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The caloron correspondence and higher string classes for loop groups

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ABSTRACT

We review the caloron correspondence between G-bundles on $M \times S^1$ and ΩG-bundles on M, where ΩG is the space of smooth loops in the compact Lie group G. We use the caloron correspondence to define characteristic classes for ΩG-bundles, called string classes, by transgression of characteristic classes of G-bundles. These generalise the string class of Killingback to higher-dimensional cohomology.

1. Introduction

The caloron correspondence was first introduced in [1] as a bijection between isomorphism classes of G-instantons on $\mathbb{R}^3 \times S^1$ (calorons) and ΩG-monopoles on $\mathbb{R}^3$, where ΩG is the loop group of based loops in G. The motivation in that case was the study of monopoles for loop groups, in particular, their twistor theory. It was subsequently [2] applied to the case of instantons on the four-sphere and the four-sphere minus a two-sphere and loop group monopoles on hyperbolic three-space. The motivation for the present work however was [3], which used the caloron correspondence to relate string structures on loop group bundles and the Pontryagin class of G-bundles. In particular, it calculated an explicit de Rham representative for Killingback's string class [4] using bundle gerbes. We adopt a similar approach, without using gerbes, to define higher classes of ΩG-bundles which we call string classes and discuss their properties.

We begin in Section 2 with a brief review of Chern–Weil theory for G-bundles and characteristic classes. In Section 3 we explain the caloron correspondence which transforms a framed G-bundle over $M \times S^1$ to an ΩG-bundle on M and vice versa. We show that this is an equivalence of categories between the category of framed G-bundles over manifolds of the form $M \times S^1$ and the category of ΩG-bundles. When the framed G-bundle has a (framed) connection the appropriate objects to consider on the ΩG-bundle are a connection and a suitably defined Higgs field [3] which are introduced in Section 3.3.

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In Section 4 we present our main results, we define (higher) string classes of ΩG-bundles, show that they are characteristic classes and give an explicit formula for them. This is a generalisation of [3] where only the case of the three-dimensional string class was discussed. A central role in this discussion is played by the path fibration $PG \to G$ which is well known to be the universal ΩG-bundle. Less well known, but also important, is the corresponding G-bundle on $G \times S^1$ introduced in [5] which plays the role of a universal bundle for G-bundles over spaces of the form $M \times S^1$. We use the Higgs field to construct for any ΩG-bundle a classifying map generalising results of [6] where the classifying map was defined only for the case of ΩG-bundles arising by taking loops in a G-bundle.

Throughout this paper, G will be a compact, connected Lie group and all cohomology groups will use real coefficients.

2. Characteristic classes and Chern–Weil theory

2.1. Classifying maps and characteristic classes

In the interests of being self-contained we shall begin by giving a short overview of the theory of classifying maps and characteristic classes before moving on to the specific case we are interested in. For details see the standard texts such as [7]. Recall that there is a *universal G-bundle* $EG \to BG$ with the property that for any G-bundle over M there is a so-called *classifying map* $f : M \to BG$ such that P is isomorphic to the pull back of $EG \to BG$ by f. The homotopy class of the classifying map is uniquely determined by the bundle and this construction establishes a bijection between isomorphism classes of G-bundles on M and homotopy classes of maps from M to BG. The universal bundle is characterised (up to homotopy equivalence) by the fact that it is a principal G-bundle and that EG is a contractible space.

A *characteristic class* associates to a G-bundle $P \to M$ a class $c(P)$ in $H^*(M)$ which is natural with respect to pulling back in the sense that if $g : N \to M$ is a smooth map then $c(g^*P) = g^*c(P)$. Since all G-bundles are pulled back from the universal bundle $EG \to BG$ and homotopic maps induce equal maps on cohomology, we conclude that characteristic classes are in bijective correspondence with elements of the cohomology group $H^*(BG)$.

2.2. The Chern–Weil homomorphism

One method of constructing characteristic classes is Chern–Weil theory. Denote by $I^k(\mathfrak{g})$ the algebra of all multilinear, symmetric, ad-invariant functions of degree k on $\mathfrak{g}$. Elements of $I^k(\mathfrak{g})$ are called invariant polynomials. If A is a connection on a G-bundle $P \to M$ with curvature F then the $2k$-form $cw_f(A) = f(F, \dots, F)$ descends to M and defines a $2k$-form which we denote by the same symbol. We have the well-known

Theorem 2.1 (*Chern–Weil Homomorphism*)**.** *Let $P \to M$ be a G-bundle with connection A and curvature F and let f be an invariant polynomial of degree k on $\mathfrak{g}$. Then the form $f(F, \dots, F)$ on P descends to a $2k$-form on M which is closed and whose de Rham class is independent of the choice of A.*

We denote the form on M by $cw_f(A)$, its de Rham class by $cw_f(P) \in H^{2k}(M)$ and the Chern–Weil homomorphism $f \mapsto cw_f(P)$ by

$$cw(P) : I^k(\mathfrak{g}) \to H^{2k}(M).$$

Notice that it follows easily from the construction that if $\psi : N \to M$ and we endow $\psi^*P \to N$ with the pullback connection ψ^*A whose curvature is ψ^*F then we have

$$\psi^*(cw_f(A)) = cw_f(\psi^*A)$$

and thus

$$\psi^*(cw_f(P)) = cw_f(\psi^*P)$$

so that $P \mapsto cw_f(P)$ is a characteristic class. In fact if G is compact the Chern–Weil homomorphism

$$cw(EG) : I^k(\mathfrak{g}) \to H^{2k}(BG)$$

is an isomorphism [8] which extends to an algebra isomorphism

$$cw(EG) : I^*(\mathfrak{g}) \to H^*(BG).$$

The proof of these results can be found in many standard places such as [8,9] and we will not repeat them here. However as a final remark we shall record here an important result about invariant polynomials which we will use in Section 4.3 when we examine an analogue of the Chern–Weil homomorphism for loop groups. The derivative of the ad-invariance condition on a polynomial $f \in I^k(\mathfrak{g})$ gives

Lemma 2.2. *Let $f \in I^k(\mathfrak{g})$ and $\alpha_1, \dots, \alpha_k$ be $\mathfrak{g}$-valued forms of degree $q_1, \dots, q_k$ respectively. Then if A is a $\mathfrak{g}$-valued p-form, we have*

$$f([\alpha_1, A], \alpha_2, \dots, \alpha_k) = f(\alpha_1, [A, \alpha_2], \dots, \alpha_k) + (-1)^{pq_2} f(\alpha_1, \alpha_2, [A, \alpha_3], \dots, \alpha_k) + \cdots + (-1)^{p(q_2 + \cdots + q_{k-1})} f(\alpha_1, \dots, \alpha_{k-1}, [A, \alpha_k]).$$

3. The Caloron correspondence

The idea of transforming G bundles over $M \times S^1$ to bundles over M whose structure group is the loop group LG first arose in [1,2] in the context of monopoles and calorons and for the case of $M = \mathbb{R}^3$. In [3] this was generalised to the setting of a general manifold M and applied to string structures. The current work makes two further generalisations. The first is that we want to work with the group ΩG of based loops. This necessitates a careful use of framings for the G-bundle. The second is to note that it is common to all these constructions that the inverse correspondence is only an inverse 'up to isomorphism'. In category theory this is what is called a pseudo-inverse and for this reason we have found it useful to discuss the caloron transform as a functor between categories

Before discussing the caloron correspondence we need some definitions.

3.1. Looping bundles

We regard the circle S^1 as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ and hence denote the identity in S^1 as 0. Let ΩG be the group of all smooth maps from S^1 into G whose value at 0 is the identity.

Definition 3.1. Let $P \to X$ be a G-bundle and $X_0 \subset X$ a submanifold. We say that P is *framed* (over X_0) if we have chosen a section $s_0 \in \Gamma(X_0, P)$. Denote by $P_0 \subset P$ the image of s_0.

The two particular examples of this general concept that we will be using are:

Example 3.1. If X is a pointed space, that is a space with a point $x_0 \in X$ chosen, then a framed bundle is a *pointed* bundle, that is a bundle $Q \to X$ with a point q_0 chosen in the fibre over x_0.

Example 3.2. When $\widetilde{P} \to M \times S^1$ is a G-bundle over $M \times S^1$ we will always frame with respect to the submanifold $M_0 = M \times \{0\}$.

If $P \to X$ is a framed bundle let $\Omega_{P_0}(P)$ be all smooth maps from S^1 into P whose value at 0 is in $P_0 = s_0(X_0)$ and similarly let $\Omega_{X_0}(X)$ be all smooth maps from S^1 into X whose value at 0 is in X_0. Note that $\Omega_{P_0}(P) \to \Omega_{X_0}(X)$ is an ΩG-bundle. We do not discuss the Fréchet principal bundle structure of $\Omega_{P_0}(P) \to \Omega_{X_0}(X)$ here, details can be found in [10–12]. Note however that we need G connected so that any G-bundle over the circle is trivial and thus $\Omega_{P_0}(P) \to \Omega_{X_0}(X)$ is onto. We call $\Omega_{P_0}(P) \to \Omega_{X_0}(X)$ a *loop* bundle. Of course not all ΩG-bundles over M are loop bundles because not all M are of the form $\Omega_{X_0}(X)$. However we prove below the useful fact that every ΩG-bundle is the pullback of a loop bundle.

Remark 3.1. If $Q \to X$ is a pointed bundle then instead of $\Omega_{\{q_0\}}(Q) \to \Omega_{\{m_0\}}(M)$ we use the notation $\Omega(Q) \to \Omega(M)$.

3.2. The caloron correspondence

Let $\mathrm{Bun}_{\Omega G}$ be the category whose objects are ΩG-bundles and morphisms are ΩG-bundle maps and let $\mathrm{Bun}_{\Omega G}(M)$ be the groupoid of all ΩG-bundles $P \to M$ with morphisms those bundle maps covering the identity map on M, that is the group of gauge transformations of P.

Let frBun_G be the category of all framed G-bundles $\widetilde{P} \to M \times S^1$ with morphisms only those G-bundle maps which preserve the framing and cover a map $N \times S^1 \to M \times S^1$ of the form $f \times \mathrm{id}$ for some $f : N \to M$. Note that such a map sends $N_0 = N \times \{0\}$ to $M_0 = M \times \{0\}$. In both cases there are projection functors Π to the category of manifolds Man defined by $\Pi(P \to M) = M$ and $\Pi(\widetilde{P} \to M \times S^1) = M$ and in the obvious way on morphisms.

Define a map $\eta : M \to \Omega_{M_0}(M \times S^1)$ by $\eta(m)(\theta) = (m, \theta)$. Notice that $\eta(m)(0) = (m, 0) \in M_0$ so this is well defined. If $\widetilde{P} \to M \times S^1$ is a framed G-bundle then $\Omega_{\widetilde{P}_0}(\widetilde{P}) \to \Omega_{M_0}(M \times S^1)$ is an ΩG-bundle and we can pull it back with η to form an ΩG-bundle $\mathcal{F}(\widetilde{P}) = \eta^*(\Omega_{\widetilde{P}_0}(\widetilde{P})) \to M$. It is straightforward to check that this defines a functor

$$\mathcal{F} : \mathrm{frBun}_G \to \mathrm{Bun}_{\Omega G}$$

which commutes with the projection functor. We will show that this functor is an equivalence of categories. This means we can find a functor

$$\mathcal{C} : \mathrm{Bun}_{\Omega G} \to \mathrm{frBun}_G$$

and natural isomorphisms

$$\alpha : \mathcal{F} \circ \mathcal{C} \cong \mathrm{id}_{\mathrm{Bun}_{\Omega G}} \quad \text{and} \quad \beta : \mathcal{C} \circ \mathcal{F} \cong \mathrm{id}_{\mathrm{frBun}_G}.$$

We shall call the functor $\mathcal{C}$ the *caloron transform*. For simplicity, from now on we shall write $\mathcal{C}^{-1}$ for $\mathcal{F}$ and call it the *inverse caloron transform*. Note however that strictly speaking it is only a pseudo-inverse, that is it inverts $\mathcal{C}$ only up to natural isomorphisms.

To construct the caloron transform we follow [3]. Suppose we have an ΩG-bundle $P \to M$ and consider the ΩG-bundle $P \times S^1 \to M \times S^1$ where the ΩG action is trivial on the S^1 factor. Then use the evaluation map $\mathrm{ev} : \Omega G \times S^1 \to G$ to form the associated G-bundle $\widetilde{P} \to M \times S^1$. That is, define $\widetilde{P}$ by

$$\widetilde{P} = (P \times G \times S^1)/\Omega G$$

where ΩG acts on $P \times G \times S^1$ by $(p, g, \theta)\gamma = (p\gamma, \gamma(\theta)^{-1}g, \theta)$. Then there is a right G action on $\widetilde{P}$ given by $[p, g, \theta]h = [p, gh, \theta]$ (where square brackets denote equivalence classes) and a projection $\tilde{\pi} : \widetilde{P} \to M \times S^1$ given by $\tilde{\pi}([p, g, \theta]) = (\pi(p), \theta)$. This action is free and transitive on the fibres (which are the orbits of the G action) and $\widetilde{P} \to M \times S^1$ is a principal G-bundle. Notice that over $M_0 = M \times \{0\}$ we have the well-defined framing

$$s_0(m, 0) = [p, 1, 0]$$

where p is any point in the fibre of P over m and $1 \in G$ is the identity. We define $\widetilde{P}$ with this framing over M_0 to be $\mathcal{C}(P)$.

To see that the inverse caloron transform inverts the caloron transform up to a natural isomorphism we first define

$$\hat{\eta} : P \to \Omega_{\widetilde{P}_0}(\widetilde{P}) = \Omega_{\widetilde{P}_0}((P \times G \times S^1)/\Omega G)$$

by $\hat{\eta}(p)(\theta) = [p, 1, \theta]$. Notice that

$$\begin{aligned} \hat{\eta}(p\gamma)(\theta) &= [p\gamma, 1, \theta] \\ &= [p\gamma, \gamma^{-1}(\theta)\gamma(\theta), \theta] \\ &= [p, \gamma(\theta), \theta] \\ &= (\hat{\eta}(p)\gamma)(\theta). \end{aligned}$$

Thus we have the bundle map

$$\begin{array}{ccc} P & \xrightarrow{\hat{\eta}} & \Omega_{\widetilde{P}_0}(\widetilde{P}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\eta} & \Omega_{M_0}(M \times S^1) \end{array}$$

which defines an isomorphism $\alpha_P : \mathcal{C}^{-1}(\mathcal{C}(P)) = \mathcal{C}^{-1}(\widetilde{P}) = \eta^*(\Omega_{\widetilde{P}_0}(\widetilde{P})) \simeq P$ and the natural isomorphism $\alpha : \mathcal{C}^{-1} \circ \mathcal{C} \cong \mathrm{id}_{\mathrm{Bun}_{\Omega G}}$.

For the second isomorphism we start with a G-bundle $\widetilde{P} \to M \times S^1$ and note that the construction of $P = \mathcal{C}^{-1}(\widetilde{P})$ is such that the fibre of P over m is given by

$$P_m = \{f : S^1 \to \widetilde{P} \mid \tilde{\pi}(f(\theta)) = (m, \theta), f(0) \in \widetilde{P}_0\}.$$

The ΩG action is the pointwise action of a loop. The fibre of $\mathcal{C}(P)$ over $(m, \theta) \in M \times S^1$ is given by

$$(P_m \times G \times \{\theta\})/\Omega G = \{f : S^1 \to \widetilde{P} \mid \tilde{\pi}(f(\theta)) = (m, \theta)\} \times G \times \{\theta\}$$

and we have the obvious map

$$\beta_{\widetilde{P}} : [f, g, \theta] \mapsto f(\theta)g \in \widetilde{P}_{(m,\theta)}$$

which is a well-defined isomorphism of G-bundles and the required natural isomorphism. We need to check that this preserves the framing. Let $\widetilde{P}_0$ be the framing of $\widetilde{P} \to M \times S^1$. The framing of $\mathcal{C}(P)$ over a point $(m, 0)$ is given by $[f, 1, 0]$ which maps under $\beta_{\widetilde{P}}$ to $f(0) \in P_0$ so that the natural isomorphism preserves the framings.

We have now proved:

Proposition 3.2 ([1,3]). *The caloron correspondence is an equivalence of categories between* frBun_G *and* $\mathrm{Bun}_{\Omega G}$ *commuting in both cases with the projections to* Man.

Remark 3.2. Note that being an equivalence means in particular that the caloron correspondence is a bijection between isomorphism classes of framed G-bundles on $M \times S^1$ and ΩG-bundles on M. Moreover it behaves naturally with respect to maps between these bundles.

We also have

Corollary 3.3. *Let $P \to M$ be an ΩG-bundle and let $\eta : M \to \Omega_{M_0}(M \times S^1)$ be defined by $\eta(m)(\theta) = (m, \theta)$. Then there exists a framed G-bundle $\widetilde{P} \to M \times S^1$ with the property that P is isomorphic to the pull back of the loop bundle $\Omega_{\widetilde{P}_0}(\widetilde{P}) \to \Omega_{M_0}(M \times S^1)$.*

Example 3.3. Let $Q \to X$ be a pointed G-bundle and $\Omega(Q) \to \Omega(X)$ its loop bundle. The inverse caloron transform associates to $\Omega(Q) \to \Omega(X)$ a framed G-bundle over $\Omega(X) \times S^1$. It is straightforward to see that this is the pullback of Q by the evaluation map $\mathrm{ev} : \Omega(X) \times S^1 \to X$ which sends $(\gamma, 0) \mapsto \gamma(0)$. The framing is defined by noting that $(\mathrm{ev}^*Q)_{(\gamma,0)} = Q_{\gamma(0)}$ so we can define $s_0(\gamma, 0) = q_0 \in Q_x$, the point of Q, because the loop γ satisfies $\gamma(0) = x_0$, the point of X.

Example 3.4 (*The Path Fibration*). Following [13] let PG be the space of paths in G, that is smooth maps $p : \mathbb{R} \to G$ such that $p(0)$ is the identity and $p^{-1}\partial p$ is periodic. Then this is acted on by ΩG and

$$\begin{array}{ccc} \Omega G & \longrightarrow & PG \\ & & \downarrow \\ & & G \end{array}$$

is an ΩG-bundle called the *path fibration*, where the projection π sends a path p to its value at 2π. PG is contractible and so the path fibration is a model for the universal ΩG-bundle and we have $B\Omega G = G$.

The caloron transform of the path fibration must be a G-bundle on $G \times S^1$. This bundle has the following simple description [5]: Start with $G \times G \times \mathbb{R}$ and define an action of $\mathbb{Z}$ by $(g, h, t)n = (g, g^n h, t + 2\pi n)$. Denote the quotient by $\widetilde{PG}$. Notice that it is a principal G-bundle with the G action given by $[g, h, t]k = [g, hk, t]$ and the projection $\widetilde{PG} \to G \times S^1$ defined by $[g, h, t] \mapsto (g, [t])$. To see that $\widetilde{PG}$ really is $\mathcal{C}(PG)$ note that we can view the latter as $(PG \times G \times \mathbb{R})/(\Omega G \times \mathbb{Z})$ where the action is given by $(p, g, t)(\gamma, n) = (p\gamma, \gamma(t)^{-1}g, t + 2\pi n)$. Then an isomorphism $\mathcal{C}(PG) \xrightarrow{\sim} \widetilde{PG}$ is given by $[p, g, t] \mapsto [p(2\pi), p(t)g, t]$. Here we use the fact that $p\colon \mathbb{R} \to G$ satisfies $p^{-1}\partial p$ is periodic if and only if it satisfies $p(t + 2\pi n) = p(2\pi)^n p(t)$.

It is well known that $PG \to G$ is a universal ΩG-bundle and we construct a classifying map for any ΩG-bundle $P \to M$ below. Being universal in categorical language means that $PG \to G$ is a terminal object in the category $\mathrm{Bun}^h_{\Omega G}$ where morphisms are replaced by homotopy classes of morphisms. As the caloron transformations are an equivalence of categories it follows that $\widetilde{PG} = \mathcal{C}(\Omega G) \to G \times S^1$ is a terminal object in the category frBun^h_G of framed G-bundles $\widetilde{P} \to M \times S^1$ with morphisms given by homotopy classes of maps $f \times \mathrm{id}_{S^1}\colon N \times S^1 \to M \times S^1$ where the allowable homotopies are those of the form $H \times \mathrm{id}_{S^1}\colon [0, 1] \times N \times S^1 \to M \times S^1$ for $H\colon [0, 1] \times N \to M$ is a homotopy between maps from N to M.

3.3. Higgs fields and the caloron correspondence

Importantly for our purposes we can extend the caloron correspondence to bundles with connection. More precisely, we have a correspondence between framed G-bundles on $M \times S^1$ with framed connection and ΩG-bundles on M with connection and Higgs field (Definition 3.6).

Definition 3.4. Let $P \to X$ be a framed bundle with framing $s_0 \in \Gamma(X_0, P)$. A framed connection is a connection A on P such that $s_0^*(A) = 0$.

Lemma 3.5. *Framed connections exist on framed bundles.*

Proof. As $X_0 \subset X$ is a submanifold we can choose an open cover $\{U_\alpha\}_{\alpha\in I}$ such that $U_\alpha \cong U_\alpha \cap X_0 \times V_\alpha$ for some open ball V_α in $\mathbb{R}^d$ where $d = \dim(X) - \dim(X_0)$. Moreover there is a section $s_\alpha\colon U_\alpha \to P$. On each $U_\alpha \cap X_0$ we can choose $g_\alpha\colon U_\alpha \cap X_0 \to G$ such that $g_\alpha s_\alpha$ takes values in P_0 and we can extend g_α to all of U_α by making it constant in the V_α directions. In other words we can just assume that s_α restricted to $U_\alpha \cap X_0$ takes values in P_0. We can now take the flat connection induced by each s_α and combine these with a partition of unity. The result is a framed connection. □

The next concept we need is that of a *Higgs field* for an ΩG-bundle. Let $\Omega\mathfrak{g}$ be the Lie algebra of all smooth maps from S^1 into $\mathfrak{g}$, the Lie algebra of G, whose value at 0 is zero. Of course, $\Omega\mathfrak{g}$ is the Lie algebra of ΩG.

Definition 3.6. A *Higgs field* for an ΩG-bundle $P \to M$ is a map $\Phi\colon P \to \Omega\mathfrak{g}$ satisfying the (twisted) equivariance condition

$$\Phi(p\gamma) = \mathrm{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma,$$

for $p \in P$ and $\gamma \in \Omega G$.

Lemma 3.7 (*cf. [3]*)**.** *Higgs fields exist.*

Proof. This is a standard construction relying on the fact that a convex combination of Higgs fields is a Higgs field so that one can use partitions of unity to patch together locally defined Higgs fields. See the discussion on page 551 of [3]. □

Example 3.5 (*The Path Fibration*)**.** A connection for the path fibration is given in [14, Section 2, page 53]. Let α be a smooth real-valued function on $\mathbb{R}$ such that $\alpha(t) = 0$ for $t \le 0$ and $\alpha(t) = 1$ for all $t \ge 2\pi$. Then a connection in PG is given by

$$A = \Theta - \alpha\, \mathrm{ad}(p^{-1})\pi^*\widehat{\Theta},$$

where Θ is the (left-invariant) Maurer–Cartan form on G and $\widehat{\Theta}$ is the *right*-invariant Maurer–Cartan form. The curvature of this connection is

$$F = \frac{1}{2}\left(\alpha^2 - \alpha\right)\mathrm{ad}(p^{-1})[\pi^*\widehat{\Theta}, \pi^*\widehat{\Theta}].$$

A Higgs field for PG is given by

$$\Phi(p) = p^{-1}\partial p.$$

We call these the *standard* connection and Higgs field for the path fibration.

Proposition 3.8. *Let $P \to X$ be a framed bundle with framed connection. Then the loop bundle $\Omega_{P_0}(P) \to \Omega_{X_0}(X)$ has a connection and Higgs field.*

Proof. If A is the connection on P then $\Phi(q) = A(\partial q)$ defines a Higgs field

$$\Phi\colon \Omega_{P_0}(P) \to \Omega\mathfrak{g}.$$

As in [3] we can use the connection to define a connection on the loop bundle by acting pointwise. □

Proposition 3.9. *If $\widetilde{P} \to M \times S^1$ is a framed bundle with framed connection then the ΩG-bundle $\mathcal{C}^{-1}(P) \to M$ has a connection and Higgs field.*

Proof. It suffices to note that connections and Higgs fields pull back. □

Suppose instead we are given an ΩG-bundle P with connection A and Higgs field Φ. Then we can define a form on $P \times G \times S^1$ by

$$\tilde{A} = \mathrm{ad}(g^{-1})A(\theta) + \Theta + \mathrm{ad}(g^{-1})\Phi\, d\theta.$$

This form descends to a form on $\widetilde{P}$ and the connection (also called $\tilde{A}$) is given by this equation considered as a form on $(P \times G \times S^1)/\Omega G$. To show that this is well defined, we need to check that it is independent of the lift of a vector in $\widetilde{P}$. That is, if $\hat{X}$ and $\hat{X}'$ are two lifts of the vector $X \in T_{[p,g,\theta]}\widetilde{P}$ to the fibre in $P \times G \times S^1$ above $[p, g, \theta]$, then $\tilde{A}(\hat{X}) = \tilde{A}(\hat{X}')$. Suppose then, that $\hat{X} \in T_{(p,g,\theta)}(P \times G \times S^1)$ and $\hat{X}' \in T_{(p,g,\theta)\gamma}(P \times G \times S^1)$. Then $\hat{X}\gamma \in T_{(p,g,\theta)\gamma}(P \times G \times S^1)$, and $\hat{X}'$ and $\hat{X}\gamma$ differ by a vertical vector (with respect to the LG action) at $(p, g, \theta)\gamma = (p\gamma, \gamma(\theta)^{-1}g, \theta)$ and so it is sufficient to show that $\tilde{A}$ is zero on vertical vectors and invariant under the LG action (since then $\tilde{A}(\hat{X}') = \tilde{A}(\hat{X}\gamma + \text{vertical}) = \tilde{A}(\hat{X})$). The vertical vector at (p, g, θ) generated by $\xi \in \Omega\mathfrak{g}$ is

$$V = (\iota_p(\xi), -\xi(\theta)g, 0),$$

where $\iota_p(\xi)$ represents the fundamental vector field at p generated by the Lie algebra element ξ. Therefore,

$$\begin{aligned}\tilde{A}(V) &= \mathrm{ad}(g^{-1})A(\iota_p(\xi))(\theta) - g^{-1}\xi(\theta)g\\ &= g^{-1}\xi(\theta)g - g^{-1}\xi(\theta)g\\ &= 0.\end{aligned}$$

So $\tilde{A}$ is zero on vertical vectors. Now, suppose $\hat{X} = (X, g\zeta, x_\theta)$ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\gamma_X(t), g\exp(t\zeta), \theta + tx),$$

where $\gamma_X(t)$ is a path in P whose tangent vector at 0 is X and where ζ and x are elements of the Lie algebras of G and S^1 respectively. Then

$$\hat{X}\gamma = (X\gamma, \gamma(\theta)g(\zeta + x\mathrm{ad}(g^{-1})\gamma(\theta)^{-1}\partial\gamma(\theta)), x).$$

So

$$\begin{aligned}\tilde{A}_{(p\gamma,\gamma(\theta)^{-1}g,\theta)}(\hat{X}\gamma) &= \tilde{A}_{(p\gamma,\gamma(\theta)^{-1}g,\theta)}(X\gamma, \gamma(\theta)g(\zeta + x\mathrm{ad}(g^{-1})\gamma(\theta)^{-1}\partial\gamma(\theta)), x)\\ &= \mathrm{ad}((\gamma(\theta)^{-1}g)^{-1})A(X\gamma) + \zeta + x\mathrm{ad}(g^{-1})\gamma(\theta)^{-1}\partial\gamma(\theta) + \mathrm{ad}((\gamma(\theta)^{-1}g)^{-1})x\Phi(p\gamma)\\ &= \mathrm{ad}(g^{-1})\mathrm{ad}(\gamma)\mathrm{ad}(\gamma^{-1})A(X)(\theta) + \zeta + x\mathrm{ad}(g^{-1})\gamma(\theta)^{-1}\partial\gamma(\theta)\\ &\quad + \mathrm{ad}(g^{-1})x\mathrm{ad}(\gamma)(\mathrm{ad}(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma)\\ &= \mathrm{ad}(g^{-1})A(X)(\theta) + \zeta + \mathrm{ad}(g^{-1})x\Phi(p).\end{aligned}$$

Therefore $\tilde{A}$ is invariant under the LG action and so defines a form on $\widetilde{P}$. It is a connection form since if $[X, g\zeta, x_\theta]$ is a vector at $[p, g, \theta]$, (so $X \in T_pP$, $\zeta \in \mathfrak{g}$ and $x_\theta \in T_\theta S^1$) then $[X, g\zeta, x_\theta]h = [X, gh\,\mathrm{ad}(h^{-1})\zeta, x_\theta]$ and so

$$\begin{aligned}\tilde{A}([X, g\zeta, x_\theta]h) &= \mathrm{ad}(h^{-1}g^{-1})A(X)(\theta) + \mathrm{ad}(h^{-1})\zeta + \mathrm{ad}(h^{-1}g^{-1})x\Phi(p)\\ &= \mathrm{ad}(h^{-1})\tilde{A}([X, g\zeta, x_\theta])\end{aligned}$$

and further, the vertical vector at $[p, g, \theta]$ generated by $\zeta \in \mathfrak{g}$ is given by

$$\begin{aligned}V_\zeta &= \frac{\mathrm{d}}{\mathrm{d}t}_{|0}[p, g\exp(t\zeta), \theta]\\ &= [0, g\zeta, 0]\end{aligned}$$

and so $\tilde{A}(V_\zeta) = \zeta$. Note that the connection defined above is a framed connection. To see this, recall from Section 3.2 that $\widetilde{P}_0$ is given by equivalence classes of the form $[p, 1, 0]$. A tangent vector to $[p, 1, 0] \in \widetilde{P}_0$ is therefore of the form $[X, 0, 0]$, for X a vector field along p. Thus, if $(v, x) \in T_{\{m\}\times\{0\}}(M \times S^1)$ and p is in the fibre above m, we have $s_0^*(\tilde{A})_{(m,0)}(v, x) = \tilde{A}_{[p,1,0]}([X, 0, 0]) = A(X)(0) = 0$. Therefore $\tilde{A}$ is a framed connection.

We extend our earlier notation by defining $\mathrm{Bun}^c_{\Omega G}$ be the category whose objects are ΩG-bundles with connection and Higgs field and morphisms are ΩG-bundle maps preserving connections and Higgs fields. Let frBun^c_G be the category whose objects are framed G-bundles $\widetilde{P} \to M \times S^1$ with framed connections and with morphisms only those G-bundle maps which preserve the framing and connection and cover a map $N \times S^1 \to M \times S^1$ of the form $f \times \mathrm{id}$ for some $f : N \to M$. We have

Proposition 3.10. *The caloron correspondence*

$$\mathcal{C} : \mathrm{Bun}^c_{\Omega G} \to \mathrm{frBun}^c_G$$

and

$$\mathcal{C}^{-1} : \mathrm{frBun}^c_G \to \mathrm{Bun}^c_{\Omega G}$$

is an equivalence of categories with the same natural isomorphisms as before.

Proof. It suffices to show that the natural transformations preserve the connections.

Suppose we have the G-bundle $\widetilde{P} \to M \times S^1$ with connection $\tilde{A}$. Applying the caloron construction twice gives us the G-bundle $\mathcal{C}(\mathcal{C}^{-1}(\widetilde{P}))$ and according to the discussion above the connection on this bundle is given by

$$\mathcal{C}(\mathcal{C}^{-1}(\tilde{A}))_{[p,g,\theta]} = \mathrm{ad}(g^{-1})\tilde{A}_{p(\theta)} + \Theta + \mathrm{ad}(g^{-1})\tilde{A}(\partial p)d\theta.$$

Now $\mathcal{C}(\mathcal{C}^{-1}(\widetilde{P}))$ is isomorphic to $\widetilde{P}$ via the map $\beta_{\widetilde{P}} : [f, g, \theta] \mapsto f(\theta)g$. The pushforward of this on a tangent vector $[X, g\zeta, x_\theta]$ at $[p, g, \theta]$ is given by

$$\beta_{\widetilde{P}*}[X, g\zeta, x_\theta] = X(\theta)g + \iota_{p(\theta)g}(\zeta) + \partial p(\theta)xg.$$

(Note here that X is a vector tangent to the point $p \in \mathcal{C}^{-1}(\widetilde{P}) = \eta^*\Omega(\widetilde{P})$, which means it is a vector field along the loop p.) Using this it is easy to see that $\beta^*_{\widetilde{P}}\mathcal{C}(\mathcal{C}^{-1}(\tilde{A})) = \tilde{A}$.

Suppose, on the other hand, we had started with an ΩG-bundle P with connection A and constructed $\mathcal{C}^{-1}(\mathcal{C}(P))$. Then the connection $\mathcal{C}^{-1}(\mathcal{C}(A))$ is given in terms of the connection on $\mathcal{C}(P)$ by acting pointwise and since the isomorphism $\alpha_P : P \xrightarrow{\sim} \mathcal{C}^{-1}(\mathcal{C}(P))$ is essentially given by $p \mapsto (\theta \mapsto [p, 1, \theta])$ we clearly see that $\alpha^*_P\mathcal{C}^{-1}(\mathcal{C}(A)) = A$. □

4. Higgs fields and characteristic classes

4.1. Higgs fields and the string class

To illustrate the caloron correspondence let us briefly outline an application, that of *string structures*. This will serve not only to give an example of the correspondence above but also as motivation for the next section in which our main results will be in some sense an extension of those presented below. The material in this section is taken from [3].

String structures were introduced by Killingback as the string theory analogue of spin structures [4]. Suppose we have an ΩG-bundle $P \to M$. Since ΩG has a central extension by the circle (see, for example, [12] for details) we can consider the problem of lifting the structure group of P to the central extension $\widehat{\Omega G}$ of ΩG. Physically, this is related to the problem of defining a Dirac–Ramond operator in string theory.[1] Mathematically, one has an obstruction to doing this—a certain degree three cohomology class on the base of the bundle. This class is called the *string class* of the bundle and we write $s(P) \in H^3(M)$. In [3] Murray and Stevenson give a formula for a de Rham representative of this class which, adapted from the case of free loops to based loops, is given by:

Theorem 4.1 (*Cf. [3, Theorem 5.1]*). *Let $P \to M$ be a principal ΩG-bundle. Let A be a connection on P with curvature F and let Φ be a Higgs field for P. Then the string class of P is represented in de Rham cohomology by the form*

$$-\frac{1}{4\pi^2}\int_{S^1}\langle\nabla\Phi, F\rangle\, d\theta,$$

where $\langle\ ,\ \rangle$ is an invariant inner product on $\mathfrak{g}$ normalised so the longest root has length squared equal to 2 and $\nabla\Phi = d\Phi + [A, \Phi] - \frac{\partial A}{\partial\theta}$.

In the case where $P \to M$ is given by loops in a G-bundle $Q \to X$ (so $P = \Omega(Q)$ and $M = \Omega(X)$) Killingback's result (also proved in [6,3]) is to relate this class to the first Pontryagin class of Q. In particular if $p_1(Q)$ is the first Pontryagin class of Q then $s(\Omega(Q))$ is given by transgressing $p_1(Q)$ to $\Omega(X)$:

$$s(\Omega(Q)) = \int_{S^1}\mathrm{ev}^* p_1(Q),$$

[1] In fact, the case that Killingback considered originally was that of a free loop bundle. Here we shall concern ourselves with the more general case of a loop group bundle which is not necessarily a loop bundle but restrict our interest to the case of based loops.

where $\mathrm{ev}\colon \Omega(X) \times S^1 \to X$ is the evaluation map. In order to obtain an analogue of this result for ΩG-bundles which are not loop bundles, since we do not have the option of transgressing the Pontryagin class, we use the caloron correspondence. Specifically, the string class of an ΩG-bundle $P \to M$ is given by integrating over the circle the first Pontryagin class of the caloron transform of P which is a G-bundle $\widetilde{P} \to M \times S^1$. To see this consider the caloron transform connection $\tilde{A}$ on $\widetilde{P}$ as in the previous section. A calculation shows that the curvature of this connection is given by

$$\begin{aligned}\tilde{F} &= d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] \\ &= \mathrm{ad}(g^{-1})(F + \nabla\Phi\, d\theta),\end{aligned}$$

for F the curvature of the connection A on P and $\nabla\Phi$ the covariant derivative of the Higgs field as above. The first Pontryagin class of $\widetilde{P}$ is then given by

$$\begin{aligned}p_1(\tilde{P}) &= -\frac{1}{8\pi^2}\langle\tilde{F}, \tilde{F}\rangle \\ &= -\frac{1}{8\pi^2}(\langle F, F\rangle + 2\langle F, \nabla\Phi\rangle\, d\theta).\end{aligned}$$

Hence integrating over S^1 yields

Theorem 4.2 (*Cf.* [3, Theorem 6.1])**.** *Let $P \to M$ be an ΩG-bundle and $\widetilde{P} \to M \times S^1$ its caloron transform. Then the string class of P is given by integrating over the circle the first Pontryagin class of $\widetilde{P}$. That is,*

$$s(P) = \int_{S^1} p_1(\widetilde{P}).$$

The important point to note here is that the string class is canonically associated to a characteristic class for G-bundles, namely the first Pontryagin class. Furthermore, the string class is itself a characteristic class for the ΩG-bundle P (see [6, 15][2]). So using the caloron correspondence we have calculated a characteristic class of P. In the next section we shall extend this idea to higher degree classes and we will see that, in fact, it is possible to construct characteristic classes for loop group bundles for *any* characteristic class for G-bundles.

4.2. Classifying maps for ΩG-bundles

Since we wish to calculate characteristic classes of loop group bundles, it seems natural to try to find a classifying theory for these bundles.

The case where the loop group bundle arises as a loop bundle is covered in [6] which considers a pointed bundle $Q \to X$: To write down the classifying map of the bundle $\pi : \Omega(Q) \to \Omega(X)$ choose a connection for $Q \to X$. Then take a loop $\gamma \in \Omega(Q)$ (so $\gamma(0) = q_0$) and project it down to $\pi \circ \gamma \in \Omega(X)$. Lift this back up to a horizontal path γ_h in Q starting at q_0. That is, γ_h is horizontal, $\gamma_h(0) = q_0$ and $\pi \circ \gamma = \pi \circ \gamma_h$. Then the *holonomy*, $\mathrm{hol}(\gamma) \in PG$ is determined by $\gamma = \gamma_h \mathrm{hol}(\gamma)$. This covers the usual holonomy $\mathrm{hol}\colon \Omega(X) \to G$ and defines a bundle map:

$$\begin{array}{ccc} P = \Omega(Q) & \xrightarrow{\mathrm{hol}} & PG \\ \downarrow & & \downarrow \\ M = \Omega(X) & \xrightarrow{\mathrm{hol}} & G \end{array}$$

Thus hol is a classifying map for the bundle $\Omega(Q) \to \Omega(X)$.

We can extend this to the case of a general ΩG-bundle $P \to M$ as follows. Consider the ΩG-bundle $P \to M$. Choose a Higgs field $\Phi\colon P \to \Omega\mathfrak{g}$ for P. The equation $\Phi(p) = g^{-1}\partial g$ for $g \in PG$ has a unique solution and we define the *Higgs field holonomy*, hol_Φ, by $\mathrm{hol}_\Phi(p) = g$ where g solves this equation. Note that

$$\Phi(ph) = \mathrm{ad}(h^{-1})\Phi(p) + h^{-1}\partial h$$

and

$$(gh)^{-1}\partial(gh) = \mathrm{ad}(h^{-1})g^{-1}\partial g + h^{-1}\partial h,$$

so that $\mathrm{hol}_\Phi(p \cdot h) = \mathrm{hol}_\Phi(p)h$ and hence hol_Φ descends to a map (also called hol_Φ) from M to G and we have

Proposition 4.3. *If $P \to M$ is an ΩG-bundle with connection Φ then* $\mathrm{hol}_\Phi\colon M \to G$ *is a classifying map.*

Remark 4.1. Recall our comment that the G-bundle $\widetilde{PG} \to G \times S^1$ was universal for framed G-bundles over manifolds of the form $M \times S^1$. Proposition 4.3 also implies how to construct a classifying map $M \times S^1 \to G \times S^1$ for any G-bundle $\widetilde{P} \to M \times S^1$.

2 In fact, this will follow from our work in Section 4.3.

That is pick a connection $\tilde{A}$ for $P \to M \times S^1$ and define $h\colon M \to G$ by sending $m \in M$ to the holonomy of A around the loop $\theta \mapsto (m, \theta)$ computed relative to the framing. The classifying map is then $h \times \mathrm{id}_{S^1}$. Of course if $\widetilde{P}$ is the caloron transform of an ΩG-bundle $P \to M$ with connection A and Higgs field Φ we have that $h = \mathrm{hol}_\Phi$.

A natural question arises at this point: If $Q \to M$ is a G-bundle with connection A then we can define the holonomy of a loop $\gamma \in \Omega(Q)$. However, since the loop bundle $\Omega(Q) \to \Omega(M)$ is an ΩG-bundle, we can also choose a Higgs field for it and define the Higgs field holonomy of a loop γ in this bundle. Can we choose the Higgs field Φ such that $\mathrm{hol}_\Phi = \mathrm{hol}$? Define Φ in terms of A as in the proof of Proposition 3.8 by

$$\Phi(\gamma) = A(\partial\gamma).$$

Using $\gamma = \gamma_h \mathrm{hol}(\gamma)$, we find

$$\partial\gamma = \partial\gamma_h \cdot \mathrm{hol}(\gamma) + \iota_{\gamma_h}(\mathrm{hol}(\gamma)^{-1}\partial\mathrm{hol}(\gamma)).$$

Since γ_h is horizontal (in the sense that all its tangent vectors are horizontal), applying the connection form A gives

$$A(\partial\gamma) = \mathrm{hol}(\gamma)^{-1}\partial\mathrm{hol}(\gamma).$$

Therefore, $\mathrm{hol}_\Phi = \mathrm{hol}$.

Recall from Section 3.2 that the inverse caloron transform of a G-bundle $\widetilde{P}$ is given by the pullback $\eta^*\Omega_{P_0}(\widetilde{P})$, for $\eta\colon M \to \Omega_{M_0}(M \times S^1)$ defined by $\eta(m)(\theta) = (m, \theta)$. Proposition 3.2 then implies that every ΩG-bundle is (isomorphic to) the pullback of a loop bundle. Namely, the loop bundle $\Omega_{P_0}(\widetilde{P}) \to \Omega_{M_0}(M \times S^1)$ where $\widetilde{P} \to M \times S^1$ is the caloron transform of P. This suggests that there should be a relationship between hol_Φ and hol in general. We have

Lemma 4.4. *Let $P \to M$ be an ΩG-bundle with connection A and Higgs field Φ, $\widetilde{P} \to M \times S^1$ its caloron transform and η as above. Then* $\mathrm{hol}_\Phi = \mathrm{hol} \circ \eta$.

Proof. If $\tilde{A}$ is the connection form on $\widetilde{P}$ then $\tilde{\Phi}\colon \Omega_{P_0}(\widetilde{P}) \to \Omega\mathfrak{g}$ defined by

$$\tilde{\Phi}(\gamma) = \tilde{A}(\partial\gamma)$$

gives us that

$$\mathrm{hol}_{\tilde{\Phi}} = \mathrm{hol}$$

as above. Therefore we need only show that $\mathrm{hol}_\Phi = \mathrm{hol}_{\tilde{\Phi}} \circ \hat{\eta}$, where $\hat{\eta}\colon P \to \Omega_{P_0}(\widetilde{P})$ is the bundle map which covers $\eta\colon M \to \Omega_{M_0}(M \times S^1)$.

Let $p \in P$. Consider the unique horizontal path $\hat{\eta}(p)_h$ such that

$$\tilde{\pi}(\hat{\eta}(p)) = \tilde{\pi}(\hat{\eta}(p)_h)$$

given by projecting $\hat{\eta}(p)$ to $\Omega_{M_0}(M \times S^1)$ and lifting horizontally back to $\Omega_{P_0}(\widetilde{P})$. The tangent vector to the loop $\hat{\eta}(p)$ at the point θ is given by the derivative $\partial\hat{\eta}(p)_\theta$ and since $\hat{\eta}(p)_h$ is horizontal we have that

$$\tilde{A}(\partial\hat{\eta}(p)_{h,\theta}) = 0.$$

Now, $\hat{\eta}(p)_\theta = [p, 1, \theta]$, so we can explicitly calculate $\partial\hat{\eta}(p)_\theta$:

$$\frac{\partial}{\partial\theta}\hat{\eta}(p)_\theta = [0, 0, 1].$$

Recall that the connection $\tilde{A}$ is given in terms of the connection A and Higgs field Φ for P as

$$\tilde{A} = \mathrm{ad}(g^{-1})A(\theta) + \Theta + \mathrm{ad}(g^{-1})\Phi\, d\theta.$$

Therefore, we have $\tilde{A}(\partial\hat{\eta}(p)) = \Phi(p)$. Or, in terms of the Higgs field for $\Omega_{P_0}(\widetilde{P})$,

$$\Phi = \tilde{\Phi} \circ \hat{\eta}.$$

As above, we have

$$\tilde{\Phi}(\hat{\eta}(p)) = \mathrm{hol}(\hat{\eta}(p))^{-1}\partial\mathrm{hol}(\hat{\eta}(p)),$$

and therefore $\mathrm{hol}_\Phi = \mathrm{hol}_{\tilde{\Phi}} \circ \eta$. □

4.3. Higher string classes

In this section we shall present our main results (Theorem 4.13). As mentioned in the Introduction we are interested in developing a method for geometrically constructing characteristic classes for ΩG-bundles. We will accomplish this by passing to the corresponding G-bundle and then writing the result in terms of data on the original loop group bundle. Many of the calculations in this section appear in more detail in the second author's Ph.D. thesis [15].

Definition 4.5. If $P \to M$ is an ΩG-bundle with connection A and Higgs field Φ and $f \in I^k(\mathfrak{g})$ we define the *string form* by

$$s_f(A, \Phi) = \int_{S^1} cw_f(\tilde{A}) \in \Omega^{2k-1}(M)$$

where $\tilde{A}$ is the connection defined by the caloron transform on the G-bundle $\mathcal{C}(P) \to M \times S^1$.

While this is a definition we need a formula for the string form to be able to work with it. The Chern–Weil theory tells us that if we start with an invariant polynomial $f \in I^k(\mathfrak{g})$ then the element in $H^{2k}(M \times S^1)$ that we end up with is $f(\tilde{F}, \ldots, \tilde{F})$ where $\tilde{F}$ is the curvature of the G-bundle $\widetilde{P}$ on $M \times S^1$. Note that if we write out $f(\tilde{F}, \ldots, \tilde{F})$ in terms of the curvature and Higgs field on the corresponding ΩG-bundle $P \to M$, we get

$$\begin{aligned} cw_f(\tilde{A}) &= f(\tilde{F}, \ldots, \tilde{F}) \\ &= f(F + \nabla\Phi\, d\theta, \ldots, F + \nabla\Phi\, d\theta) \\ &= f(F, \ldots, F) + kf(\nabla\Phi\, d\theta, F, \ldots, F) \end{aligned}$$

since f is multilinear and symmetric and all terms with more than one $d\theta$ will vanish. From now on we will adopt the convention that whenever f has repeated entries they will be ordered at the end and we will write them only once. That is, whatever appears as the last entry in f is repeated however many times required to fill the remaining slots. (For example, $f(F) = f(F, \ldots, F)$ and $f(\nabla\Phi, F)\, d\theta = f(\nabla\Phi, F, \ldots, F)\, d\theta$.) So integrating this over the circle gives

$$\int_{S^1} cw_f(\tilde{A}) = k \int_{S^1} f(\nabla\Phi, F)\, d\theta$$

and we conclude

Proposition 4.6. *If $P \to M$ is an ΩG-bundle with connection A and Higgs field Φ then the string form is given by*

$$s_f(A, \Phi) = k \int_{S^1} f(\nabla\Phi, F)\, d\theta.$$

We can now prove

Proposition 4.7. *The string form is closed.*

Proof. This can be proved directly from the formula

$$s_f(A, \Phi) = k \int_{S^1} f(\nabla\Phi, F)\, d\theta.$$

using the same methods as in Chern–Weil theory [9] but it is simpler to just note that $cw_f(\tilde{A})$ is closed and that integration over the fibre commutes with the exterior derivative so that $s_f(A, \Phi)$ is closed. □

We can now consider the de Rham cohomology class of $s_f(A, \Phi)$ in $H^{2k-1}(M)$ and we have

Proposition 4.8. *The class of the string form is independent of the choice of the connection and Higgs field.*

Proof. Again this can be proved directly but we can also note that if (A, Φ) and (A', Φ') are two connections and Higgs fields for $P \to M$ then we have two corresponding connections $\tilde{A}$ and $\tilde{A}'$ for the bundle $\mathcal{C}(P) \to M \times S^1$. We know from standard Chern–Weil theory that $cw_f(\tilde{A}') = cw_f(\tilde{A}) + d\beta$ for a form $\beta \in \Omega^{2k-1}(M \times S^1)$. So we have

$$s_f(A', \Phi') = s_f(A, \Phi) + d \int_{S^1} \beta. \quad \square$$

An explicit formula for β is given in [15, Proposition 3.2.4]: Let α and φ be the difference between the two connections and Higgs fields, respectively. So $\alpha = A' - A$ and $\varphi = \Phi' - \Phi$. Then, since the space of connections is an affine space (and the same is true for Higgs fields [3]), we can define a one parameter family of connections and Higgs fields by

$$A_t = A + t\alpha, \qquad \Phi_t = \Phi + t\varphi$$

for $t \in [0, 1]$. We let $\tilde{\alpha} = \mathrm{ad}(g^{-1})(\alpha + \varphi\, d\theta)$. Now consider the corresponding connection $\tilde{A}_t$ on $\mathcal{C}(P)$. If $\tilde{F}_t$ is the curvature of this connection, then a calculation shows that the β is given by

$$\beta = k \int_0^1 f(\tilde{\alpha}, \tilde{F}_t)\, dt.$$

We now define

Definition 4.9. If $P \to M$ is an ΩG-bundle and $f \in I^k(\mathfrak{g})$ we define the *string class* of P, $s_f(P) \in H^{2k}(M)$, to be the de Rham class of $s_f(A, \Phi)$ for any choice of connection and Higgs field.

It follows immediately from the formula

$$s_f(A, \Phi) = k \int_{S^1} f(\nabla\Phi, F)\, d\theta$$

that if $\psi : N \to M$ then

$$\begin{aligned} \psi^*(s_f(A, \Phi)) &= k \int_{S^1} \psi^*(f(\nabla\Phi, F))\, d\theta \\ &= k \int_{S^1} f(\psi^*(\nabla\Phi), \psi^*(F))\, d\theta \\ &= s_f(\psi^*(A), \psi^*(\Phi)) \end{aligned}$$

and we have

Proposition 4.10. *Both the string form and the string class are natural with respect to pulling back ΩG-bundles with connection and Higgs field. In particular, the string class defines a characteristic class for ΩG-bundles.*

Recall that in Example 3.5 we defined a connection A and Higgs field Φ on the path fibration which we called the standard connection and Higgs field. In this case we have

Proposition 4.11 (*[15]*)**.** *The string form of the standard connection and Higgs field of the path fibration over G is*

$$s_f(A, \Phi) = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta], \ldots, [\Theta, \Theta]),$$

where Θ is the usual left-invariant Maurer–Cartan form on G. Hence the string class $s_f(PG)$, which is independent of the choice of connection and Higgs field is the class of $s_f(A, \Phi)$.

Proof. Evaluating the expression for the string form on the standard connection and Higgs field for the path fibration given in Example 3.5 gives

$$\begin{aligned} k \int_{S^1} f(\nabla\Phi, F)\, d\theta &= f(\Theta, [\Theta, \Theta]) \left(\frac{1}{2}\right)^{k-1} k \int_{S^1} \left(\alpha^2 - \alpha\right)^{k-1} \partial\alpha\, d\theta \\ &= f(\Theta, [\Theta, \Theta]) \left(\frac{1}{2}\right)^{k-1} k \int_{S^1} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} \alpha^{2i} \alpha^{k-1-i} \partial\alpha\, d\theta \\ &= f(\Theta, [\Theta, \Theta]) \left(-\frac{1}{2}\right)^{k-1} k \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \frac{1}{k+i}. \end{aligned}$$

We can write the coefficient above without the sum [16]:

$$k \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{k+i} = \frac{k!(k-1)!}{(2k-1)!}.$$

Therefore, we have

$$k \int_{S^1} f(\nabla\Phi, F)\, d\theta = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta], \ldots, [\Theta, \Theta]). \quad \square$$

Remark 4.2. We remarked earlier that if G is compact we have an isomorphism $I^k(\mathfrak{g}) \simeq H^{2k}(BG)$ given by $f \mapsto cw_f(EG)$. Because EG is contractible we can *transgress* this class. That is, we pull it back to EG, solve for $\pi^*(cw_f(EG)) = d\rho$, and then the restriction of ρ to a fibre of $EG \to BG$ defines an element $\tau(f)$ in $H^{2k-1}(G)$ which is well known (see for example [17,18]) to be the class of the form above. That is

$$\tau(f) = \left(-\frac{1}{2}\right)^{k-1} \frac{k!(k-1)!}{(2k-1)!} f(\Theta, [\Theta, \Theta], \ldots, [\Theta, \Theta]).$$

Before continuing we need a simple result about integration over the fibre:

Lemma 4.12. *If $\psi : N \to M$ is a smooth function then pullback and integration over the fibre form a commuting diagram as follows:*

$$\begin{array}{ccc} \Omega^q(M\times S^1) & \xrightarrow{(\psi\times \mathrm{id})^*} & \Omega^q(N\times S^1) \\ \downarrow{\scriptstyle \int_{S^1}} & & \downarrow{\scriptstyle \int_{S^1}} \\ \Omega^q(M) & \xrightarrow{\psi^*} & \Omega^q(N) \end{array}$$

It follows that we have a commuting diagram also on cohomology.

$$\begin{array}{ccc} H^{2k}(M\times S^1) & \xrightarrow{(\psi\times \mathrm{id})^*} & H^{2k}(N\times S^1) \\ \downarrow{\scriptstyle \int_{S^1}} & & \downarrow{\scriptstyle \int_{S^1}} \\ H^{2k-1}(M) & \xrightarrow{\psi^*} & H^{2k-1}(N) \end{array}$$

In particular if $\psi = \mathrm{hol}_\Phi \colon M \to G$ is the classifying map of an ΩG-bundle with connection and Higgs field (A, Φ) we have

$$\begin{array}{ccc} H^{2k}(G\times S^1) & \xrightarrow{(\mathrm{hol}_\Phi\times \mathrm{id})^*} & H^{2k}(M\times S^1) \\ \downarrow{\scriptstyle \int_{S^1}} & & \downarrow{\scriptstyle \int_{S^1}} \\ H^{2k-1}(G) & \xrightarrow{\mathrm{hol}_\Phi^*} & H^{2k-1}(M) \end{array}$$

Composing this with the results we have already established for the path fibration we have a commuting diagram

$$\begin{array}{ccccc} & \overset{cw(\widetilde{PG})}{\nearrow} & H^{2k}(G\times S^1) & \xrightarrow{(\mathrm{hol}_\Phi\times \mathrm{id})^*} & H^{2k}(M\times S^1) \\ I^k(\mathfrak{g}) & & \downarrow{\scriptstyle \int_{S^1}} & & \downarrow{\scriptstyle \int_{S^1}} \\ & \underset{\tau}{\searrow} & H^{2k-1}(G) & \xrightarrow{\mathrm{hol}_\Phi^*} & H^{2k-1}(M) \end{array}$$

This gives us

Theorem 4.13. *If $P \to M$ is an ΩG-bundle and*

$$s(P) \colon I^k(\mathfrak{g}) \to H^{2k-1}(M)$$

is the map which associates to any invariant polynomial f the string class of P, that is $s(P)(f) = s_f(P)$, then the following diagram commutes

$$\begin{array}{ccc} I^k(\mathfrak{g}) & \xrightarrow{cw(\widetilde{P})} & H^{2k}(M\times S^1) \\ \downarrow{\scriptstyle \tau} & \overset{s(P)}{\searrow} & \downarrow{\scriptstyle \int_{S^1}} \\ H^{2k-1}(G) & \xrightarrow{\mathrm{hol}_\Phi^*} & H^{2k-1}(M) \end{array}$$

Notice that although the string form is natural we would not expect the diagram in Theorem 4.13 to commute at the level of forms unless the connection and Higgs field on P are the pullback of the connection and Higgs field on the path fibration. While it is straightforward to see that this is true for the Higgs field it is not true for the connection. We can however calculate what happens directly as follows.

If we start with the G-bundle $\widetilde{P} \to M \times S^1$ we can pull back by the evaluation map $\mathrm{ev} \colon [0, 1] \times \Omega_{M_0}(M \times S^1) \to M \times S^1$ to get a trivial bundle $\mathrm{ev}^*\widetilde{P}$ over $[0, 1] \times \Omega_{M_0}(M \times S^1)$. A section is given by

$$h \colon [0, 1] \times \Omega_{M_0}(M \times S^1) \to \mathrm{ev}^*\widetilde{P}; \quad (t, \gamma) \mapsto \hat{\gamma}(t),$$

where $\hat{\gamma}$ is the horizontal lift of γ. If $\tilde{A}$ is the connection in $\widetilde{P}$ we can pull it back to $\mathrm{ev}^*\widetilde{P}$ and then back to $[0, 1] \times \Omega_{M_0}(M \times S^1)$ to obtain

$$\tilde{A}' := h^*\mathrm{ev}^*\tilde{A}.$$

We can calculate the curvature $\tilde{F}$ of $\tilde{A}$ and pull it back by ev to $[0, 1] \times \Omega_{M_0}(M \times S^1)$ and because this is a product manifold we can decompose it into parts with a dt and parts without a dt. Under this decomposition, we have

$$\mathrm{ev}^*\tilde{F} = -\frac{\partial}{\partial t}\tilde{A}' \wedge dt + \tilde{F}',$$

where we call the component without a dt $\tilde{F}'$ since if we view the form $\tilde{A}'$ for fixed t_0 as a connection form on $\Omega_{M_0}(M \times S^1)$ then its curvature is $\tilde{F}'$ evaluated at t_0.

Now, we want to calculate $\int_{S^1} f(\tilde{F})$. The following result is straightforward and allows us to write this integral in terms of the pull back by the evaluation map

Lemma 4.14 ([15]). *Let $\eta\colon M \to \Omega_{M_0}(M \times S^1)$ be as in Section 3.2. For differential q-forms on $M \times S^1$ we have*

$$\eta^* \int_{S^1} \mathrm{ev}^* = \int_{S^1},$$

or equivalently the following diagram commutes

$$\begin{array}{ccc} \Omega^q(M \times S^1) & \xrightarrow{\mathrm{ev}^*} & \Omega^q(\Omega_{M_0}(M \times S^1) \times S^1) \\ \downarrow{\textstyle\int_{S^1}} & & \downarrow{\textstyle\int_{S^1}} \\ \Omega^q(M) & \xleftarrow[\eta^*]{} & \Omega^q(\Omega_{M_0}(M \times S^1)) \end{array}$$

Therefore for a general ΩG-bundle $P \to M$, we have

$$\begin{aligned} \int_{S^1} f(\tilde{F}) &= \eta^* \int_{S^1} \mathrm{ev}^* f(\tilde{F}) \\ &= \eta^* \int_{S^1} f(\mathrm{ev}^* \tilde{F}). \end{aligned}$$

So we wish to calculate explicitly $\int_{S^1} f(\mathrm{ev}^*\tilde{F})$. To avoid many factors of 2π we will, for this proof, regard the circle as the interval [0, 1] with endpoints identified. Then we can write

$$\int_{S^1} f(\mathrm{ev}^*\tilde{F}) = \int_{[0,1]} f(\mathrm{ev}^*\tilde{F})$$

and so we have

$$\begin{aligned} k\int_{S^1} f(\nabla\Phi, F)d\theta &= \eta^* \int_{S^1} f(\mathrm{ev}^*\tilde{F}) \\ &= \eta^* \int_{[0,1]} f\left(-\frac{\partial}{\partial t}\tilde{A}' \wedge dt + \tilde{F}'\right) \\ &= \eta^* \int_{[0,1]} f(\tilde{F}') - k\eta^* \int_{[0,1]} f\left(-\frac{\partial}{\partial t}\tilde{A}', \tilde{F}'\right) dt \\ &= -k\eta^* \int_{[0,1]} f\left(-\frac{\partial}{\partial t}\tilde{A}', \tilde{F}'\right) dt. \end{aligned}$$

Using the formula $\tilde{F}' = d\tilde{A}' + \frac{1}{2}[\tilde{A}', \tilde{A}']$, we can write this as:

$$\begin{aligned} -k\eta^* \Bigg\{ &\int_{[0,1]} f(\partial\tilde{A}', d\tilde{A}')dt + (k-1)\frac{1}{2}\int_{[0,1]} f(\partial\tilde{A}', d\tilde{A}', \dots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt + \cdots \\ &+ \binom{k-1}{k-2}\left(\frac{1}{2}\right)^{k-2} \int_{[0,1]} f(\partial\tilde{A}', d\tilde{A}', [\tilde{A}', \tilde{A}'])dt + \left(\frac{1}{2}\right)^{k-1} \int_{[0,1]} f(\partial\tilde{A}', [\tilde{A}', \tilde{A}'])dt \Bigg\} \end{aligned}$$

where we have written $\partial\tilde{A}'$ for $\partial\tilde{A}'/\partial t$. Thus we need to work with the general term

$$\binom{k-1}{i}\left(\frac{1}{2}\right)^i \int_{[0,1]} f(\partial\tilde{A}', \underbrace{d\tilde{A}', \dots, d\tilde{A}'}_{k-i-1}, \underbrace{[\tilde{A}', \tilde{A}'], \dots, [\tilde{A}', \tilde{A}']}_{i})dt.$$

To deal with these terms we shall use integration by parts and the ad-invariance of f (Lemma 2.2). We are now in a position to prove

Proposition 4.15.

$$s_f(A, \Phi) = \mathrm{hol}_\Phi^* \tau(f) + d\chi$$

for some $(2k-2)$ form χ.

Proof. To calculate the general term given above, we integrate by parts in the $\Omega_{M_0}(M \times S^1)$ and t directions giving

$$\int_{[0,1]} f_i dt = \int_{[0,1]} f(d\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt + i\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', d[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt - d\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt$$

and

$$\int_{[0,1]} f_i dt = f(\tilde{A}'_1, d\tilde{A}'_1, \ldots, d\tilde{A}'_1, [\tilde{A}'_1, \tilde{A}'_1]) - f(\tilde{A}'_0, d\tilde{A}'_0, \ldots, d\tilde{A}'_0, [\tilde{A}'_0, \tilde{A}'_0]) - (k-1-i)\int_{[0,1]} f(\tilde{A}', \partial d\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt - i\int_{[0,1]} f(\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', \partial[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt$$

where we have written f_i for the integrand of the general term given earlier. Combining these gives

$$(k-i)\int_{[0,1]} f_i dt = f_{i,1} - f_{i,0} - i\int_{[0,1]} f(\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', \partial[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt + i(k-1-i)\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', d[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt - (k-1-i)d\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt$$

where we have written $f_{i,1}$ and $f_{i,0}$ for f_i evaluated at $t = 1$ and 0 respectively. Using ad-invariance, the term on the middle line simplifies to

$$\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', d[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt = \int_{[0,1]} f(d\tilde{A}', \partial[\tilde{A}', \tilde{A}'], \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt - 2\int_{[0,1]} f(\partial\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt - (k-2-i)\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', d[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt$$

and so

$$(k-1-i)\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', d[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt = \int_{[0,1]} f(\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', \partial[\tilde{A}', \tilde{A}'], [\tilde{A}', \tilde{A}'])dt - 2\int_{[0,1]} f(\partial\tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt.$$

Inserting this into the formula for $\int f_i dt$ gives

$$(k-i)\int_{[0,1]} f_i dt = f_{i,1} - f_{i,0} - 2i\int_{[0,1]} f_i dt - (k-1-i)d\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt$$

and hence

$$(k+i)\int_{[0,1]} f_i dt = f_{i,1} - f_{i,0} - (k-1-i)d\int_{[0,1]} f(\partial\tilde{A}', \tilde{A}', d\tilde{A}', \ldots, d\tilde{A}', [\tilde{A}', \tilde{A}'])dt.$$

So we have the following expression for $s_f(A, \Phi)$:

$$k\int_{S^1} f(\nabla\Phi, F)d\theta = -k\eta^*\left\{\sum_{i=0}^{k-1}\binom{k-1}{i}\left(\frac{1}{2}\right)^i\frac{1}{k+i}\left(f_{i,1} - f_{i,0} - (k-i-1)dc_i\right)\right\}$$

where c_i is the last integral in the equation above (with i $[\tilde{A}', \tilde{A}']$'s).

Now since $\tilde{A}'_0 = 0$ and $h(0, \gamma) = h(1, \gamma)\mathrm{hol}(\gamma)$ (where h is the section from earlier), we have that

$$\tilde{A}'_0 = \mathrm{ad}(\mathrm{hol}^{-1})\tilde{A}'_1 + \mathrm{hol}^{-1}d\mathrm{hol}$$

and so

$$\tilde{A}_1' = -d\text{hol}\,\text{hol}^{-1}.$$

Therefore we have that $f_{i,0} = 0$ and we can calculate $f_{i,1}$ in terms of $f_{0,1}$ as follows:

$$\begin{aligned} f_{0,1} &= f(\tilde{A}_1', d\tilde{A}_1') \\ &= f(-d\text{hol}\,\text{hol}^{-1}, d(-d\text{hol}\,\text{hol}^{-1})) \\ &= (-1)^k \left(\frac{1}{2}\right)^{k-1} \text{hol}^* f(\Theta, [\Theta, \Theta]) \end{aligned}$$

and in general,

$$\begin{aligned} f_{i,1} &= f(\tilde{A}_1', d\tilde{A}_1', \dots, d\tilde{A}_1', [\tilde{A}_1', \tilde{A}_1']) \\ &= (-1)^{k-i} \left(\frac{1}{2}\right)^{k-1-i} \text{hol}^* f(\Theta, [\Theta, \Theta]) \\ &= (-1)^i 2^i f_{0,1} \end{aligned}$$

using the fact that $d(-d\text{hol}\,\text{hol}^{-1}) = -\frac{1}{2}[d\text{hol}\,\text{hol}^{-1}, d\text{hol}\,\text{hol}^{-1}]$.

Therefore we have

$$\begin{aligned} k\int_{S^1} f(\nabla\Phi, F)d\theta = &\left(-\frac{1}{2}\right)^{k-1} k\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{k+i} \text{hol}_\Phi^* f(\Theta, [\Theta, \Theta]) \\ &+ k\sum_{i=0}^{k-i} \binom{k-1}{i} \left(\frac{1}{2}\right)^i \frac{1}{k+i}(k-i-1)dc_i. \end{aligned}$$

Recall from the proof of Proposition 4.11 that the coefficient above is equal to the coefficient in the definition of the transgression map:

$$k\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{k+i} = \frac{k!(k-1)!}{(2k-1)!}.$$

So we see that the pull back of $\tau(f)$ is cohomologous to the string form. □

5. Conclusion

There is an immediate natural generalisation of what we have done which is to replace $M \times S^1$ by an S^1-bundle $Y \to M$. In this case the caloron correspondence has been used in [19] in an application to string theory and in [20] in an application to T-duality. String classes in this case have been constructed in [15] and will appear in [21].

The results we have presented are one way of defining characteristic classes for infinite-dimensional bundles by essentially using the caloron correspondence to avoid the infinite dimensionality. Another approach would be to deal with the infinite dimensionality directly by extending the notion of invariant polynomials to a genuinely infinite-dimensional setting. Paycha, Rosenberg and collaborators have done this by using the Wodzicki residue as a trace on the Lie algebra of the group of invertible, zeroth-order pseudo-differential operators on a vector bundle over a compact space (see for example [22] or [23] and the references therein). In the case of a trivial, rank n real vector bundle over the circle this group contains the loop group of $O(n)$ as the subgroup of multiplication operators. The Wodzicki characteristic classes defined in this way vanish on bundles whose structure group has a reduction to the loop group [24].

Finally we note two unanswered questions. Firstly we know from [3] that the three-dimensional string class is the obstruction to lifting the structure group of a loop group bundle to the Kac–Moody group. The geometric significance of the higher string classes is an open question. Secondly if we regard ΩG as based gauge transformations of a bundle over the circle, how much of our work can be generalised to an arbitrary group of gauge transformations?

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References

[1] H. Garland, M.K. Murray, Kac–Moody monopoles and periodic instantons, Comm. Math. Phys. 120 (2) (1988) 335–351.
[2] H. Garland, M.K. Murray, Why instantons are monopoles, Comm. Math. Phys. 121 (1) (1989) 85–90.
[3] M.K. Murray, D. Stevenson, Higgs fields, bundle gerbes and string structures, Comm. Math. Phys. 243 (3) (2003) 541–555.

[4] T.P. Killingback, World-sheet anomalies and loop geometry, Nuclear Phys. B 288 (3–4) (1987) 578–588.
[5] A.L. Carey, S. Johnson, M.K. Murray, D. Stevenson, B.-L. Wang, Bundle gerbes for Chern-Simons and Wess–Zumino–Witten theories, Comm. Math. Phys. 259 (3) (2005) 577–613.
[6] A.L. Carey, M.K. Murray, String structures and the path fibration of a group, Comm. Math. Phys. 141 (3) (1991) 441–452.
[7] D. Husemoller, Fibre Bundles, in: Graduate Texts in Mathematics, Springer-Verlag, 1993.
[8] J.L. Dupont, Curvature and Characteristic Classes, in: Lecture Notes in Mathematics, vol. 640, Springer-Verlag, Berlin, 1978.
[9] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. II, Interscience Publishers, John Wiley & Sons, Inc., 1969.
[10] R.S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (N.S.) 7 (1) (1982) 65–222.
[11] J. Milnor, Remarks on infinite-dimensional Lie groups, in: Relativity, Groups and Topology, II (Les Houches, 1983), North-Holland, Amsterdam, 1984, pp. 1007–1057.
[12] A. Pressley, G. Segal, Loop Groups, in: Oxford Mathematical Monographs, Oxford University Press, New York, 1986.
[13] A.L. Carey, J. Mickelsson, A gerbe obstruction to quantization of fermions on odd-dimensional manifolds with boundary, Lett. Math. Phys. 51 (2) (2000) 145–160.
[14] A.L. Carey, J. Mickelsson, The universal gerbe, Dixmier-Douady class, and gauge theory, Lett. Math. Phys. 59 (1) (2002) 47–60.
[15] R.F. Vozzo, Loop Groups, Higgs Fields and Generalised String Classes. Ph.D. Thesis, University of Adelaide, 2009. math.DG/0906.4843.
[16] B. Sury, T. Wang, F.-Z. Zhao, Identities involving reciprocals of binomial coefficients, J. Integer Seq. 7 (2) (2004) Article 04.2.8, 12 pp. (electronic).
[17] S.S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. of Math. (2) 99 (1974) 48–69.
[18] J.L. Heitsch, H.B. Lawson Jr., Transgressions, Chern-Simons invariants and the classical groups, J. Differential Geom. 9 (1974) 423–434.
[19] A. Bergman, U. Varadarajan, Loop groups, Kaluza-Klein reduction and M-theory, J. High Energy Phys., (6) 043 (2005) 28 pp (electronic).
[20] P. Bouwknegt, V. Mathai, T-duality as a duality of loop group bundles, J. Phys A: Math. Theory, 2009. arXiv:0902.4341.
[21] M.K. Murray, R.F. Vozzo, Circle actions, central extensions and string structures, 2010. arXiv:1004.0779.
[22] S. Paycha, S. Rosenberg, Traces and characteristic classes on loop spaces, in: Infinite Dimensional Groups and Manifolds, in: IRMA Lect. Math. Theor. Phys., Vol. 5, de Gruyter, Berlin, 2004, pp. 185–212.
[23] S. Paycha, Chern–Weil calculus extended to a class of infinite dimensional manifolds, 2007. math.DG/0706.2554.
[24] S. Rosenberg, Private communication and unpublished manuscript. 2009.