{\mathrm{Aut}(Q)} Q)) \to M$ with connection A' and Higgs field $(\Phi')^r$ where

$$A' = \eta^*(\mathrm{Eq}(Q, \widetilde{A})) + (\Phi')^r(\pi^* a).$$

Each $p \in P$ defines a natural map $f_p : Q \to P \times_{\mathrm{Aut}(Q)} Q$ by $f_p(q) = [p, q]$ and $\chi(p) = f_p$ is the isomorphism of Aut(Q)-bundles defined in Section 3.2:

$$\chi : P \to \eta^*(\mathrm{Eq}(Q, P \times_{\mathrm{Aut}(Q)} Q)).$$

We want to show that under this isomorphism A' and Φ' pullback to A and Φ.

First we consider Φ'. From Proposition 4.5 we have that at p the pullback of Φ' is $f_p^*(\widetilde{A})$. The frame $f_p : Q \to P \times_{\mathrm{Aut}(Q)} Q$ lifts to $Q \to P \times Q$ sending $q \mapsto (p, q)$ and the tangent to this is $\zeta \mapsto (0, \zeta)$. Applying $\widetilde{\omega}$ gives $\Phi(p)_q(\zeta)$ as required.

To calculate the pullback of A' by χ we need to consider the tangent map to χ applied to $\xi \in T_pP$. The result is a section in $\Gamma_G(Q, f_p^*(P \times_{\mathrm{Aut}(Q)} Q))$ which we can lift to $P \times Q$ and realise as $q \mapsto (\xi, 0) \in T_pP \times T_qQ$. Considering the first term in A' we have

$$\eta^*(\mathrm{Eq}(Q, \widetilde{A}))_{(p,q)}(\xi, 0) = \Phi(p)_q(A_p(\xi)(q) + 0)$$

so that

$$A'_p(\xi) = \Phi(p)(A_p(\xi)) + \Phi^r(p)(a(\pi_*(\xi, 0))).$$

But the connection a is the projection of the connection A so we must have

$$A'_p(\xi) = \Phi(p)(A_p(\xi)) + \Phi^r(p)(\beta(A_p(\xi))).$$

The relation in Eq. (4.4) then tells us that

$$A'_p(\xi) = A_p(\xi)$$

as required. □

4.4. The unframed caloron correspondence for products with connections and Higgs fields

We consider Theorem 4.8 and show how to reduce it in this case. Let $\widetilde{P} \to M \times X$ be a G-bundle of type $Q \to X$ with connection $\widetilde{A}$. Take as a, the flat connection on $X \times M \to M$. We have seen that the Aut(Q)-bundle can be reduced to a $\mathcal{G}$-bundle $P \to M$, whose fibre at $m \in M$ is all diffeomorphisms $f : Q \to \widetilde{P}_m$ covering $X \to X \times M$ given by $x \mapsto (x, m)$. Therefore, in the notation of Theorem 4.8, $\bar{f}(x) = (x, m)$. Let ρ be a tangent vector at f. Then $\bar{\rho}(x) = (0, v) \in T_xX \times T_mM$ for some constant vector $v \in T_mM$ and thus $v_a(\bar{\rho}(x)) = 0$ and we have the following reduction of Theorem 4.8.

Theorem 4.11. *Let $\widetilde{P} \to X \times M$ be a G-bundle of type $Q \to X$ with connection $\widetilde{A}$. Then the $\mathcal{G}$-bundle $P \to M$ has connection A and Higgs field Φ defined as follows.*

- *Let $f \in P$ so that $f : Q \to \widetilde{P}_m$ for some $m \in M$ covering $x \mapsto (x, m)$, then $\Phi(f) = f^*(\widetilde{A})$.*
- *If $\rho \in \Gamma(Q, f^*(T\widetilde{P}))$, then $A_f(\rho) \in \Gamma_G(Q, TQ)$ is given by*

$$A_f(\rho)(q) = \widetilde{A}_{f(q)}(\rho(q)).$$

Going in the other direction we have a $\mathcal{G}$-bundle $P \to M$ with connection A and Higgs field Φ and define $\widetilde{P} = P \times_{\mathcal{G}} Q \to M \times X$. In this case A is a Lie$(\mathcal{G})$-valued one-form on P so that if $\xi \in T_pP$, then $A_p(\xi) \in \Gamma_G(Q, \mathfrak{g})$ and applying the Higgs field has no effect. The formula in the fibration case therefore reduces to

$$\widetilde{\omega}_{(p,q)}(\xi, \zeta) = A_p(\xi)(q) + \Phi(p)_q(\zeta)$$

and we have the following theorem.

Theorem 4.12. *The one-form*

$$\widetilde{\omega}_{(p,q)}(\xi, \zeta) = A_p(\xi)(q) + \Phi(p)_q(\zeta)$$

descends to a connection one-form $\widetilde{A}$ on the G-bundle $\widetilde{P} \to X \times M$.

4.5. The framed caloron correspondences with connections and Higgs fields

We leave it to the interested reader to show that in the case of framings the connections we have defined already respect the framings and the Higgs fields take values in the appropriate space of framed connections.

5. Characteristic classes for gauge group bundles

In this section we shall use the caloron correspondence to calculate characteristic classes for $\mathcal{G}$-bundles, following a similar approach as in [3]. We begin with a brief review of characteristic classes and the Chern–Weil theory for G-bundles for the convenience of the reader.

5.1. Review of the Chern–Weil theory

Let $EG \to BG$ denote the universal bundle, with the property that any principal G-bundle over M is isomorphic to the pullback of $EG \to BG$ by a classifying map $f : M \to BG$. Up to homotopy equivalence, the universal bundle is fully characterised by the fact that it is a principal G-bundle and EG is a contractible space. A characteristic class for a G-bundle over M is a class in $H^*(M)$ obtained by pulling back elements in the cohomology group $H^*(BG)$. The Chern–Weil theory provides a mechanism for producing characteristic classes in de Rham cohomology. Let $I^k(\mathfrak{g})$ denote the algebra of multilinear, symmetric, ad-invariant functions on k copies of the Lie algebra $\mathfrak{g}$. Elements of $I^k(\mathfrak{g})$ are called invariant polynomials. The Chern–Weil homomorphism is a map

$$cw : I^k(\mathfrak{g}) \to H^{2k}(M)$$

defined by $f \mapsto cw_f(A) = f(F, \ldots, F)$, for A a connection on the G-bundle with curvature F. The class $cw_f(A) \in H^{2k}(M)$ is independent of the choice of connection and represents a characteristic class of the bundle. For compact Lie groups G, the Chern–Weil homomorphism applied to the universal bundle is an isomorphism, which extends to an algebra isomorphism $cw\colon I^*(\mathfrak{g}) \xrightarrow{\sim} H^*(BG)$. The proof of these results can be found in many standard references such as [11].

Since f is multilinear and symmetric, we will adopt the convention that whenever f has repeated entries they will be collected into one slot and written as a power, for instance $f(\underbrace{A, \ldots, A}_{k}, \underbrace{B, \ldots, B}_{l}) = f(A^k B^l)$.

5.2. The curvature of the caloron connection

The first step towards calculating characteristic classes is to determine the curvature of the caloron connection given in Theorem 4.12. Recall that $\tilde{A}$ is expressed in terms of a connection A and Higgs field Φ as follows. For a tangent vector $(\xi, \zeta) \in T_{[p,q]}\widetilde{P}$ we have

$$\tilde{A}_{(p,q)}(\xi, \zeta) = A_p(\xi)(q) + \Phi(p)_q(\zeta).$$

The curvature $\tilde{F}$ of $\tilde{\omega}$ is given by the formula

$$\tilde{F} = d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}].$$

For a pair of tangent vectors $V_1 = (\xi_1, \zeta_1)$, $V_2 = (\xi_2, \zeta_2) \in T_{[p,q]}\widetilde{P}$, the commutator term is given by

$$\begin{aligned}\frac{1}{2}[\tilde{A}_{(p,q)}(V_1), \tilde{A}_{(p,q)}(V_2)] &= \frac{1}{2}[A_p(\xi_1), A_p(\xi_2)](q) + \frac{1}{2}[A_p(\xi_1)(q), \Phi(p)_q(\xi_2)] \\ &\quad + \frac{1}{2}[\Phi(p)_q(\zeta_1), A_p(\zeta_2)(q)] + \frac{1}{2}[\Phi(p)_q(\zeta_1), \Phi(p)_q(\zeta_2)] \\ &= \frac{1}{2}[A, A]_p(\xi_1, \xi_2)(q) + \frac{1}{2}[\Phi, \Phi](p)_q(\zeta_1, \zeta_2) + [A, \Phi]_{(p,q)}(V_1, V_2),\end{aligned}$$

while the differential

$$d\tilde{A}(V_1, V_2) = \frac{1}{2}\big(V_1(\tilde{A}(V_2)) - V_2(\tilde{A}(V_1)) - \tilde{A}([V_1, V_2])\big)$$

can be expressed as a sum of four terms,

$$d\tilde{A} = d_P A + d_Q \Phi + d_P \Phi + d_Q A.$$

Here we are considering A and Φ variously as forms on P and Q and as maps from these spaces. More specifically, we have

- $d_P A$ is the derivative of A considered as a one-form on P:

$$d_P A_{(p,q)}(V_1, V_2) = \frac{1}{2}\Big((\xi_1 A)_p(\xi_2) - (\xi_2 A)_p(\xi_1) - A_p([\xi_1, \xi_2])\Big)(q)$$

- $d_Q \Phi$ is the derivative of the Higgs field considered as a one-form on Q, i.e. an element of $\mathcal{A}$:

$$d_Q \Phi_{(p,q)}(V_1, V_2) = \frac{1}{2}\Big(\zeta_1(\Phi(p))_q(\zeta_2) - \zeta_2(\Phi(p))_q(\zeta_1) - \Phi(p)_q([\zeta_1, \zeta_2])\Big)$$

- $d_P \Phi$ is the derivative of the Higgs field considered as a map $P \to \mathcal{A}$:

$$d_P \Phi_{(p,q)}(V_1, V_2) = \frac{1}{2}\Big((\xi_1 \Phi)(p)_q(\zeta_2) - (\xi_2 \Phi)(p)(\zeta_1)\Big)$$

- $d_Q A$ is the derivative of A considered as an element of $\mathrm{Lie}(\mathcal{G})$, i.e. a G-equivariant map $Q \to \mathfrak{g}$:

$$d_Q A_{(p,q)}(V_1, V_2) = \frac{1}{2}\Big(\zeta_1(A_p(\xi_2)) - \zeta_2(A_p(\xi_1))\Big)(q).$$

Putting this all together, we have

$$\tilde{F} = F_A + F_\Phi + \nabla\Phi,$$

where F_A is the curvature of A, F_Φ is the curvature of Φ considered as a connection on Q and $\nabla\Phi = d_P\Phi + [A, \Phi] + d_Q A$.

Remark 5.1. Comparing this with the caloron correspondence for loop groups [3], the Higgs field there is a flat connection on the trivial bundle over the circle and hence $F_\Phi = 0$.

5.3. Caloron classes

We can now proceed to define characteristic classes for the $\mathcal{G}$-bundle $P \to M$ using the caloron correspondence. Let f be an invariant polynomial in $I^k(\mathfrak{g})$. The Chern–Weil homomorphism for the caloron transform $\widetilde{P} \to M \times X$ determines a $2k$-form representing a class $cw_f(\widetilde{A}) \in H^{2k}(M \times X)$. Integrating this form over the d-dimensional manifold X yields a closed $(2k-d)$-form on M, which we call the *caloron class* of P. In short, we have

$$I^k(\mathfrak{g}) \xrightarrow{cw_f(\widetilde{A})} H^{2k}(M \times X) \xrightarrow{\int_X} H^{2k-d}(M),$$

and we denote by $\varsigma_{2k-d}(P)$ the resulting caloron class in $H^{2k-d}(M)$.

Note that in order to define a degree r characteristic class for P, we must pick $k = (d+r)/2$, which in particular requires r and d to have the same parity.[1] An explicit formula for the caloron classes of P is obtained by calculating $\int_X cw(\widetilde{A})$. Given a connection A and Higgs field Φ for P, we know from Section 5.2 that the curvature $\tilde{F}$ of the caloron connection $\tilde{A}$ on $\widetilde{P}$ splits into a sum of three terms,

$$\tilde{F} = F_A + F_\Phi + \nabla\Phi.$$

Inserted as arguments into an invariant polynomial $f \in I^k(\mathfrak{g})$, we have

$$cw_f(\widetilde{A}) = f(\tilde{F}^k) = f\big((F_A + F_\Phi + \nabla\Phi)^k\big).$$

Note that the base is a product and the forms F_A, F_Φ and $\nabla_A\Phi$ are of type $(2, 0)$, $(0, 2)$ and $(1, 1)$ respectively. Integrating over X picks out only the terms of the type $(2k-d, d)$. Hence, we have

$$\varsigma_{2k-d}(P) = \int_X f\big((F_A + F_\Phi + \nabla\Phi)^k\big)_{[2k-d,d]}$$

which for the lowest values of the pair (d, k) yields Table 1.

In particular, for $X = S^1$, we recover the string classes introduced in [2,3],

$$\varsigma_{2k-1}(P) = k \int_{S^1} f\left(F_A^{k-1}\nabla\Phi\right).$$

More generally for a manifold X of dimension d, the lowest degree caloron classes are given by

$$\varsigma_0(P) = \int_X f\left(F_\Phi^{d/2}\right),$$

$$\varsigma_1(P) = \frac{d+1}{2}\int_X f\left(\nabla\Phi F_\Phi^{\frac{d-1}{2}}\right),$$

$$\varsigma_2(P) = \frac{d+2}{2}\int_X f\left(F_A F_\Phi^{d/2} + \frac{1}{2}\sum_{j=0}^{\frac{d-2}{2}} \nabla\Phi F_\Phi^{\frac{d-2}{2}-j}\nabla\Phi F_\Phi^j\right),$$

$$\varsigma_3(P) = \frac{d+3}{2}\int_X f\left(\sum_{j=0}^{\frac{d-1}{2}} F_A F_\Phi^{\frac{d-1}{2}-j}\nabla\Phi F_\Phi^j + \frac{1}{3}\sum_{j=0}^{\frac{d-3}{2}}\sum_{l=0}^{j} \nabla\Phi F_\Phi^{\frac{d-3}{2}-j}\nabla\Phi F_\Phi^{j-l}\nabla\Phi F_\Phi^l\right),$$

$$\begin{aligned}\varsigma_4(P) = \frac{d+4}{2}\int_X f\Bigg(&\frac{1}{2}\sum_{j=0}^{\frac{d}{2}} F_A F_\Phi^{\frac{d}{2}-j} F_A F_\Phi^j + \sum_{j=0}^{\frac{d-2}{2}}\sum_{l=0}^{j} F_A F_\Phi^{\frac{d-2}{2}-j}\nabla\Phi F_\Phi^{j-l}\nabla\Phi F_\Phi^l \\ &+ \frac{1}{4}\sum_{j=0}^{\frac{d-4}{2}}\sum_{l=0}^{j}\sum_{m=0}^{l} \nabla\Phi F_\Phi^{\frac{d-4}{2}-j}\nabla\Phi F_\Phi^{j-l}\nabla\Phi F_\Phi^{l-m}\nabla\Phi F_\Phi^m\Bigg).\end{aligned}$$

The formulae get increasingly more complicated, involving more nested sums, as one goes to higher degrees in cohomology. However, in the special case when G is abelian a straightforward calculation leads to the following general formula:

$$\varsigma_{2k-d}(P) = \sum_{i=\frac{d}{2}}^{\min\{d,k\}} \binom{k}{i}\binom{i}{d-i}\int_X f\left(F_A^{k-i}F_\Phi^{d-i}(\nabla_A\Phi)^{2i-d}\right).$$

[1] In the loop group case X was the circle, so $\dim(X) = 1$. Hence the string classes are all of odd degree.

6. Universal caloron classes

In this final section we restrict our attention to the framed and product case of the constructions in Section 2. There is a natural model for the universal $\mathcal{G}_0$-bundle which allows for explicit expressions for the caloron classes.

The group of based gauge transformations $\mathcal{G}_0$ acts freely on the space of connections $\mathcal{A}$ and the quotient is a smooth tame Fréchet manifold [12], which can be identified with the classifying space $B\mathcal{G}_0$. We want to calculate the caloron classes explicitly for the universal bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}_0$. We do this by choosing a connection and Higgs field; the connection is the one described in [13] whose horizontal subspaces are given by $\ker d_\omega^*$ and the Higgs field is simply $\Phi : \mathcal{A} \xrightarrow{id} \mathcal{A}$. The operator d_ω^* is the adjoint of the covariant exterior differential d_ω with respect to the inner product

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta$$

for $\alpha, \beta \in \Omega^p(X, \mathrm{ad}\, Q)$. Because $\mathcal{A}$ is an affine space, the tangent space to $\mathcal{A}$ at ω is $\Omega^1(X, \mathrm{ad}\, Q)$. The map into the vertical tangent space is

$$d_\omega : \Gamma_0(X, \mathrm{ad}\, Q) \to T_\omega\mathcal{A} = \Omega^1(X, \mathrm{ad}\, Q)$$

and we have a splitting $T_\omega\mathcal{A} = \mathrm{im}\, d_\omega \oplus \ker d_\omega^*$ into orthogonal subspaces. Since the gauge transformations are based, the Laplacian

$$d_\omega^* d_\omega : \Gamma_0(X, \mathrm{ad}\, Q) \to \Gamma_0(X, \mathrm{ad}\, Q)$$

is invertible and we denote by $G_\omega = (d_\omega^* d_\omega)^{-1}$ the corresponding Green's operator. The connection form can now be expressed as $G_\omega d_\omega^*$ and by Theorem 4.12 the caloron connection on $\tilde{\mathcal{A}} = (\mathcal{A} \times Q)/\mathcal{G}_0$ over $\mathcal{A}/\mathcal{G}_0 \times X$ is given by

$$\tilde{A}_{(\omega,q)}(\xi, \zeta) = G_\omega d_\omega^*(q)(\xi) + \omega_q(\zeta)$$

for a tangent vector $(\xi, \zeta) \in T_{[\omega,q]}\widetilde{\mathcal{A}}$, where $G_\omega d_\omega^*(\xi)$ is interpreted as a G-equivariant map $Q \to \mathfrak{g}$. Note that this caloron connection is framed. Recall that the curvature of $\tilde{A}$ consists of three terms,

$$\tilde{F} = F_A + F_\Phi + \nabla\Phi.$$

Since the Higgs field maps a connection $\omega \in \mathcal{A}$ onto itself, clearly F_Φ evaluated on a pair of tangent vectors $V_1 = (\xi_1, \zeta_1)$ and $V_2 = (\xi_2, \zeta_2)$ at $(\omega, q) \in \tilde{\mathcal{A}}$ is given by the curvature $F_\omega(q)(\zeta_1, \zeta_2)$ of ω.

Next let us consider the term $\nabla\Phi$. Recall that as with any curvature we only need to evaluate it on horizontal vectors. Indeed if V_1 and V_2 are horizontal, then $\tilde{F}(V_1, V_2) = d\tilde{A}(V_1, V_2)$ so the commutator term does not contribute. Let therefore $\zeta_1, \zeta_2 \in T_qQ$ be horizontal for ω and $\xi_1, \xi_2 \in \Omega^1(X, \mathrm{ad}\, Q)$ be horizontal (i.e. $d_\omega^*\xi = 0$). As $\mathcal{A}$ is an affine space we can extend ξ_1 and ξ_2 to constant vector fields whose Lie bracket must vanish. We also extend ζ_1 and ζ_2 to vector fields. From Section 5.2 we have

$$\nabla\Phi_{(\omega,q)}(V_1, V_2) = d_P\Phi_{(\omega,q)}(V_1, V_2) + d_Q\left(G_\omega d_\omega^*\right)(q)(V_1, V_2),$$

where

$$d_P\Phi_{(\omega,q)}(V_1, V_2) = -\frac{1}{2}\omega_q([\zeta_1, \zeta_2])$$

since Φ is the identity map, and

$$\begin{aligned} d_Q\left(G_\omega d_\omega^*\right)(q)(V_1, V_2) &= \frac{1}{2}\zeta_1\left(q \mapsto G_\omega d_\omega^*(q)(\xi_2)\right) - \zeta_2\left(q \mapsto G_\omega d_\omega^*(q)(\xi_1)\right) \\ &= \frac{1}{2}\left(t_0\left(t \mapsto \omega_q(\xi_2) + t\xi_1(\zeta_2)(q)\right) - t_0\left(t \mapsto \omega_q(\xi_1) + t\xi_2(\zeta_1)(q)\right)\right) \\ &= \frac{1}{2}\left(\xi_1(\zeta_2)(q) - \xi_2(\zeta_1)(q)\right), \end{aligned}$$

where we have abused the notation here by treating ω both as a constant and a variable. Hence the $(1, 1)$ component of the curvature applied to horizontal vector fields V_1 and V_2 is

$$\nabla\Phi_{(\omega,q)}(V_1, V_2) = \frac{1}{2}\left(\xi_1(\zeta_2)(q) - \xi_2(\zeta_1)(q) - \omega_q([\zeta_1, \zeta_2])\right).$$

Finally we calculate

$$F_{A(\omega,q)}(V_1, V_2) = \frac{1}{2}\left(\xi_1(\omega \mapsto G_\omega d_\omega^*)(q)(\xi_2) - \xi_2(\omega \mapsto G_\omega d_\omega^*)(q)(\xi_1)\right).$$

Table 1
Caloron classes for small values of d and k.

d	$k=1$	$k=2$	$k=3$
1	$\int_X f(\nabla\Phi)$	$\int_X f\left(2F_A\nabla\Phi\right)$	$\int_X f\left(3F_A^2\nabla\Phi\right)$
2	$\int_X f(F_\Phi)$	$\int_X f\left(\nabla\Phi^2+2F_AF_\Phi\right)$	$\int_X f\left(3F_A\nabla\Phi^2+3F_A^2F_\Phi\right)$
3	–	$\int_X f\left(2\nabla\Phi F_\Phi\right)$	$\int_X f\left(\nabla\Phi^3+3F_A\nabla\Phi F_\Phi+3F_AF_\Phi\nabla\Phi\right)$
4	–	$\int_X f\left(F_\Phi^2\right)$	$\int_X f\left(3\nabla\Phi^2F_\Phi+3F_AF_\Phi^2\right)$
5	–	–	$\int_X f\left(3\nabla\Phi F_\Phi^2\right)$
6	–	–	$\int_X f\left(F_\Phi^3\right)$

Consider the first term which is

$$\begin{aligned} t_0\Big(t\mapsto G_{\omega+t\xi_1}d^*_{\omega+t\xi_1}(\xi_2)\Big) &= t_0\Big(t\mapsto G_{\omega+t\xi_1}\Big)d^*_\omega(\xi_2)+G_\omega t_0\Big(t\mapsto d^*_{\omega+t\xi_1}(\xi_2)\Big)\\ &= t_0\Big(t\mapsto G_{\omega+t\xi_1}\Big)0+G_\omega t_0\Big(t\mapsto d^*_\omega(\xi_2)+t\,\mathrm{ad}^*_{\xi_1}(\xi_2)\Big)\\ &= G_\omega\,\mathrm{ad}^*_{\xi_1}(\xi_2). \end{aligned}$$

Taking the adjoint is linear and therefore commutes with differentiation. Finally we note that $\mathrm{ad}^*_{\xi_1}(\xi_2)=-\mathrm{ad}^*_{\xi_2}(\xi_1)$ because

$$\int_X\langle\mathrm{ad}^*_{\xi_1}(\xi_2),\xi_3\rangle\,\mathrm{vol}_X=\int_X\langle\xi_2,\mathrm{ad}_{\xi_1}(\xi_3)\rangle\,\mathrm{vol}_X=-\int_X\langle\mathrm{ad}^*_{\xi_2}(\xi_1),\xi_3\rangle\,\mathrm{vol}_X$$

and the inner product on $\mathfrak{g}$ is invariant. Hence we conclude that the curvature F_A applied to V_1 and V_2 is $G_\omega\,\mathrm{ad}^*_{\xi_1}(\xi_2)$. Putting this all together we have

$$\tilde{F}_{(\omega,q)}(V_1,V_2)=G_\omega\,\mathrm{ad}^*_{\xi_1}(\xi_2)+F_\omega(q)(\zeta_1,\zeta_2)+\frac{1}{2}\Big(\xi_1(\zeta_2)(q)-\xi_2(\zeta_1)(q)-\omega_q([\zeta_1,\zeta_2])\Big).$$

Using this $\tilde{F}$ in the formulae in Table 1 for $d=4$ and $k=3$ we reproduce the results of [13] and for $d=k=3$ the results of [14].

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References

[1] H. Garland, M.K. Murray, Kac–Moody monopoles and periodic instantons, Comm. Math. Phys. 120 (2) (1988) 335–351.
[2] M.K. Murray, D. Stevenson, Higgs fields, bundle gerbes and string structures, Comm. Math. Phys. 243 (3) (2003) 541–555.
[3] M.K. Murray, R.F. Vozzo, The caloron correspondence and higher string classes for loop groups, J. Geom. Phys. 60 (9) (2010) 1235–1250.
[4] M.K. Murray, R.F. Vozzo, Circle actions, central extensions and string structures, Int. J. Geom. Methods Mod. Phys. 7 (6) (2010) 1065–1092.
[5] A. Bergman, U. Varadarajan, Loop groups, Kaluza–Klein reduction and *M*-theory, J. High Energy Phys. 043 (6) (2005) 28 pp. (electronic).
[6] P. Bouwknegt, V. Mathai, *T*-duality as a duality of loop group bundles, J. Phys. A: Math. Theory 42 (16) (2009).
[7] R.F. Vozzo, Loop groups, string classes and equivariant cohomology, J. Aust. Math. Soc. 90 (1) (2011).
[8] W.D. Curtis, The automorphism group of a compact group action, Trans. Amer. Math. Soc. 203 (1975) 45–54.
[9] Richard S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (N.S.) 7 (1) (1982) 65–222.
[10] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957) 181–207.
[11] J.L. Dupont, Curvature and Characteristic Classes, in: Lecture Notes in Mathematics, vol. 640, Springer-Verlag, Berlin, 1978.
[12] M.C. Abbati, R. Cirelli, A. Manià, The orbit space of the action of gauge transformation group on connections, J. Geom. Phys. 6 (4) (1989) 537–557.
[13] M.F. Atiyah, I.M. Singer, Dirac operators coupled to vector potentials, Proc. Natl. Acad. Sci. USA 81 (8, Phys. Sci.) (1984) 2597–2600.
[14] A. Carey, J. Mickelsson, M. Murray, Index theory, gerbes, and Hamiltonian quantization, Comm. Math. Phys. 183 (3) (1997) 707–722.