

CIRCLE ACTIONS, CENTRAL EXTENSIONS AND STRING STRUCTURES

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The caloron correspondence can be understood as an equivalence of categories between G -bundles over circle bundles and $LG \rtimes_{\rho} S^1$ -bundles where LG is the group of smooth loops in G . We use it, and lifting bundle gerbes, to derive an explicit differential form based formula for the (real) string class of an $LG \rtimes_{\rho} S^1$ -bundle.

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Contents

1	Introduction	1066
2	The Caloron Correspondence for $LG \rtimes_{\rho} S^1$ -Bundles	1067
2.1	Semi-direct correspondence	1067
2.2	Caloron correspondence	1070
2.3	Higgs fields and $LG \rtimes_{\rho} S^1$ -bundles	1072
3	The String Class	1077
3.1	The lifting problem	1077
3.2	The string class	1077
4	Central Extensions	1079
4.1	Preliminaries	1079
4.2	Circle actions and semi-direct products	1079
4.3	Central extension of loop group and semi-direct product	1081
5	Lifting Bundle Gerbes	1083
6	The Real String Class of an $LG \rtimes_{\rho} S^1$ -Bundle	1085
6.1	The real string class of an $LG \rtimes_{\rho} S^1$ -bundle	1085
6.1.1	A connection for the lifting bundle gerbe	1085

6.1.2 A curving for the lifting bundle gerbe 1088
 6.1.3 The real string class of an $LG \rtimes_{\rho} S^1$ -bundle 1090
 6.2 The real string class and the first Pontrjagyn class 1091

1. Introduction

The caloron correspondence was first introduced in [1] as a bijection between isomorphism classes of G -instantons on $\mathbb{R}^3 \times S^1$ (calorons) and ΩG -monopoles on \mathbb{R}^3 , where ΩG is the group of based loops in G . The motivation in that case was the study of monopoles for loop groups, in particular, their twistor theory. It was subsequently [2] applied to the case of instantons on the four-sphere and the four-sphere minus a two-sphere and loop group monopoles on hyperbolic three-space. The motivation for the present work however was [3], which used the caloron correspondence to relate string structures on loop group bundles and the Pontrjagyn class of G -bundles. In particular it calculated an explicit de Rham representative for Killingback’s string class [4] using bundle gerbes. In [5] we followed a similar approach to define higher classes of ΩG -bundles which we called string classes and discussed their properties. The present work generalises [3] in a different direction replacing the space $M \times S^1$ by a principal S^1 -bundle $Y \rightarrow M$ and deriving a formula for the image in de Rham cohomology of the string class of an $LG \rtimes_{\rho} S^1$ -bundle $P \rightarrow M$ where LG is the space of free loops in G and $LG \rtimes_{\rho} S^1$ the semi-direct product where the circle acts by rotating the loop.

We begin by discussing the caloron correspondence in this broader context. This was first done in [6] and later in [7] in a string theory context. It is a correspondence between G -bundles $\tilde{P} \rightarrow Y$ over a circle bundle $Y \rightarrow M$ and $LG \rtimes_{\rho} S^1$ -bundles $P \rightarrow M$. It proves useful to introduce an intermediate step which are ρ -equivariant LG -bundles $\bar{P} \rightarrow Y$. We also consider the action of the caloron correspondence on connections which necessitates the introduction of Higgs fields.

With the caloron correspondence complete we review basic material on lifting problems for principal bundles and then introduce the central extension of $LG \rtimes_{\rho} S^1$ proving that it must have the form $\widehat{LG} \rtimes_{\hat{\rho}} S^1$ where $\widehat{LG} \rightarrow LG$ is the Kac–Moody central extension of LG .

The final sections review the notion of a lifting bundle gerbe and extend the results of [3] to the case of the string class of an $LG \rtimes_{\rho} S^1$ -bundle. Our central results are Theorems 6.2 and 6.3:

Theorem. *Let $P \rightarrow M$ be a principal $LG \rtimes_{\rho} S^1$ -bundle and let Φ be a Higgs field for P and (A, a) be a connection for P with curvature (F, f) . Then the real string class of P , is represented in de Rham cohomology by*

$$-\frac{1}{4\pi^2} \int_{S^1} \langle F + f\Phi, \nabla\Phi \rangle d\theta,$$

where

$$\nabla\Phi = d\Phi + [A, \Phi] - \partial A - a\partial\Phi.$$

Theorem. *Let $P \rightarrow M$ be a principal $LG \rtimes_{\rho} S^1$ -bundle and $\tilde{P} \rightarrow Y \rightarrow M$ be the corresponding G -bundle over an S^1 -bundle. Then the real string class of P is given by the integration over the fibre of the first Pontrjagyn class of \tilde{P} . That is,*

$$s(P) = \int_{S^1} p_1(\tilde{P}).$$

Throughout this paper, G will be a compact, connected Lie group and all cohomology groups will use real coefficients.

2. The Caloron Correspondence for $LG \rtimes_{\rho} S^1$ -Bundles

Recall that in [5] we explained the caloron correspondence between ΩG -bundles on a manifold M and certain framed G -bundles on $M \times S^1$. We want to extend this to the case of LG -bundles and also replace $M \times S^1$ by a circle bundle $Y \rightarrow M$. To do this it is useful to first introduce a correspondence between semi-direct product bundles.

2.1. Semi-direct correspondence

We need some results on equivariant bundles but first we need to make conventions for left and right circle actions. One circle action we will be interested in will be that on a right principal S^1 -bundle. However we will also want to consider actions of the circle on Lie groups K arising from automorphisms $\rho: S^1 \rightarrow \text{Aut}(K)$. In this case the natural thing to do is to make the action on the left. So our convention will be that groups are acted on on the left via homomorphisms to the automorphism group and spaces are acted on on the right. We then have

Definition 2.1. Let $\overline{P} \rightarrow M$ be a K -bundle and $\rho: S^1 \rightarrow \text{Aut}(K)$ a homomorphism. We say that $\overline{P} \rightarrow M$ is a ρ -equivariant bundle if S^1 acts on \overline{P} covering an action on M such that for all $p \in \overline{P}, k \in K$ and $\theta \in S^1$ we have $R_{\theta}(pk) = R_{\theta}(p)\rho_{-\theta}(k)$. Here $p \mapsto R_{\theta}(p)$ denotes the action of θ on p and $k \mapsto \rho_{\theta}(k)$ the action of θ on k .

Remark 2.1. As we will see in the next example the sign change here is a result of mixing left and right actions.

Example 2.1. If X is a space let LX be the space of all smooth maps $\gamma: S^1 \rightarrow X$. We make S^1 act on LX by defining $R_{\theta}(\gamma)(\phi) = \gamma(\phi + \theta)$. If $P \rightarrow M$ is a G -bundle we can form the LG -bundle $LP \rightarrow LM$ and note that this is a ρ -equivariant LG -bundle where $\rho: S^1 \rightarrow \text{Aut}(LG)$ is the action $\rho_{\theta}(g)(\phi) = g(\phi - \theta)$. We call this a *loop bundle*.

Example 2.2. The discussion above defines an action $\rho: S^1 \rightarrow LG$ by $\rho_{\theta}(g)(\phi) = g(\phi + \theta)$. There is a natural notion of a homomorphism of groups with actions of the circle on them. In particular if $\rho: S^1 \rightarrow G$ is an action consider the map $\hat{\rho}: G \rightarrow LG$

defined by $\hat{\rho}(g)(\phi) = \rho_\phi(g)$. Then this is a homomorphism of groups and in fact a homomorphism of groups with circle actions as

$$\hat{\rho}(\rho_\theta(g))(\phi) = \rho_\phi(\rho_\theta(g)) = \rho_{\theta+\phi}(g) = \hat{\rho}(g)(\theta + \phi) = \rho_\theta(\hat{\rho}(g))(\phi).$$

Recall that if S^1 acts on a group K we form the semi-direct product $K \rtimes_\rho S^1 = K \times S^1$ with the product

$$(k, \theta)(h, \phi) = (k\rho_\theta(h), \theta + \phi).$$

Note that there is a short exact sequence

$$1 \rightarrow K \rightarrow K \rtimes_\rho S^1 \rightarrow S^1 \rightarrow 1$$

so that if we have a principal $K \rtimes_\rho S^1$ -bundle $P \rightarrow M$ it induces an S^1 -bundle $P(S^1) \rightarrow M$.

Proposition 2.2 (Semi-direct correspondence). *Let $Y \rightarrow M$ be an S^1 -bundle and $\rho: S^1 \rightarrow \text{Aut}(K)$ a homomorphism. Then there is a bijective correspondence between*

- (1) *Principal K -bundles $\bar{P} \rightarrow Y$ which are ρ -equivariant for the circle action on Y , and*
- (2) *Principal $K \rtimes_\rho S^1$ -bundles $P \rightarrow M$ with a circle bundle isomorphism from $P(S^1)$ to Y .*

Proof. Let $\bar{P} \rightarrow Y$ be ρ -equivariant so that we have a circle action on \bar{P} covering the principal bundle circle action on Y such that $R_\theta(pk) = R_\theta(p)\rho_{-\theta}(k)$. Define an action of $K \rtimes_\rho S^1$ on the right of \bar{P} by $p(k, \theta) = R_\theta(pk)$. Then we have

$$\begin{aligned} (p(k, \theta))(h, \phi) &= (R_\theta(pk))(h, \phi) = R_\phi(R_\theta(pk)h) \\ &= R_{\theta+\phi}(pk)\rho_{-\phi}(h) = R_{\theta+\phi}(p)\rho_{-\theta-\phi}(k)\rho_{-\phi}(h) \end{aligned}$$

and thus

$$\begin{aligned} p((k, \theta)(h, \phi)) &= p(k\rho_\theta(h), \theta + \phi) = R_{\theta+\phi}(pk\rho_\theta(h)) \\ &= R_{\theta+\phi}(p)\rho_{-\theta-\phi}(k)\rho_{-\theta-\phi}(\rho_\theta(h)) \\ &= R_{\theta+\phi}(p)\rho_{-\theta-\phi}(k)\rho_{-\phi}(h) = (p(k, \theta))(h, \phi) \end{aligned}$$

so this is a right action. Local triviality and freeness of the action are straightforward to check as is the fact that the orbits are the fibres of the composed map $\bar{P} \rightarrow M$ making \bar{P} a $K \rtimes_\rho S^1$ -bundle over M . The induced S^1 -bundle is the bundle $\bar{P} \times_{K \rtimes_\rho S^1} S^1$ where equivalence classes satisfy $[p, \phi] = [p(k, \theta), -\phi + \theta]$. Let $\pi: \bar{P} \rightarrow Y$ be the projection and define $\chi: \bar{P} \times_{K \rtimes_\rho S^1} S^1 \rightarrow Y$ by $\chi([p, \theta]) = \pi(\rho_\theta(p)) = \rho_\theta(\pi(p))$. Then $\chi(\rho_\phi([p, \theta])) = \chi([p, \theta + \phi]) = \rho_{\phi+\theta}(\pi(p)) = \rho_\phi(\rho_\theta(\pi(p))) = \rho_\phi(\chi([p, \theta]))$. So χ is an S^1 -bundle isomorphism.

Consider the converse. Define an action of K on P by composing with the projection $K \rightarrow K \rtimes S^1$, that is $pk = p(k, 0)$. It is straightforward to check that

these are the fibres of the projection $P \rightarrow P \times_{K \rtimes S^1} S^1$. Define a circle action on P by $R_\theta(p) = p(1, \theta)$. We have

$$\begin{aligned} R_\theta(pk) &= p(k, 0)(1, \theta) = p(1, \theta)(1, -\theta)(k, 0)(1, \theta) \\ &= R_\theta(p)(\rho_{-\theta}(k), 0) = R_\theta(p)\rho_{-\theta}(k) \end{aligned}$$

so that P is a ρ -equivariant K -bundle $P \rightarrow P \times_{K \rtimes S^1} S^1$. Now just pull P back by the inverse of the S^1 -equivariant isomorphism $\chi: P \times_{K \rtimes S^1} S^1 \rightarrow Y$. \square

Remark 2.2. Notice that if $P \rightarrow M$ is a $K \rtimes_\rho S^1$ -bundle over M and $P(S^1)$ is trivial we can consider the subset of P of all elements mapping to the image of the trivialisation. It is easy to see that this is a reduction of P to K . Conversely if $\overline{P} \rightarrow M$ is a K -bundle and we induce a $K \rtimes S^1$ -bundle $P = \overline{P} \times_K (K \rtimes S^1)$ then it has $P(S^1)$ trivial and its reduction to K is naturally isomorphic to \overline{P} via the map $p \mapsto [p, (1, 0)]$.

Denote by $\text{Bun}_K^\rho / \text{Bun}_{S^1}$ the category whose objects are triples (\overline{P}, Y, M) where M is a manifold, $Y \rightarrow M$ an S^1 -bundle and $\overline{P} \rightarrow Y$ a ρ -equivariant bundle and whose morphisms from (\overline{P}, Y, M) to (\overline{P}', Y', M') are triples (f, g, h) where $g: Y \rightarrow Y'$ is an S^1 -bundle map covering $h: M \rightarrow M'$ and $f: \overline{P} \rightarrow \overline{P}'$ is a K -bundle map covering g and commuting with the circle action. Also denote by $\text{Bun}_{K \rtimes_\rho S^1}$ the category of $K \rtimes_\rho S^1$ -bundles $P \rightarrow M$ with bundle maps as morphisms. In both cases we have functors

$$\Pi_{S^1}: \text{Bun}_K^\rho / \text{Bun}_{S^1} \rightarrow \text{Bun}_{S^1}$$

and

$$\Pi_{S^1}: \text{Bun}_{K \rtimes_\rho S^1} \rightarrow \text{Bun}_{S^1}$$

to the category of S^1 -bundles. The first is the obvious projection and the second is the functor $\Pi_{S^1}(P) = P(S^1)$. What we have described in the proposition above are two functors

$$\mathcal{F}: \text{Bun}_K^\rho / \text{Bun}_{S^1} \rightarrow \text{Bun}_{K \rtimes_\rho S^1}$$

and

$$\mathcal{K}: \text{Bun}_{K \rtimes_\rho S^1} \rightarrow \text{Bun}_K^\rho / \text{Bun}_{S^1}$$

which in fact are pseudo-inverses. This means that there are natural isomorphisms

$$\mathcal{K} \circ \mathcal{F} \simeq \text{id}_{\text{Bun}_K^\rho / \text{Bun}_{S^1}} \quad \text{and} \quad \mathcal{F} \circ \mathcal{K} \simeq \text{id}_{\text{Bun}_{K \rtimes_\rho S^1}}.$$

so we have

Proposition 2.3. *The semi-direct product correspondence defines an equivalence of categories*

$$\text{Bun}_K^\rho / \text{Bun}_{S^1} \rightarrow \text{Bun}_{K \rtimes_\rho S^1}$$

2.2. Caloron correspondence

If $Y \rightarrow M$ is an S^1 -bundle denote by $L_{\text{eq}}Y \subset LY$ those loops $f: S^1 \rightarrow Y$ which are S^1 -equivariant in the sense that $f(\theta + \phi) = R_\phi(f(\theta))$. Define $\eta: Y \rightarrow LY$ by $\eta(y)(\phi) = R_\phi(y)$ and note that $\eta(y)(\theta + \phi) = R_{\theta+\phi}(y) = R_\phi(R_\theta(y)) = R_\phi(\eta(y)(\theta))$. Notice also that η is equivariant in the sense that $\eta(R_\theta(y)) = R_\theta(\eta(y))$. In fact we have $\eta(R_\theta(y))(\phi) = R_\phi(R_\theta(y)) = R_{\theta+\phi}(y) = \eta(y)(\theta + \phi) = R_\theta(\eta(y))(\phi)$.

Consider now a G -bundle $\tilde{P} \rightarrow Y$ where $Y \rightarrow M$ is an S^1 -bundle. We can then form the ρ -equivariant LG -bundle $L\tilde{P} \rightarrow LY$. If we pull this back with η to $\eta^*(L\tilde{P}) \rightarrow Y$ we obtain a ρ -equivariant LG -bundle over Y . It is easy to check that this defines a functor

$$\mathcal{R}: \text{Bun}_G / \text{Bun}_{S^1} \rightarrow \text{Bun}_{LG}^\rho / \text{Bun}_{S^1}.$$

This functor is also an equivalence which we can see by constructing a pseudo-inverse

$$\mathcal{S}: \text{Bun}_{LG}^\rho / \text{Bun}_{S^1} \rightarrow \text{Bun}_G / \text{Bun}_{S^1}$$

If $\overline{P} \rightarrow Y$ is a ρ -equivariant LG -bundle we can use the evaluation homomorphism $\text{ev}_0: LG \rightarrow G$, whose kernel is the based loop group ΩG , to form the associated G -bundle $\overline{P} \times_{LG} G = \overline{P}/\Omega G \rightarrow Y$ which is $\mathcal{S}(\overline{P})$.

Proposition 2.4. *The functors \mathcal{R} and \mathcal{S} are pseudo-inverses of each other.*

Proof. Start with a ρ -equivariant LG -bundle $\overline{P} \rightarrow Y$ over a circle bundle $Y \rightarrow M$. Applying \mathcal{S} yields $\overline{P}/\Omega G \rightarrow Y$. Applying \mathcal{R} we first construct $L(\overline{P}/\Omega G) \rightarrow LY$ and then pull back with $\eta: Y \rightarrow LY$. We want to construct an isomorphism $\overline{P} \rightarrow \eta^*(L(\overline{P}/\Omega G))$ of ρ -equivariant LG -bundles over Y . We do this by defining a ρ -equivariant map $\hat{\eta}: \overline{P} \rightarrow L(\overline{P}/\Omega G)$ covering η by $\hat{\eta}(p)(\theta) = R_\theta(p)\Omega G$. We leave the reader to check that this defines an isomorphism of bundles which is moreover a natural transformation from $\mathcal{R} \circ \mathcal{S}$ to the identity functor on $\text{Bun}_{LG}^\rho / \text{Bun}_{S^1}$.

In the other direction we start with a G -bundle $\tilde{P} \rightarrow Y$ over a circle bundle Y . First we form the LG -bundle $L\tilde{P} \rightarrow LY$, pull-back with $\eta: Y \rightarrow LY$ and then $(\mathcal{S} \circ \mathcal{R})(P) = \eta^*(L\tilde{P})/\Omega G$. The evaluation map $\text{ev}_0: \eta^*(L\tilde{P}) \rightarrow \tilde{P}$ is an isomorphism of $LG/\Omega G = G$ -bundles giving the required result. Again we leave the requirements of a natural transformation for the reader to check. □

We define the *caloron transform* to be the composition of functors

$$\mathcal{C} = \mathcal{S} \circ \mathcal{K}: \text{Bun}_{LG \rtimes_\rho S^1} \rightarrow \text{Bun}_G / \text{Bun}_{S^1}$$

and deduce that

Proposition 2.5 (Caloron correspondence). *The caloron transform is an equivalence of categories.*

Remark 2.3. The caloron correspondence in this form first appeared in [6] and later in [7]. We have introduced the equivariant bundles so we can proceed in a

similar manner to [5] and essentially reduce the inverse caloron transform to a pull-back which makes it simpler to define on connections.

It will be useful to write down a formula for the caloron transform of an $LG \rtimes_{\rho} S^1$ -bundle $P \rightarrow Y$ directly. Note that this is given by the constructions outlined above as $\mathcal{C}(P) = P \times_{LG} S^1$ where the LG action on P is factored through the map $LG \rightarrow LG \rtimes_{\rho} S^1$ (Proposition 2.2 and the discussion preceding Proposition 2.4). The following Proposition gives an alternative description:

Proposition 2.6 ([7, 8]). *The caloron transform of the $LG \rtimes_{\rho} S^1$ -bundle $P \rightarrow Y$ is isomorphic to the G -bundle*

$$\tilde{P} = (P \times G \times S^1)/(LG \rtimes_{\rho} S^1),$$

where the action of $LG \rtimes_{\rho} S^1$ on $P \times G \times S^1$ is given by

$$(p, g, \theta)(\gamma, \phi) = (p(\gamma, \theta), \gamma(\theta)^{-1}g, \theta - \phi)$$

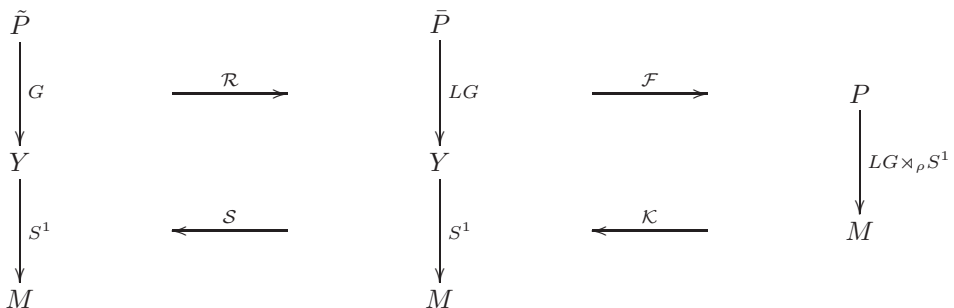
and the action of $h \in G$ on $[p, g, \theta] \in \tilde{P}$ is given by $[p, g, \theta]h = [p, gh, \theta]$.

Proof. In [7] and [8] the caloron correspondence is developed using the bundle \tilde{P} above. Here we show that the map $\hat{\psi}: P \times G \times S^1 \rightarrow \tilde{P}$ given by $(p, g, \theta) \mapsto R_{\theta}(pg) = p(g, \theta)$ descends to an isomorphism of G -bundles $\psi: \tilde{P} \xrightarrow{\sim} \tilde{P}/\Omega G$. To see this, note that if we chose a different representative of the equivalence class of (p, g, θ) , say $(p(\gamma, \phi), \gamma(\theta)^{-1}g, \theta - \phi)$, then

$$\begin{aligned} \hat{\psi}(p(\gamma, \phi), \gamma(\theta)^{-1}g, \theta - \phi) &= p(\gamma, \phi)(\gamma(\theta)^{-1}g, \theta - \phi) = p(\gamma\gamma(\theta)^{-1}g, \theta) \\ &= p(g, \theta)(g^{-1}, -\theta)(\gamma\gamma(\theta)^{-1}g, \theta) \\ &= p(g, \theta)(g^{-1}\rho_{-\theta}(\gamma)\gamma(\theta)^{-1}g, 0) \end{aligned}$$

and $g^{-1}\rho_{-\theta}(\gamma)\gamma(\theta)^{-1}g$ is a based loop in G . Clearly ψ commutes with the G action and so is an isomorphism of G -bundles. □

We can summarise the results of this section with the following diagram:



2.3. Higgs fields and $LG \rtimes_{\rho} S^1$ -bundles

We want to first discuss how the semi-direct correspondence in Proposition 2.2 can be extended to bundles with connections. We find along the way that we need to introduce Higgs fields.

Definition 2.7. Let $\rho: S^1 \rightarrow \text{Aut}(K)$ be a homomorphism and $\overline{P} \rightarrow Y$ a ρ -equivariant bundle. A connection $\overline{A} \in \Omega^1(\overline{P}, \mathfrak{k})$ is called ρ -equivariant if $R_{\theta}^*(\overline{A}) = \rho_{-\theta}(\overline{A})$ where we abuse notation and denote by $\rho_{\theta} \in \text{Aut}(\mathfrak{k})$ the automorphism of Lie algebras induced by $\rho_{\theta} \in \text{Aut}(K)$.

Example 2.3. We have seen in Example 2.1 that if $\tilde{P} \rightarrow Y$ is a G -bundle then $L\tilde{P} \rightarrow LY$ is a ρ -equivariant LG -bundle. If \tilde{A} is a connection on \tilde{P} we can define a connection $L\tilde{A}$ on $L\tilde{P} \rightarrow LY$ as follows. Let $\gamma \in L\tilde{P}$ and note that a tangent vector to $L\tilde{P}$ at γ is a section of the tangent bundle to \tilde{P} pulled-back by γ . Let $\xi \in T_{\gamma}L\tilde{P} = \Gamma([0, 1], \gamma^*(T\tilde{P}))$ we define $L\tilde{A}_{\gamma}(\xi) \in L\mathfrak{g}$ by $L\tilde{A}_{\gamma}(\xi)(\theta) = \tilde{A}_{\gamma(\theta)}(\xi(\theta))$. It is straightforward to check that this is a ρ -equivariant connection.

It is not obvious that ρ -equivariant connections exist but we will prove this below in the case of interest.

We need some additional notation. Let $\delta(p) \in T_pP$ be the tangent vector to the circle action and let $\iota_p: \mathfrak{k} \rightarrow T_pP$ be the derivative of the map $k \mapsto R_k(p)$. If $\gamma: S^1 \rightarrow K$ denote by $\partial\gamma(\theta) \in T_{\gamma(\theta)}K$ the image of $\partial/\partial\theta$ under the tangent to γ . Denote by $\gamma^{-1}\partial\gamma(\theta)$ the left-translate of this to \mathfrak{k} so that $\gamma^{-1}\partial\gamma$ is a loop in \mathfrak{k} . In particular $\theta \mapsto \rho_{\theta}(k)$ defines a map $\rho_k: S^1 \rightarrow K$ and thus $\rho_k^{-1}\partial\rho_k(0) \in \mathfrak{k}$.

Example 2.4. If $\gamma \in K = LG$ then $\rho_{\gamma}^{-1}\partial\rho_{\gamma}(0) = \gamma^{-1}\partial\gamma$.

We also note here the data which corresponds to a connection on a $K \rtimes_{\rho} S^1$ -bundle P . It is straightforward using the definition of connection and the semi-direct product action to show that

Proposition 2.8. A connection on a $K \rtimes_{\rho} S^1$ -bundle $P \rightarrow M$ is equivalent to a pair (A, a) , where A is \mathfrak{k} valued and a is \mathbb{R} valued, satisfying

$$\begin{aligned} a(\delta) &= 1 \\ A_p(\iota_p(\xi)) &= \xi \end{aligned}$$

and

$$\begin{aligned} R_k^*(A) &= \text{ad}(k^{-1})A - a\rho_k^{-1}\partial\rho_k(0) \\ R_k^*(a) &= a \\ R_{\theta}^*(A) &= \rho_{-\theta}(A) \\ R_{\theta}^*(a) &= a. \end{aligned}$$

We also need the notion of a Higgs field.

Definition 2.9. Let $P \rightarrow M$ be a $K \rtimes_{\rho} S^1$ -bundle. A *Higgs field* for P is a function $\Phi: P \rightarrow \mathfrak{k}$ satisfying:

$$\Phi(p(k, \theta)) = \rho_{-\theta}(\text{ad}(k^{-1})\Phi(p) + \rho_k^{-1}\partial\rho_k(0))$$

for all $k \in K, \theta \in S^1$ and $p \in P$.

Proposition 2.10. *Higgs fields exist.*

Proof. First notice that a convex combination of Higgs fields is a Higgs field. We can then combine locally defined Higgs fields with a partition of unity. So assume now that $P = M \times K \rtimes_{\rho} S^1$ and then define

$$\Phi((k, \theta)) = \rho_k^{-1}\partial\rho_k(0)$$

We leave it as an exercise to show that this is a Higgs field. □

Proposition 2.11. *Let $Y \rightarrow M$ be an S^1 -bundle with connection α and $\rho: S^1 \rightarrow \text{Aut}(K)$ a homomorphism. Then there is a bijective correspondence between*

- (1) *Principal K -bundles $\bar{P} \rightarrow Y$ which are ρ -equivariant for the circle action on Y equipped with a ρ -equivariant connection \bar{A} on $\bar{P} \rightarrow Y$ and*
- (2) *Principal $K \rtimes_{\rho} S^1$ -bundles $P \rightarrow M$ with a circle bundle isomorphism from $P(S^1)$ to Y and a pair $((A, a), \Phi)$ consisting of a connection (A, a) which projects to α on Y under the isomorphism and a Higgs field Φ on $P \rightarrow M$.*

Proof. Note that as a manifold $\bar{P} = P$. It is easy to show that

$$\bar{A} = A + a\Phi$$

defines a ρ -equivariant connection on $\bar{P} \rightarrow Y$.

Conversely given the ρ -equivariant connection on \bar{P} we can define a Higgs field $\Phi(p) = A_p(\delta_p)$ for $p \in P = \bar{P}$. Then the connection (A, a) is defined by

$$(A, a) = (\bar{A} + \pi^*(\alpha)\Phi, \pi^*(\alpha))$$

where $\pi: \bar{P} \rightarrow Y$. □

These constructions allow us to extend the functors defined in the previous section to act on bundles with connections and Higgs fields. We define a new category $\text{cBun}_K^{\rho} / \text{cBun}_{S^1}$ to consist of objects (\bar{P}, Y, M) in $\text{Bun}_K^{\rho} / \text{Bun}_{S^1}$ with the addition of a connection α on $Y \rightarrow M$ and a ρ -equivariant connection \bar{A} on $\bar{P} \rightarrow Y$. Morphisms are just the morphisms in $\text{Bun}_K^{\rho} / \text{Bun}_{S^1}$ which preserve the connections. Similarly an object in $\text{cBun}_{K \rtimes_{\rho} S^1}$ is an object $P \rightarrow M$ in $\text{Bun}_{K \rtimes_{\rho} S^1}$

with a connection and Higgs field $((A, a), \Phi)$ again morphisms preserve connections and Higgs fields. Then we have functors

$$\mathcal{F}: \text{cBun}_K^\rho / \text{cBun}_{S^1} \rightarrow \text{cBun}_{K \rtimes_\rho S^1}$$

and

$$\mathcal{K}: \text{cBun}_{K \rtimes_\rho S^1} \rightarrow \text{cBun}_K^\rho / \text{cBun}_{S^1}$$

which are pseudo-inverses.

To define the caloron transform for bundles with connections (and Higgs fields) we have to extend the second part of the correspondence

$$\mathcal{R}: \text{Bun}_G / \text{Bun}_{S^1} \rightarrow \text{Bun}_{LG}^\rho / \text{Bun}_{S^1}$$

and

$$\mathcal{S}: \text{Bun}_{LG}^\rho / \text{Bun}_{S^1} \rightarrow \text{Bun}_G / \text{Bun}_{S^1}$$

to act appropriately. First we need to define the corresponding categories. We define $\text{cBun}_G / \text{cBun}_{S^1}$ to consist of G -bundles $\tilde{P} \rightarrow Y$ with connection \tilde{A} and a connection α on $Y \rightarrow M$. Morphisms are those morphisms preserving the connections. The definition of $\text{cBun}_{LG}^\rho / \text{cBun}_{S^1}$ follows from the definitions already given.

To extend the correspondence note first that if we start with a ρ -equivariant LG -bundle $\bar{P} \rightarrow Y$ with a ρ -equivariant connection \bar{A} then there is an induced connection on the G -bundle $\bar{P} \times_{LG} G = \bar{P}/\Omega G$. In the other direction if $\tilde{P} \rightarrow Y$ is a G -bundle and \tilde{A} is a connection on $\tilde{P} \rightarrow Y$ and α a connection on $Y \rightarrow M$ then there is a loop connection $L\tilde{A}$ on $L\tilde{P} \rightarrow LY$ which is an LG connection. This pulls back to give a connection $\eta^*(L\tilde{A})$. It remains to check that this is ρ -equivariant but as η is ρ -equivariant it suffices to check this for $L\tilde{A}$ which we noted in Example 2.3 is straightforward.

It is now easy to define the inverse caloron correspondence with connection as it is the result of applying Proposition 2.11 and noting that equivariant connections and Higgs fields pull-back. We have

Proposition 2.12. *The caloron correspondence extends to an equivalence of categories between G -bundles with connection over S^1 -bundles with connection and $LG \rtimes_\rho S^1$ -bundles with connection and Higgs field.*

Proof. We need to show the constructions in Proposition 2.4 preserve the connections. Recall that if $Q \rightarrow X$ is a LG -bundle then $Q/\Omega G = Q \times_{LG} G$ is a G -bundle. Moreover if $\text{ev}_0: LG \rightarrow G$ is the evaluation map whose kernel is ΩG and $p: Q \rightarrow Q/\Omega G$ is the projection then given a connection one-form A on Q there is a unique connection one-form B on $Q/\Omega G$ satisfying $p^*B = \text{ev}_0(A)$ where $\text{ev}_0: \mathfrak{Lg} \rightarrow \mathfrak{g}$ is the induced evaluation map on Lie algebras (see [9] for a proof in a more general setting).

Recall from Proposition 2.4 the constructions involved. We start with a ρ -equivariant LG -bundle $\bar{P} \rightarrow Y$ over a circle bundle $Y \rightarrow M$. Then we construct

an isomorphism $\bar{P} \rightarrow \eta^*(L(\bar{P}/\Omega G))$ of ρ -equivariant LG -bundles over Y by defining a ρ -equivariant map $\hat{\eta}: \bar{P} \rightarrow L(\bar{P}/\Omega G)$ covering η by $\hat{\eta}(p)(\theta) = R_\theta(p)\Omega G$. Notice that we can lift $\hat{\eta}$ to $\tilde{\eta}: \bar{P} \rightarrow L\bar{P}$ by letting $\tilde{\eta}(p)(\theta) = R_\theta(p)$. Consider a ρ -equivariant connection \bar{A} on \bar{P} . Let B be the induced connection on $\bar{P}/\Omega G$. Then if $p: \bar{P} \rightarrow \bar{P}/\Omega G$ we have $p^*B = \text{ev}_0(\bar{A})$ from the discussion above. The connection on $L(\bar{P}/\Omega G)$ is LB and we need to show that $\hat{\eta}^*LB = \bar{A}$. We have

$$\hat{\eta}^*LB = \tilde{\eta}^*p^*LB = \tilde{\eta}^*L(\text{ev}_0(\bar{A})).$$

Let ξ be a tangent vector at $q \in \bar{P}$ and note that the ρ -equivariance of \bar{A} implies that $R_\theta^*\bar{A} = \rho_{-\theta}(\bar{A})$ so that

$$\bar{A}(R_\theta(p))(R_\theta(\xi))(\phi) = \bar{A}(p)(\xi)(\phi + \theta)$$

so that

$$\bar{A}(R_\theta(p))(R_\theta(\xi))(0) = \bar{A}(p)(\xi)(\theta)$$

and thus

$$\tilde{\eta}^*(L(\text{ev}_0(\bar{A}))) (p)(\xi)(\theta) = \bar{A}(R_\theta(p))(R_\theta(\xi))(0) = \bar{A}(p)(\xi)(\theta)$$

so that

$$\hat{\eta}^*LB = \tilde{\eta}^*L(\text{ev}_0(\bar{A})) = \bar{A}$$

as required.

In the other direction we start with a G -bundle $\tilde{P} \rightarrow Y$ over a circle bundle Y and use the evaluation map $\text{ev}_0: \eta^*(L\tilde{P})/\Omega G \rightarrow \tilde{P}$ to define an isomorphism of $LG/\Omega G = G$ -bundles. If \tilde{A} is a connection on \tilde{P} then $\eta^*L\tilde{A}$ is a connection on $\eta^*(L\tilde{P})$ and the connection B on $\eta^*(L\tilde{P})/\Omega G$ satisfies $p^*B = \text{ev}_0(\eta^*L\tilde{A})$ where $p: \eta^*(L\tilde{P}) \rightarrow \eta^*(L\tilde{P})/\Omega G$. Denoting $\tilde{\text{ev}}_0: \eta^*(L\tilde{P}) \rightarrow \tilde{P}$ where $\tilde{\text{ev}}_0 = \text{ev}_0 \circ p$ it suffices to show that $\tilde{\text{ev}}_0^*\tilde{A} = \text{ev}_0(\eta^*L\tilde{A})$. If ξ is a tangent vector to $\gamma \in L\tilde{P}$ then $(\tilde{\text{ev}}_0^*\tilde{A})(\gamma)(\xi) = \tilde{A}(\gamma(0))(\xi(0)) = \text{ev}_0(\theta \mapsto \tilde{A}(\gamma(\theta))(\xi(\theta))) = \text{ev}_0(\eta^*L\tilde{A})(\gamma)(\xi)$ as required. \square

Recall that in Proposition 2.6 we gave a formula for the caloron transform \tilde{P} of P . Namely, $\tilde{P} = (P \times G \times S^1)/(LG \times_\rho S^1)$. We can also give a formula for the connection on this bundle induced by the functor \mathcal{C}

Proposition 2.13. *Let P be an $LG \times_\rho S^1$ -bundle with connection (A, a) and Higgs field Φ . Then the connection form on the bundle \tilde{P} (considered as a form on $P \times G \times S^1$ which descends to the quotient) is given by*

$$\tilde{A} = \text{ad}(g^{-1})A(\theta) + \Theta + \text{ad}(g^{-1})\Phi(a + d\theta),$$

where Θ is the Maurer–Cartan form on G .

Proof. Recall from Proposition 2.6 that we have a bundle map

$$\begin{array}{ccc}
 P \times G \times S^1 & \xrightarrow{\widehat{\psi}} & \bar{P} \\
 \downarrow \bar{\pi} & & \downarrow \bar{\pi} \\
 P \times G \times S^1 & \xrightarrow{\psi} & \bar{P}/\Omega G \\
 LG \rtimes_{\rho} S^1 & &
 \end{array}$$

given by $\widehat{\psi}(p, g, \theta) = p(g, \theta)$. We know that the connection on the equivariant LG -bundle \bar{P} is given in terms of the pair $((A, a), \Phi)$ on P by $\bar{A} = A + a\Phi$ (Proposition 2.11) and that the connection B on $\bar{P}/\Omega G$ satisfies $\bar{\pi}^*B = \text{ev}_0(\bar{A}) = \text{ev}_0(A + a\Phi)$ (from the proof of Proposition 2.12). Therefore, since the diagram above commutes, to give the connection on \bar{P} as a form on $P \times G \times S^1$ it is enough for us to calculate $\widehat{\psi}^* \text{ev}_0(A + a\Phi)$. Note that if Q is a principal G -bundle and $R: Q \times G \rightarrow Q$ is the right action of G on Q then for tangent vectors $X \in T_qQ$ and $g\xi \in T_gG$ (where $\xi \in \mathfrak{g}$ and here we are identifying T_gG with \mathfrak{g} via left multiplication on G) the push-forward $R_*: T_qQ \times T_gG \rightarrow T_{qg}Q$ is given (using the Liebnitz rule) by $(X, g\xi) \mapsto (R_{g, \theta})_*X + \iota_{qg}\xi$. In this case, if $(X, g\xi, x_{\theta}) \in T_{(p, g, \theta)}(P \times G \times S^1)$ is a tangent vector then the derivative of $\widehat{\psi}$ is given by

$$\widehat{\psi}_*(X, g\xi, x_{\theta}) = (R_{(g, \theta)})_*X + \iota_{p(g, \theta)}(\xi, x),$$

where $x \in \mathbb{R}$ is the Lie algebra element corresponding to the tangent vector $x_{\theta} \in T_{\theta}S^1$. (Note that here we are using the fact that \bar{P} is really the same as the bundle P .) We have

$$\begin{aligned}
 & (\widehat{\psi}^* \text{ev}_0(A + a\Phi))_{(p, g, \theta)}(X, g\xi, x_{\theta}) \\
 &= \text{ev}_0(A + a\Phi)_{p(g, \theta)}((R_{(g, \theta)})_*X + \iota_{p(g, \theta)}(\xi, x)) \\
 &= \text{ev}_0(A + a\Phi)_{p(g, \theta)}((R_{(g, \theta)})_*X) + \text{ev}_0(A + a\Phi)_{p(g, \theta)}(\iota_{p(g, \theta)}(\xi, x)) \\
 &= \text{ad}(g, \theta)^{-1} \text{ev}_0(A + a\Phi)_p(X) + \text{ev}_0(\xi + x\Phi(p(g, \theta))) \\
 &= \rho_{-\theta}(\text{ad}(g^{-1})A(X)(0) + a(X) \text{ad}(g^{-1})\Phi(p)(0)) \\
 &\quad + \xi + x\rho_{-\theta}(\text{ad}(g^{-1})\Phi(p)(0)) \\
 &= \text{ad}(g^{-1})A(X)(\theta) + a(X) \text{ad}(g^{-1})\Phi(p)(\theta) + \xi + x \text{ad}(g^{-1})\Phi(p)(\theta).
 \end{aligned}$$

That is,

$$\widehat{\psi}^* \text{ev}_0(A + a\Phi) = \text{ad}(g^{-1})A(\theta) + \Theta + \text{ad}(g^{-1})\Phi(a + d\theta).$$

□

3. The String Class

3.1. The lifting problem

In this paper, we shall be primarily concerned with the so-called lifting problem for a principal bundle whose structure group has a central extension. (In particular, we will be concerned with the central extension of the semi-direct product $LG \rtimes_{\rho} S^1$.)

Suppose we are given a K -bundle $P \rightarrow M$ and a central extension

$$U(1) \rightarrow \widehat{K} \rightarrow K. \tag{1}$$

We would like to know when the bundle P lifts to a \widehat{K} -bundle \widehat{P} . If we denote by \underline{H} the sheaf of smooth maps into a Lie group H then the exact sequence (1) gives rise to a corresponding short exact sequence of sheaves of groups and that in turn gives rise to a long exact sequence in cohomology part of which is

$$\dots \rightarrow H^1(M, \underline{U}(1)) \rightarrow H^1(M, \underline{\widehat{K}}) \rightarrow H^1(M, \underline{K}) \rightarrow \dots$$

Note, however, that since K and \widehat{K} are in general nonabelian, this is an exact sequence of *pointed sets* rather than groups. Since $U(1)$ is central in \widehat{K} we can extend this sequence one step to the right to obtain (see [10])

$$H^1(M, \underline{U}(1)) \rightarrow H^1(M, \underline{\widehat{K}}) \rightarrow H^1(M, \underline{K}) \rightarrow H^2(M, \underline{U}(1)).$$

To define the coboundary map $H^1(M, \underline{K}) \rightarrow H^2(M, \underline{U}(1))$ take transition functions $g_{\alpha\beta}$ for the K -bundle P relative to some good cover $\{U_{\alpha}\}$ and lift them to maps $\hat{g}_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \rightarrow \widehat{K}$ such that $p(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. These are our candidate transition functions for \widehat{P} . However, transition functions are required to satisfy the cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on triple overlaps but the lifts $\hat{g}_{\alpha\beta}$ only satisfy

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}\hat{g}_{\alpha\gamma}$$

for some $U(1)$ -valued function $\epsilon_{\alpha\beta\gamma}$. It can be shown that the function $\epsilon_{\alpha\beta\gamma}$ defines a cocycle in $H^2(M, \underline{U}(1))$ and this cocycle defines the image of $g_{\alpha\beta}$ under the coboundary $H^1(M, \underline{K}) \rightarrow H^2(M, \underline{U}(1))$. We see that the $\hat{g}_{\alpha\beta}$'s are transition functions precisely when this cocycle is trivial. Note that there is an isomorphism $H^2(M, \underline{U}(1)) \simeq H^3(M, \mathbb{Z})$ induced by the exact sequence

$$\mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1).$$

Therefore, under this isomorphism, we have that the obstruction to lifting P to \widehat{P} is a class in $H^3(M, \mathbb{Z})$.

3.2. The string class

As a precursor to the problem we really want to consider — that of lifting a principal $LG \rtimes_{\rho} S^1$ -bundle — we shall quickly describe the simpler case [3] of which our work is a generalisation. Namely that of a *string structure* for an LG -bundle. The loop group of a compact, simple, simply connected Lie group G has a well-known central extension \widehat{LG} (see [11] for details) and so the theory we have described in

the previous section can be applied. In particular, given an LG -bundle $P \rightarrow M$ there is a class in $H^3(M)$, called the *string class*, which represents the obstruction to lifting P to an \widehat{LG} -bundle. If such a lifting exists (i.e. if the string class vanishes) then P is said to have a string structure. We shall always be interested in the image of the string class in real cohomology which we shall also call the string class or occasionally the real string class. Of course if M has no three-dimensional torsion the string class and the real string class are equivalent information. String structures were first introduced by Killingback in [4] as a string theory analogue of spin structures and studied further in [12–14, 3]. Our work is closely related to that in [3]. In that case, the first author together with Stevenson have given an explicit differential form based formula for the string class which we review. It involves a Higgs field for the bundle. A Higgs field for an LG -bundle is a map $\Phi: P \rightarrow L\mathfrak{g}$ which satisfies

$$\Phi(p\gamma) = \text{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma,$$

for all $\gamma \in LG$. They then prove

Theorem 3.1 ([3]). *Let $P \rightarrow M$ be a principal LG -bundle. Let A be a connection on P with curvature F and let Φ be a Higgs field for P . Then the real string class of P is represented in de Rham cohomology by the form*

$$-\frac{1}{4\pi^2} \int_{S^1} \langle F, \nabla\Phi \rangle d\theta,$$

where \langle, \rangle is a suitably normalised invariant inner product on \mathfrak{g} and

$$\nabla\Phi = d\Phi + [A, \Phi] - \partial A.$$

Additionally in [3], a formula is given for the real string class in terms of the Pontrjagyn class of the caloron transform of P . This formula generalises Killingback’s original result which states that in the special case that $P \rightarrow M$ is a loop bundle (so $P = LQ$ and $M = LX$ for $Q \rightarrow X$ a G -bundle) then the string class of P is given by pulling-back the first Pontrjagyn class of $Q, p_1(Q)$, to $LX \times S^1$ and integrating over the circle:

$$s(LQ) = \int_{S^1} \text{ev}^* p_1(Q),$$

where we have written $s(P)$ for the string class of P and $\text{ev}: LX \times S^1 \rightarrow X$ is the evaluation map which evaluates a loop at a point in S^1 . In order to generalise this result to general LG -bundles (which are not necessarily loop bundles) Murray and Stevenson use the caloron correspondence to relate the real string class to a class on a G -bundle. The generalisation of Killingback’s result then is

Theorem 3.2 ([3]). *If $\tilde{P} \rightarrow M \times S^1$ is the G -bundle corresponding to the LG -bundle $P \rightarrow M$ then the real string class of P is given by*

$$s(P) = \int_{S^1} p_1(\tilde{P}).$$

4. Central Extensions

4.1. Preliminaries

In this section we shall review some general results from [15, 3] on constructing central extensions of Lie groups before discussing the case where we have the circle acting and central extensions of loop groups. As in (1) let \widehat{K} be a central extension of K by $U(1)$. Choose a connection A for $\widehat{K} \rightarrow K$ thought of as $U(1)$ -bundle and denote its curvature by $R \in \Omega^2(K)$. Let $\widehat{m}: \widehat{K} \times \widehat{K} \rightarrow \widehat{K}$ be the multiplication and denote

$$\delta(A) = \pi_1^*(A) - m^*(A) + \pi_2^*(A).$$

It is shown in [15, 3] that $\delta(A) = \pi^*(\alpha)$ for $\alpha \in \Omega^1(K \times K)$ and that moreover $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$. Here we use $\delta(K^p) \rightarrow \delta(K^{p+1})$ to denote the alternating sum

$$\delta = \sum_{i=0}^p (-1)^i d_i^*.$$

where $d_i: K^{p+1} \rightarrow K^p$ is defined by

$$d_i(k_1, \dots, k_{p+1}) = \begin{cases} (k_2, \dots, k_{p+1}), & i = 0 \\ (k_1, \dots, k_{i-1}k_i, k_{i+1}, \dots, k_{p+1}), & 1 \leq i \leq p - 1 \\ (k_1, \dots, k_p), & i = p \end{cases}$$

Conversely choose a pair of forms (R, α) such that $R \in \Omega^2(K)$, $\alpha \in \Omega^2(K \times K)$, $dR = 0$, $2\pi iR$ is integral, $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$. Then we can explicitly construct a central extension $\widehat{K} \rightarrow K$ with connection from which we can recover (R, α) by the above construction. Two pairs (R, α) and (R', α') give rise to the same central extension if and only if there is a one-form $\eta \in \Omega^1(K)$ such that $R = R' + d\eta$ and $\alpha = \alpha' + \delta\eta$.

Example 4.1. In the case that K is the loop group LG the central extension whose class is the generator of $H^2(LG, \mathbb{Z})$ is shown in [3] to be determined by

$$R = \frac{i}{4\pi} \int_{S^1} \langle \Theta, \partial\Theta \rangle d\theta, \tag{2}$$

$$\alpha = \frac{i}{2\pi} \int_{S^1} \langle d_2^*\Theta, d_0^*Z \rangle d\theta \tag{3}$$

for Θ the Maurer–Cartan form on G . The bracket here is an invariant inner product on \mathfrak{g} normalised so that the longest root has length squared equal to 2 and Z is the function on $LG; \gamma \mapsto \partial\gamma\gamma^{-1}$.

4.2. Circle actions and semi-direct products

We specialise now to the case of $LG \rtimes_{\rho} S^1$ for the usual homomorphism $\rho: S^1 \rightarrow \text{Aut}(LG)$. Let $\widehat{LG} \rightarrow LG$ be the standard central extension whose Chern class is a

generator of $H^2(LG, \mathbb{Z})$. We will show that there is a unique central extension of $LG \rtimes_{\rho} S^1$ whose Chern class is the class pulled back to $H^2(LG \rtimes_{\rho} S^1, \mathbb{Z})$.

For existence note that $\widehat{LG} \rightarrow LG$ can be constructed explicitly [3] using the (R, α) in Example 4.1 and, moreover, these (R, α) are invariant under the action given by ρ . It follows that we can lift the action ρ to an action $\hat{\rho}: S^1 \rightarrow \text{Aut}(\widehat{LG})$ and then $\widehat{LG} \rtimes_{\hat{\rho}} S^1$ is the required central extension of $LG \rtimes_{\rho} S^1$.

For uniqueness we will show that any central extension $\mathcal{H} \rightarrow K \rtimes S^1$ is determined by its Chern class as a $U(1)$ -bundle. Consider first the corresponding problem for the loop group LG . The observant reader will notice that in [3] the central extension of LG was defined by choosing an (R, α) rather than taking one of the standard constructions and calculating (R, α) . The question, of course, is to find α as R is essentially defined by specifying the Chern class of the central extension. While still avoiding the actual calculation we can resolve the question of whether we had the correct (R, α) in [3] from the following.

Proposition 4.1. *If $\mathcal{K} \rightarrow LG$ and $\mathcal{H} \rightarrow LG$ are central extensions which have the same Chern class in $H^2(LG, \mathbb{Z})$ then they are isomorphic as central extensions.*

Proof. We can assume that $\mathcal{K} \rightarrow LG$ and $\mathcal{H} \rightarrow LG$ are the same $U(1)$ -bundle with possibly different multiplications so let us denote that by $\mathcal{K} \rightarrow LG$. By choosing a connection on the bundle we can construct (R, α) and (R, β) characterising the two central extensions. We have $\delta(R) = d\alpha = d\beta$ so that $d(\alpha - \beta) = 0$. As LG is simply-connected there is a $\chi: LG \times LG \rightarrow S^1$ such that $\alpha = \beta + \chi^{-1}d\chi$. Without loss of generality we can normalise so that $\chi(e, e) = 1$. As $\delta(a) = \delta(b) = 0$ we have $\delta(\chi^{-1})d\delta(\chi) = 0$ or $\delta(\chi)$ is a constant. Evaluating at (e, e, e) we find that $\delta(\chi) = 1$ or

$$\chi(g_1, g_2) = \chi(g_2, e)^{-1}\chi(g_1, e)^{-1}$$

so that letting $\mu(g) = \chi(g, e)^{-1}$ we see that $\delta(\mu) = \chi$. If we let $\eta = \mu^{-1}d\mu$ we now have $R = R + d\mu$ and $\alpha = \beta + \delta\eta$ so the the central extensions are isomorphic. \square

Notice that any automorphism of \widehat{LG} fixes the centre so descends to an automorphism of LG so that we have a homomorphism $\text{Aut}(\widehat{LG}) \rightarrow \text{Aut}(LG)$. Let $\mathcal{H} \rightarrow LG \rtimes_{\rho} S^1$ be a central extension whose Chern class is the pull-back of the Chern class of $\widehat{LG} \rightarrow LG$. The result above shows that the restriction of \mathcal{H} to LG is isomorphic, as a central extension, to $\widehat{LG} \rightarrow LG$. Hence we have a short exact sequence of central extensions of the form:

$$\begin{CD} \widehat{LG} @>>> \mathcal{H} @>>> S^1 \\ @VVV @VVV @VVV \\ LG @>>> LG \rtimes_{\rho} S^1 @>>> S^1 \end{CD}$$

The map $z \mapsto (1, z)$ defines a homomorphism $S^1 \rightarrow LG \rtimes_{\rho} S^1$ and pulling back $\mathcal{H} \rightarrow LG \rtimes_{\rho} S^1$ defines a central extension of the circle by the circle. This is trivial by

Proposition 4.2. *Any central extension of the circle by the circle is trivial.*

Proof. For convenience let us denote the central extension by

$$U(1) \rightarrow H \rightarrow S^1$$

where $U(1) \subset H$ is in the centre of H . We prove the existence of a splitting homomorphism $S^1 \rightarrow H$.

First choose a right H invariant connection on $H \rightarrow S^1$ by translating around some splitting of the tangent space at e . Consider the homomorphism $\mathbb{R} \rightarrow S^1$ and lift it to a horizontal map $f: \mathbb{R} \rightarrow H$ with $f(0) = e$. Uniqueness for ordinary differential equations and the fact that $t \mapsto f(t + s)f(s)^{-1}$ is a horizontal lift as well shows that f is a homomorphism.

We would like $f: \mathbb{R} \rightarrow H$ to descend to $S^1 \rightarrow H$ but generally we will have $f(2\pi) \neq e$. Choose $h: \mathbb{R} \rightarrow U(1)$ a homomorphism with $h(2\pi) = f(2\pi)^{-1}$. Then, because $U(1)$ is central hf is a homomorphism which gives the required splitting. □

It follows that we have a homomorphism $\sigma: S^1 \rightarrow \mathcal{H}$ and an induced $\hat{\rho}: S^1 \rightarrow \text{Aut}(\widehat{LG})$ making \mathcal{H} a semi-direct product $\widehat{LG} \rtimes_{\hat{\rho}} S^1$. Uniqueness follows from the uniqueness of $\hat{\rho}$ which, in turn, follows from:

Proposition 4.3. *The homomorphism $\text{Aut}(\widehat{LG}) \rightarrow \text{Aut}(LG)$ is injective.*

Proof. Assume that $\phi \in \text{Aut}(\widehat{LG})$ acts trivially on LG . It follows that for any $\hat{g} \in \widehat{LG}$ we have $\phi(\hat{g}) = \hat{g}\chi(g)$ for some homomorphism $\chi: \widehat{LG} \rightarrow U(1)$. If we restrict the homomorphism χ to $U(1) \subset \widehat{LG}$ then it must take the form $\chi(z) = z^p$ for some $p \in \mathbb{Z}$. If $p = 0$ we are done. Otherwise consider the kernel $K \subset \widehat{LG}$ of χ . From [11] we know that LG is equal to its own commutator so that $[\widehat{LG}, \widehat{LG}] \subset \widehat{LG}$. But $[\widehat{LG}, \widehat{LG}] \subset K$ so that K covers LG . Thus K is a reduction of \widehat{LG} to \mathbb{Z}_p and the Chern class of \widehat{LG} is torsion which is a contradiction. So $K = \{e\}$. □

4.3. Central extension of loop group and semi-direct product

In what follows we will require an explicit construction of the central extension of $LG \rtimes_{\rho} S^1$ in a manner analogous to that in Sec. 4.1. However that construction needs a slight modification because $LG \rtimes_{\rho} S^1$ is not simply-connected. This is done [3] by replacing the two-form R with a differential character [16] for the bundle $\widehat{K} \rightarrow K$. That is, we add to our pair (R, α) a homomorphism $h: Z_1(K) \rightarrow U(1)$ satisfying

$$h(\partial\sigma) = \exp\left(\int_{\sigma} R\right)$$

for every two-cycle σ in K . We also require the compatibility condition

$$(\delta h)(\gamma) = \exp\left(\int_{\gamma} \alpha\right)$$

for every closed one-cycle γ in $K \times K$. Therefore, we need to find a triple of objects (R, α, h) as above.

Following from the result in the previous section we take as our R the pull-back of the form (2) above to $LG \rtimes_{\rho} S^1$. That is,

$$R = \frac{i}{4\pi} \int_{S^1} \langle \Theta, \partial\Theta \rangle d\theta.$$

As noted before, since we are integrating over the circle this expression is invariant under the action ρ . Now, to find α we need to calculate $\delta R = \pi_1^* R - m^* R + \pi_2^* R$, where π_i is the projection $LG \rtimes_{\rho} S^1 \times LG \rtimes_{\rho} S^1 \rightarrow LG \rtimes_{\rho} S^1$ which omits the i th factor and m is the multiplication defined above. The pull-back $\pi_i^* R$ is given by

$$\frac{i}{4\pi} \int_{S^1} \langle \pi_i^* \Theta, \partial\pi_i^* \Theta \rangle d\theta$$

and so it remains to calculate $m^* R$. From now on we shall write Θ_1 for $\pi_2^* \Theta$ and so on. Using the fact that R is ρ -invariant as well as the ad-invariance of the inner product and the identity

$$\partial(\text{ad}(\gamma^{-1})X) = \text{ad}(\gamma^{-1})[X, Z] + \text{ad}(\gamma^{-1})\partial X, \tag{4}$$

for a tangent vector X and Z the function on LG defined above, we have

$$\begin{aligned} m^* R &= \frac{i}{4\pi} \int_{S^1} \langle [\Theta_1, \Theta_1], Z_2 \rangle + \langle \Theta_1, \partial\Theta_1 \rangle + \langle \Theta_2, \partial\Theta_2 \rangle \\ &\quad + 2\langle \text{ad}(\gamma_2^{-1})\Theta_1, \partial\Theta_2^{\rho^2} \rangle - 2\langle \mu_1 \text{ad}(\gamma_2^{-1})Z_2, \partial\Theta_2^{\rho^2} \rangle - 2\langle \mu_1 Z_2, \partial\Theta_1 \rangle d\theta, \end{aligned}$$

where μ is the Maurer–Cartan form on S^1 and we have written (for example) $\xi_1^{\rho^1}$ for $\rho_{\phi_1}(\xi_1)$. Therefore

$$\begin{aligned} \delta R &= \frac{i}{2\pi} \int_{S^1} -\frac{1}{2}\langle [\Theta_1, \Theta_1], Z_2 \rangle - \langle \text{ad}(\gamma_2^{-1})\Theta_1, \partial\Theta_2^{\rho^2} \rangle \\ &\quad + \langle \mu_1 \text{ad}(\gamma_2^{-1})Z_2, \partial\Theta_2^{\rho^2} \rangle + \langle \mu_1 Z_2, \partial\Theta_1 \rangle d\theta. \end{aligned}$$

Now define

$$\alpha = \frac{i}{2\pi} \int_{S^1} \left\langle \pi_2^* \Theta^{\rho^{-1}} - \frac{1}{2} \pi_2^* \mu \pi_1^* Z, \pi_1^* Z \right\rangle d\theta. \tag{5}$$

Then it is easy to check that $d\alpha = \delta R$ and $\delta\alpha = 0$. Notice that the 2-form R is left invariant and the 1-form α is left invariant in the first slot. To find the homomorphism $h: Z_1(LG \rtimes_{\rho} S^1) \rightarrow U(1)$ we note that since $\pi_1(LG \rtimes_{\rho} S^1) = \mathbb{Z}$ any cycle $a \in Z_1(LG \rtimes_{\rho} S^1)$ can be written as $n\gamma + \partial\sigma$, for some two-cycle σ , where

γ is the generator of $H_1(LG \rtimes_{\rho} S^1)$, a loop around the S^1 factor. It is easy to see that the integral of α over the generators of $H_1(LG \rtimes_{\rho} S^1 \times LG \rtimes_{\rho} S^1)$ vanishes, that is,

$$\int_{\gamma_1} \alpha = 0 = \int_{\gamma_2} \alpha$$

for γ_1, γ_2 loops around the first and second S^1 factors respectively. This suggests that we define

$$h(a) = h(\partial\sigma) = \exp\left(\int_{\sigma} R\right). \tag{6}$$

This is well defined since if $a = n\gamma + \partial\sigma = n\gamma + \partial\sigma'$ then $\partial(\sigma - \sigma') = 0$ and so $\int_{\sigma - \sigma'} R \in 2\pi i\mathbb{Z}$ (since R is integral). Because the integral of α over the generators of $H_1(LG \rtimes_{\rho} S^1 \times LG \rtimes_{\rho} S^1)$ vanishes, it is easy to check that for any one-cycle γ we have

$$(\delta h)(\gamma) = \exp\left(\int_{\gamma} \alpha\right).$$

Thus we have proven

Proposition 4.4. *The triple (R, α, h) as above determines the central extension of the semi-direct product $LG \rtimes_{\rho} S^1$.*

5. Lifting Bundle Gerbes

In order to perform calculations with the differential forms in the previous section we shall utilise the theory of *bundle gerbes*, introduced in [17]. Let $Y \xrightarrow{\pi} M$ be a surjective submersion.

Definition 5.1 ([17]). A *bundle gerbe* over a manifold M is a pair (P, Y) where $Y \rightarrow M$ is a surjective submersion and $P \rightarrow Y^{[2]}$ is a $U(1)$ -bundle and such that there is a *bundle gerbe multiplication*, which is a smooth isomorphism

$$m: \pi_3^*P \otimes \pi_1^*P \xrightarrow{\sim} \pi_2^*P$$

of $U(1)$ -bundles over $Y^{[3]}$. Further, this multiplication is required to be associative whenever triple products are defined. We sometimes denote a bundle gerbe simply by P .

We can characterise the bundle gerbe multiplication and its associativity in a different way using sections of bundles related to P as follows. If $Q \rightarrow Y^{[p-1]}$ is a $U(1)$ -bundle, define the bundle $\delta Q \rightarrow Y^{[p]}$ as

$$\delta Q = \pi_1^*Q \otimes (\pi_2^*Q)^* \otimes \pi_3^*Q \otimes \dots$$

Then it is easy to show that $\delta\delta Q$ is canonically trivial. One can show that the bundle gerbe multiplication is equivalent to a section s of $\delta P \rightarrow Y^{[3]}$ and that the associativity condition is equivalent to the condition that $\delta s = 1$ as a section of

$\delta\delta P$ (where 1 denotes the canonical section of $\delta\delta P$). Indeed if p and q are elements of $P_{(y_1, y_2)}$ and $P_{(y_2, y_3)}$ respectively, we can define a section s of δP by

$$s(y_1, y_2, y_3) = p \otimes m(p, q)^* \otimes q,$$

then the associativity of m forces the condition $\delta s = 1$.

If \widehat{K} is a central extension of K as before, then the specific bundle gerbe we are interested in is constructed as follows. Take the principal K -bundle $P \rightarrow M$ and consider the fibre product $P^{[2]} \rightrightarrows P$ then there is a natural map $\tau : P^{[2]} \rightarrow K$, called the *difference map*, given by $p_1\tau(p_1, p_2) = p_2$. If we view \widehat{K} as a $U(1)$ -bundle over K then we can pull-back \widehat{K} by this map to obtain a $U(1)$ -bundle over $P^{[2]}$:

$$\begin{array}{ccc} \tau^*\widehat{K} & \longrightarrow & \widehat{K} \\ \downarrow & & \downarrow \\ P^{[2]} & \xrightarrow{\tau} & K \end{array}$$

where

$$\tau^*\widehat{K} = \{(p_1, p_2, \hat{g}) \mid p(\hat{g}) = \tau(p_1, p_2)\}.$$

Note that $\tau(p_1, p_2)\tau(p_2, p_3) = \tau(p_1, p_3)$ and so, because the multiplication in \widehat{K} covers that in K , we have an induced map

$$\tau^*\widehat{K}_{(p_1, p_2)} \otimes \tau^*\widehat{K}_{(p_2, p_3)} \rightarrow \tau^*\widehat{K}_{(p_1, p_3)}$$

which serves as a bundle gerbe multiplication for the bundle gerbe $(\tau^*\widehat{K}, P)$ over M . This bundle gerbe is called the *lifting bundle gerbe*.

Bundle gerbes have a characteristic class associated with them called the Dixmier–Douady class which in the case of the lifting bundle gerbe is precisely the obstruction to lifting P to a \widehat{K} -bundle.

One can write down a differential form representative of (the image in real cohomology of) the Dixmier–Douady class of a bundle gerbe in the same way that the curvature can be used to represent the Chern class of a $U(1)$ -bundle. This requires a *connection* and *curving* for the bundle gerbe.

Consider first the p -fold fibre product $Y^{[p]}$ as before. Let $\Omega^q(Y^{[p]})$ denote the space of differential q -forms on $Y^{[p]}$. Then we can define a map $\delta : \Omega^q(Y^{[p]}) \rightarrow \Omega^q(Y^{[p+1]})$ as the alternating sum of pull-backs by the projections $\pi_i : Y^{[p+1]} \rightarrow Y^{[p]}$ which omit the i th element:

$$\delta = \sum_{i=1}^{p+1} (-1)^{i-1} \pi_i^*.$$

Then $\delta^2 = 0$ and so we have a complex

$$0 \rightarrow \Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \dots$$

In [17] it is proven that this complex has no cohomology. That is, the above sequence is exact for all $q \geq 0$. We shall use this result shortly.

A bundle gerbe connection is a connection A for the $U(1)$ -bundle P that respects the bundle gerbe product in the sense that the induced connection on π_2^*P is the same as the image of the induced connection on $\pi_3^*P \otimes \pi_1^*P$ under the bundle gerbe multiplication. Equivalently using the section s we have $s^*(\delta A) = 0$. If F is the curvature of a bundle gerbe connection A viewed as a 2-form on $Y^{[2]}$, then $\delta F = s^*(\delta dA) = d(s^*(\delta A)) = 0$. This means that there is some $B \in \Omega^2(Y)$ satisfying $F = \delta B$. A choice of such a B is called a *curving* for P . Note that if B' is another choice of curving then B and B' differ by a δ -closed (and hence δ -exact) 2-form on Y . As δ and d commute, we have that $\delta(dB) = d(\delta B) = dF = 0$. Therefore there is a 3-form H on M such that $dB = \pi^*H$ (for π the projection $Y \rightarrow M$). H is called the *3-curvature* of P . It is closed and a different choice of B or H would result in a difference of an exact form. So H defines a cohomology class in $H^3(M)$. It turns out that the 3-form $H/2\pi i$ is integral and that $H/2\pi i$ is a representative of the image of the Dixmier–Douady class of P in real cohomology.

6. The Real String Class of an $LG \rtimes_{\rho} S^1$ -Bundle

6.1. The real string class of an $LG \rtimes_{\rho} S^1$ -bundle

In Sec. 3, we outlined the results from [3] which give a formula for the real string class of an LG -bundle and use the fact that LG -bundles over M correspond to G -bundles over $M \times S^1$. We have also showed that if one extends this to non-trivial S^1 -bundles over M one obtains an $LG \rtimes_{\rho} S^1$ -bundle over M . In this section we present a formula for the image in real cohomology of the obstruction to lifting a principal $LG \rtimes_{\rho} S^1$ -bundle P to a principal $\widehat{LG} \rtimes_{\hat{\rho}} S^1$ -bundle \widehat{P} , which we call the *real string class* of P .

6.1.1. A connection for the lifting bundle gerbe

Now that we have a construction of the central extension of $LG \rtimes_{\rho} S^1$ in terms of the differential forms R and α , we can consider the problem of lifting the $LG \rtimes_{\rho} S^1$ -bundle $P \rightarrow M$ to an $\widehat{LG} \rtimes_{\hat{\rho}} S^1$ -bundle $\widehat{P} \rightarrow M$. We can write down the lifting bundle gerbe for this problem, that is, the bundle gerbe $(\tau^*(\widehat{LG} \rtimes_{\hat{\rho}} S^1), P)$ over M , and we would like a connection on this bundle gerbe so we can calculate its Dixmier–Douady class.

Consider, then, the difference map $\tau: P^{[2]} \rightarrow LG \rtimes_{\rho} S^1$ as in Sec. 5. We can extend this to a map $\tau: P^{[k+1]} \rightarrow (LG \rtimes_{\rho} S^1)^k$ by defining

$$\tau(p_1, \dots, p_{k+1}) = (\tau(p_1, p_2), \dots, \tau(p_k, p_{k+1})).$$

It is shown in [3] that this kind of map is a simplicial map and thus we have commuting diagrams

$$\begin{array}{ccc} \Omega(P^{[p]}) & \xrightarrow{\delta} & \Omega(P^{[p+1]}) \\ \tau \downarrow & & \tau \downarrow \\ (LG \rtimes_{\rho} S^1)^{p-1} & \xrightarrow{\delta} & (LG \rtimes_{\rho} S^1)^p \end{array}$$

Now consider a connection ν on $\widehat{LG} \rtimes_{\hat{\rho}} S^1$ (whose curvature is the form R). The natural choice for a bundle gerbe connection would be the pull-back, $\tau^*\nu$, of this form to $\tau^*(\widehat{LG} \rtimes_{\hat{\rho}} S^1)$. However, $\tau^*\nu$ is not a bundle gerbe connection because it does not respect the product. That is, $s^*(\delta\tau^*\nu)$ is nonzero. However, $\delta(s^*(\delta\tau^*\nu)) = 0$ and so there is some form ϵ on $P^{[2]}$ such that $\delta\epsilon = s^*(\delta\tau^*\nu)$. Then $\tau^*\nu - \epsilon$ will be a bundle gerbe connection on $\tau^*(\widehat{LG} \rtimes_{\hat{\rho}} S^1)$. In fact, in this case, since $\alpha = s^*(\delta\nu)$ by definition, we have $s^*(\delta\tau^*\nu) = \tau^*\alpha$. So $\delta(s^*(\delta\tau^*\nu)) = \delta\tau^*\alpha = \tau^*\delta\alpha = 0$ as $\delta\alpha = 0$ and so ϵ satisfies $\delta\epsilon = \tau^*\alpha$. Thus it suffices to find a 1-form ϵ on $P^{[2]}$ satisfying $\delta\epsilon = \tau^*\alpha$. In fact, it is possible to write ϵ in general in terms of α [18]. We shall now demonstrate how to do this. Let P be a K -bundle with connection A . Using the equation $p_1\tau(p_1, p_2) = p_2$ and the Leibnitz rule, we find the identity

$$\pi_1^*A = \text{ad}(\tau_{12}^{-1})\pi_2^*A + \tau_{12}^*\Theta, \tag{7}$$

where we have written τ_{ij} for $\tau(p_i, p_j)$. For tangent vectors (X_1, X_2, X_3) at $(p_1, p_2, p_3) \in P^{[3]}$, we can calculate

$$\begin{aligned} &(\delta\alpha)_{(1, \tau_{12}, \tau_{23})}(A(X_1), \tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3)) \\ &= \alpha_{(\tau_{12}, \tau_{23})}(\tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3)) \\ &\quad - \alpha_{(\tau_{12}, \tau_{23})}(m_*(A(X_1), \tau_{12}(X_1, X_2)), \tau_{23}(X_2, X_3)) \\ &\quad + \alpha_{(1, \tau_{12}, \tau_{23})}(A(X_1), m_*(\tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3))) \\ &\quad - \alpha_{(1, \tau_{12})}(A(X_1), \tau_{12}(X_1, X_2)). \end{aligned}$$

Notice that the first term above is actually $\tau^*\alpha$. Since $\delta\alpha = 0$, we have

$$\begin{aligned} &(\tau^*\alpha)_{(p_1, p_2, p_3)}(X_1, X_2, X_3) \\ &= \alpha_{(\tau_{12}, \tau_{23})}(m_*(A(X_1), \tau_{12}(X_1, X_2)), \tau_{23}(X_2, X_3)) \\ &\quad - \alpha_{(1, \tau_{12}, \tau_{23})}(A(X_1), m_*(\tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3))) \\ &\quad + \alpha_{(1, \tau_{12})}(A(X_1), \tau_{12}(X_1, X_2)). \end{aligned}$$

Now, if we define ϵ in terms of α and A as

$$\epsilon_{(p_1, p_2)}(X_1, X_2) = \alpha_{(1, \tau_{12})}(A(X_1), \tau_{12}(X_1, X_2))$$

then we have

$$\begin{aligned} &(\delta\epsilon)_{(p_1, p_2, p_3)}(X_1, X_2, X_3) \\ &= \alpha_{(1, \tau_{23})}(A(X_2), \tau_{23}(X_2, X_3)) - \alpha_{(1, \tau_{13})}(A(X_1), \tau_{13}(X_1, X_3)) \\ &\quad + \alpha_{(1, \tau_{12})}(A(X_1), \tau_{12}(X_1, X_2)). \end{aligned}$$

Using the fact that $\tau_{13} = \tau_{12}\tau_{23}$, we see

$$\alpha_{(1, \tau_{13})}(A(X_1), \tau_{13}(X_1, X_2)) = \alpha_{(1, \tau_{12}\tau_{23})}(A(X_1), m_*(\tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3)))$$

and since α is left invariant in the first slot, and using equation (7) we have

$$\begin{aligned} &\alpha_{(\tau_{12}, \tau_{23})}(A(X_2), \tau_{23}(X_2, X_3)) \\ &= \alpha_{(\tau_{12}, \tau_{23})}(\tau_{12}A(X_2), \tau_{23}(X_2, X_3)) \\ &= \alpha_{(\tau_{12}, \tau_{23})}(\tau_{12} \operatorname{ad}(\tau_{12}^{-1})A(X_1) + \tau_{12}(X_1, X_2), \tau_{23}(X_2, X_3)), \end{aligned}$$

which equals

$$\alpha_{(\tau_{12}, \tau_{23})}(m_*(A(X_1), \tau_{12}(X_1, X_2)), \tau_{23}(X_2, X_3)).$$

Thus we have $\delta\epsilon = \tau^*\alpha$.

Consider now the $LG \rtimes_{\rho} S^1$ -bundle P . Choose a connection (A, a) for P , where A and a are 1-forms on P with values in $L\mathfrak{g}$ and \mathbb{R} respectively (as per Proposition 2.8). Given (A, a) then, we can write down the 1-form $\epsilon \in \Omega^1(P^{[2]})$ as above:

$$\epsilon = \frac{i}{2\pi} \int_{S^1} \left\langle \pi_2^*A - \frac{1}{2}\pi_2^*a\tau^*Z, \tau^*Z \right\rangle d\theta.$$

It is easy to check that $\delta\epsilon = \tau^*\alpha$ and so we have that $\tau^*\nu - \epsilon$ is a connection for the lifting bundle gerbe. Of course, we are concerned with finding a curving for this bundle gerbe and so we are really interested in calculating the curvature of this connection, given by $\tau^*R - d\epsilon$. For the connection (A, a) , Eq. (7) reads

$$(A_2, a_2) = (\rho_{-\tau_{S^1}}(\operatorname{ad}(\tau_{LG}^{-1})A_1 - a_1\tau_{LG}^{-1}\partial\tau_{LG}) + \tau_{LG}^*(\rho_{-\tau_{S^1}}(\Theta)), a_1 + \tau_{S^1}^*\mu)$$

where we have written the difference map τ as (τ_{LG}, τ_{S^1}) . That is, τ_{LG} is the LG part of τ and τ_{S^1} is the circle part. From now on, we will simply write τ and assume that it is clear from the context which part we mean. In particular, then, we have

$$\tau^*\rho_{-\tau}(\Theta) = A_2 - \rho_{-\tau}(\operatorname{ad}(\tau^{-1})A_1 + a_1\tau^{-1}\partial\tau). \tag{8}$$

Note that here we have used the fact that the Maurer–Cartan form on $LG \rtimes_{\rho} S^1$ is not the pair (Θ, μ) but in fact includes a rotation of Θ . So at the point (γ, ϕ) , it is given by $(\rho_{\phi^{-1}}(\Theta), \mu)$. Using Eq. (8) and writing A^ρ for $\rho(A)$ and so on as before, we calculate

$$\begin{aligned} \tau^*R &= \frac{i}{4\pi} \int_{S^1} \langle A_2, \partial A_2 \rangle - 2\langle A_2^\rho, \partial(\operatorname{ad}(\tau^{-1})A_1) \rangle + 2\langle A_2^\rho, a_1\partial(\tau^{-1}\partial\tau) \rangle \\ &\quad + \langle \operatorname{ad}(\tau^{-1})A_1, \partial(\operatorname{ad}(\tau^{-1})A_1) \rangle - 2\langle \operatorname{ad}(\tau^{-1})A_1, a_1\partial(\tau^{-1}\partial\tau) \rangle d\theta. \end{aligned}$$

For $d\epsilon$ we have:

$$\begin{aligned} d\epsilon &= \frac{i}{2\pi} d \int_{S^1} \left\langle A_1 - \frac{1}{2}a_1\tau^*Z, \tau^*Z \right\rangle d\theta \\ &= \frac{i}{2\pi} \int_{S^1} \langle dA_1, \tau^*Z \rangle - \langle A_1, d(\tau^*Z) \rangle - \frac{1}{2}\langle da_1\tau^*Z, \tau^*Z \rangle + \langle a_1\tau^*Z, d(\tau^*Z) \rangle d\theta \end{aligned}$$

which, using the identity

$$d(\tau^*Z) = \operatorname{ad}(\tau)\partial(\tau^*\Theta^\rho) \tag{9}$$

and Eq. (8), gives

$$\begin{aligned}
 d\epsilon &= \frac{i}{2\pi} \int_{S^1} \langle dA_1, \tau^* Z \rangle - \langle A_1, \text{ad}(\tau) \partial A_2^0 \rangle + \langle A_1, \text{ad}(\tau) \partial(\text{ad}(\tau^{-1}) A_1) \rangle \\
 &\quad - \langle A_1, a_1 \text{ad}(\tau) \partial(\tau^{-1} \partial \tau) \rangle - \frac{1}{2} \langle da_1 \tau^* Z, \tau^* Z \rangle \\
 &\quad + \langle a_1 \tau^* Z, \text{ad}(\tau) \partial A_2^0 \rangle - \langle a_1 \tau^* Z, \text{ad}(\tau) \partial(\text{ad}(\tau^{-1}) A_1) \rangle d\theta.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \tau^* R - d\epsilon &= \frac{i}{4\pi} \int_{S^1} \langle A_2, \partial A_2 \rangle - 2 \langle dA_1, \tau^* Z \rangle - \langle A_1, \text{ad}(\tau) \partial(\text{ad}(\tau^{-1}) A_1) \rangle \\
 &\quad + 2 \langle a_1 \tau^{-1} \partial \tau, \partial(\text{ad}(\tau^{-1}) A_1) \rangle + \langle da_1 \tau^* Z, \tau^* Z \rangle d\theta,
 \end{aligned}$$

using the ad invariance of the inner product and integration by parts. Then, using equation (4) yields

$$\begin{aligned}
 \tau^* R - d\epsilon &= \frac{i}{4\pi} \int_{S^1} \langle A_2, \partial A_2 \rangle - \langle A_1, \partial A_1 \rangle - 2 \langle dA_1, \tau^* Z \rangle - \langle [A_1, A_1], \tau^* Z \rangle \\
 &\quad + 2 \langle \tau^* Z a_1, \partial A_1 \rangle + \langle da_1 \tau^* Z, \tau^* Z \rangle d\theta.
 \end{aligned}$$

Note now that if (F, f) is the curvature of the connection (A, a) then we have

$$(F, f) = \left(dA + \frac{1}{2}[A, A] - a \wedge \partial A, da \right).$$

Therefore, the formula above for $\tau^* R - d\epsilon$ reads

$$\tau^* R - d\epsilon = \frac{i}{4\pi} \int_{S^1} \langle \pi_1^* A, \partial \pi_1^* A \rangle - \langle \pi_2^* A, \partial \pi_2^* A \rangle - 2 \left\langle \pi_2^* F - \frac{1}{2} \pi_2^* f \tau^* Z, \tau^* Z \right\rangle d\theta.$$

6.1.2. *A curving for the lifting bundle gerbe*

Recall that in order to find the 3-curvature of the lifting bundle gerbe, and hence a representative for the image in real cohomology of the Dixmier–Douady class, we need a curving for $\tau^*(\widehat{LG} \rtimes_{\hat{\rho}} S^1)$. That is, some 2-form B on P such that $\delta B = \tau^* R - d\epsilon$. Note that $\delta = \pi_1^* - \pi_2^*$ and

$$\tau^* R - d\epsilon = \delta \left(\frac{i}{4\pi} \int_{S^1} \langle A, \partial A \rangle d\theta \right) - \frac{i}{2\pi} \int_{S^1} \left\langle \pi_2^* F - \frac{1}{2} \pi_2^* f \tau^* Z, \tau^* Z \right\rangle d\theta. \quad (10)$$

To deal with the second term above, we need a Higgs field for the $LG \rtimes_{\rho} S^1$ -bundle P . Recall from Definition 2.9 that Higgs fields satisfy

$$\Phi(p(\gamma, \phi)) = \rho_{-\phi}(\text{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma).$$

Note that this condition implies that

$$\pi_1^* \Phi = \rho_{-\tau}(\text{ad}(\tau^{-1})\pi_2^* \Phi + \tau^{-1} \partial \tau)$$

or simply,

$$\text{ad}(\tau)\Phi_2^\rho = \Phi_1 + \tau^* Z. \tag{11}$$

Using this, the second term in Eq. (10) becomes

$$\frac{i}{2\pi} \int_{S^1} \left\langle F_1 - \frac{1}{2} f_1 \tau^* Z, \text{ad}(\tau)\Phi_2^\rho - \Phi_1 \right\rangle d\theta.$$

Since (F, f) is a curvature, it satisfies

$$\pi_1^*(F, f) = \text{ad}(\tau^{-1})\pi_2^*(F, f).$$

That is, $f_2 = f_1$ and

$$F_2 = \rho_{-\tau}(\text{ad}(\tau^{-1})F_1 - f_1 \tau^{-1} \partial \tau),$$

or

$$\text{ad}(\tau)F_2^\rho = F_1 - f_1 \tau^* Z.$$

Using this, we have

$$\begin{aligned} & \frac{i}{2\pi} \int_{S^1} \left\langle F_1 - \frac{1}{2} f_1 \tau^* Z, \text{ad}(\tau)\Phi_2^\rho - \Phi_1 \right\rangle d\theta \\ &= \delta \left(\frac{i}{2\pi} \int_{S^1} \langle F, \Phi \rangle d\theta \right) + \frac{i}{4\pi} \int_{S^1} 2\langle f_1 \tau^* Z, \Phi_1 \rangle + \langle f_1 \tau^* Z, \tau^* Z \rangle d\theta. \end{aligned}$$

Therefore, $\tau^* R - d\epsilon$ is equal to

$$\delta \left(\frac{i}{4\pi} \int_{S^1} \langle A, \partial A \rangle - 2\langle F, \Phi \rangle d\theta \right) - \frac{i}{4\pi} \int_{S^1} 2\langle f_1 \tau^* Z, \Phi_1 \rangle + \langle f_1 \tau^* Z, \tau^* Z \rangle d\theta.$$

So it is enough to find a $B_2 \in \Omega^2(P)$ such that

$$\delta B_2 = \frac{i}{4\pi} \int_{S^1} 2\langle f_1 \tau^* Z, \Phi_1 \rangle + \langle f_1 \tau^* Z, \tau^* Z \rangle d\theta.$$

It is easy to check that the form

$$\frac{i}{4\pi} \int_{S^1} \langle \Phi, f\Phi \rangle d\theta$$

satisfies this. Therefore, we have

Proposition 6.1. *A curving for the lifting bundle gerbe is given by*

$$B = \frac{i}{4\pi} \int_{S^1} \langle A, \partial A \rangle - 2 \left\langle F + \frac{1}{2} f\Phi, \Phi \right\rangle d\theta.$$

In [19], Gomi gives another method for calculating a curving for the lifting bundle gerbe, that of *reduced splittings*. In the interest of brevity we omit a discussion of these results, however we note that a reduced splitting for the $LG \rtimes_\rho S^1$ -bundle which leads to the same curving as in Proposition 6.1 can be found in [8].

6.1.3. *The real string class of an $LG \rtimes_{\rho} S^1$ -bundle*

The last step now that we have found a curving for the lifting bundle gerbe is to calculate the 3-curvature $H = dB$. Then $H/2\pi i$ is integral and represents the real image of the Dixmier–Douady class of $\tau^*(\widehat{LG} \rtimes_{\rho} S^1)$ (and hence the obstruction to lifting P). We have

$$dB = \frac{i}{2\pi} \int_{S^1} \langle dA, \partial A \rangle - \langle dF, \Phi \rangle - \langle F, d\Phi \rangle - \langle d\Phi, f\Phi \rangle d\theta.$$

To proceed further, we require the Bianchi identity for (F, f) . Note that

$$d(F, f) = ([dA, A] - f \wedge \partial A + a \wedge \partial(dA), d^2a).$$

In particular, this means that

$$dF = [F, A] - f \wedge \partial A + a \wedge \partial F,$$

since $[F, A] = [dA, A] - [a \wedge \partial A, A]$ and $a \wedge \partial F = a \wedge \partial(dA) + [a \wedge \partial A, A]$. Using this, and the fact that $\int_{S^1} \langle [A, A], \partial A \rangle d\theta$ and $\langle a \wedge \partial A, \partial A \rangle$ both vanish (so that $\int_{S^1} \langle dA, \partial A \rangle d\theta = \int_{S^1} \langle F, \partial A \rangle d\theta$), the expression for dB becomes

$$dB = \frac{i}{2\pi} \int_{S^1} \langle F + f\Phi, \partial A - [A, \Phi] + a\partial\Phi - d\Phi \rangle d\theta.$$

If we define the covariant derivative of Φ by

$$\nabla\Phi = d\Phi + [A, \Phi] - \partial A - a\partial\Phi,$$

then one can easily check that it is (twisted) equivariant for the adjoint action. That is,

$$\nabla\Phi(X(\gamma, \phi)) = \rho_{-\phi}(\text{ad}(\gamma^{-1})\nabla\Phi(X)),$$

for any tangent vector X . The same is true for the quantity $F + f\Phi$, and so $H = dB$ descends to a form on M . Thus we have proven

Theorem 6.2. *Let $P \rightarrow M$ be a principal $LG \rtimes_{\rho} S^1$ -bundle and let Φ be a Higgs field for P and (A, a) be a connection for P with curvature (F, f) . Then the real string class of P , is represented in de Rham cohomology by*

$$-\frac{1}{4\pi^2} \int_{S^1} \langle F + f\Phi, \nabla\Phi \rangle d\theta,$$

where

$$\nabla\Phi = d\Phi + [A, \Phi] - \partial A - a\partial\Phi.$$

Remark 6.1. Recall from Remark 2.2 that if $P(S^1) \rightarrow Y$ is trivial then P has a reduction to an LG -bundle. In such a case we choose a flat connection so that $a = 0$ and hence $f = 0$. The formulae we have derived for the real string class then reduces to that in [3] which we have given in Theorem (3.1).

6.2. The real string class and the first Pontrjagyn class

Recall that, in the case of LG -bundles, Theorem 3.2 related the real string class to the Pontrjagyn class of the corresponding G -bundle. We have already seen that Higgs fields for $LG \rtimes_{\rho} S^1$ -bundles play an integral role in the caloron correspondence for G -bundles over circle bundles and $LG \rtimes_{\rho} S^1$ -bundles. In particular, a connection on the G -bundle and circle bundle corresponds to a connection and Higgs field on the $LG \rtimes_{\rho} S^1$ -bundle. Therefore, the appearance of Higgs fields in our expression for the real string class of an $LG \rtimes_{\rho} S^1$ -bundle suggests that there should be a result similar to Theorem 3.2. In particular, we have the following theorem.

Theorem 6.3. *Let $P \rightarrow M$ be a principal $LG \rtimes_{\rho} S^1$ -bundle and $\tilde{P} \rightarrow Y \rightarrow M$ be the corresponding G -bundle over an S^1 -bundle. Then the real string class of P is given by the integration over the fibre of the first Pontrjagyn class of \tilde{P} . That is,*

$$s(P) = \int_{S^1} p_1(\tilde{P}).$$

Proof. We prove this by calculating the integral of the first Pontrjagyn class of \tilde{P} over the fibres of $Y \rightarrow M$.

Recall that the first Pontrjagyn class is given by

$$p_1 = -\frac{1}{8\pi^2} \langle \tilde{F}, \tilde{F} \rangle,$$

where $\tilde{F} = d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}]$ is the curvature of a connection \tilde{A} on \tilde{P} . Recall that in Proposition 2.13 we gave a formula for the connection on \tilde{P} corresponding to the pair $((A, a), \Phi)$ on P . The curvature of this connection is given by

$$\tilde{F} = \text{ad}(g^{-1})(F + f\Phi + \nabla\Phi \wedge (a + d\theta)).$$

So the first Pontrjagyn class is

$$p_1 = -\frac{1}{8\pi^2} (\langle F + f\Phi, F + f\Phi \rangle - 2\langle F + f\Phi, \nabla\Phi \wedge a \rangle - 2\langle F + f\Phi, \nabla\Phi \rangle d\theta).$$

Thus, integrating p_1 over the fibre, we get

$$-\frac{1}{4\pi^2} \int_{S^1} \langle F + f\Phi, \nabla\Phi \rangle d\theta,$$

which is the expression from Theorem 6.2. □

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