Higher twisted K-theory

David Brook

May 14, 2020

Thesis submitted for the degree of
Master of Philosophy
in
Pure Mathematics
at The University of Adelaide
Faculty of Engineering, Computer and Mathematical Sciences
School of Mathematical Sciences

Contents

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

Signed: .... Date: ....

Acknowledgements

First and foremost, I would like to extend my sincere gratitude to my principal supervisor Elder Professor Mathai Varghese. Without Mathai's support, I would not be the mathematician that I am today. His guidance through my project was immensely helpful, suggesting many novel ideas to research but always ensuring that I could perform research independently, and he truly pushed me to strive for the best results that I could produce. Mathai helped me to find and fostered my passion for pure mathematics and for this I will be forever grateful.

I am also deeply grateful to my secondary supervisor Dr Peter Hochs, for answering all of my questions no matter how silly and for working through many problems with me. Peter would always make me feel like I could achieve success and encouraged me to follow my passions, and for this I am truly grateful.

I would like to thank Professor Michael Murray for helpful discussions regarding my project, and Dr David Roberts for his proofreading and for listening to any mathematical ideas that I might come up with.

Of course, the late nights of work would not have been possible without friends to support me along the way. I am grateful to Johnny, Jaklyn, Taj and Hao for endless discussions, both mathematical and not, and for being the best China companions I could ask for. I thank Aline for the hours of lunches and laughter, and Haripriya for many cups of tea. I am also grateful to Mary, Mark and Izzy for the good times and the good food, as well as everybody from the "I am passed" group chat for countless good times.

Finally, I am indebted to my parents for being there for me through everything. Thank you for always encouraging me to follow my passions wherever they may lead me, and for all of your help with everything. Thank you to Pas for always supporting Mum and I, and to Fiona for always being there for me.

Dedication

To my family, friends and pets.

Abstract

Higher twisted K-theory is a recent generalisation of topological K-theory introduced by Ulrich Pennig which captures all of the homotopy-theoretic twists of topological K-theory in a geometric way. We explore a variety of basic properties belonging to higher twisted K-theory including functoriality, cohomology properties and the existence of a graded module structure, and provide an alternative formulation of the higher twisted K-theory groups from a topological perspective. We then investigate ways of producing explicit geometric representatives of the higher twists of K-theory viewed as cohomology classes in special cases using the clutching construction and when the class is decomposable. Spectral sequences are developed to allow for explicit computations to be performed, and finally a variety of computations are performed both for spaces with torsion-free cohomology – which is the case largely discussed by Pennig – as well as in the torsion case where additional difficulties are present.

Introduction

Background and motivation

Topological K-theory is a rich area of study in algebraic topology, first developed by Atiyah and Hirzebruch in the late 1950s [AH59]. It has become an indispensable tool in topology, for instance Adams and Atiyah were able to use it to provide a simple proof that S^1, S^3 and S^7 are the only spheres which can be provided with H-space structures and Adams also shows that a large amount of stable homotopy theory can be derived from topological K-theory, and has found further application in differential geometry, mathematical physics and index theory.

Since it forms a cohomology theory, there is an abstract notion of twist for K-theory which was first introduced by Donovan and Karoubi in a limited setting [DK70], before a setting with greater generality was introduced by Rosenberg [Ros89]. This was further developed by Bouwknegt, Carey, Mathai, Murray and Stevenson [BCM$^+$02], after which more work was done by Mathai and Stevenson [MS03, MS06a] as well as by Atiyah and Segal [AS04, AS06]. Twists of K-theory over a topological space X were geometrically viewed as either (isomorphism classes of) principal PU-bundles over X or (stable isomorphism classes of) bundle gerbes over X, both classified by the third-degree integral cohomology of X, and this allowed the twisted K-theory groups to be defined in a number of different ways. Work by Antieau, Gómez and Gepner [AGG14] showed that these definitions of classical twisted K-theory did indeed agree. Twisted K-theory was found to be relevant in numerous aspects of mathematical physics. In the presence of a B-field, the charges of D-branes – fundamental objects in string theory – on a spacetime take values in an appropriate twisted K-theory group of the spacetime [BM00]. Equivariant twisted K-theory has also been linked to the Verlinde ring, and allowed for a deeper understanding of this object which arises in conformal field theory [FHT11a, FHT13, FHT11b].

From a homotopy-theoretic point of view, however, the geometric twists considered up until this point did not capture the entire picture – there existed a wider class of abstract twists for which no geometric interpretation was known. This changed with the introduction of a Dixmier–Douady theory for strongly self-absorbing C^*-algebras by Pennig and Dadarlat [DP16, DP15a], analogous to the Dixmier–Douady theory for the compact operators on a Hilbert space which is integral to the construction of the classical twists. Using this theory, Pennig was able to propose a geometric model which captures all of the twists of K-theory, extending what was classically known as twisted K-theory to its greatest generality which he has termed higher twisted K-theory [Pen15]. The work in this thesis is inspired by Pennig's formulation of higher twisted K-theory, and we aim to fill in some of the details which have already been worked out in the classical case and perform computations in this new setting.

A major result of Pennig and Dadarlat's work in [DP15b] shows that the full set of twists of K-theory over a locally compact Hausdorff space X is equivalent to the set of isomorphism classes of algebra bundles over X whose fibres are isomorphic to the stabilised infinite Cuntz algebra $\mathcal{O}_\infty \otimes \mathcal{K}$, or equivalently isomorphism classes of principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over X. They also prove that when the space X has torsion-free cohomology, this set of twists can be identified with the direct sum of the odd-degree integral cohomology groups of X. These two facts are critical in our work that follows.

Thesis goals

Since higher twisted K-theory was only developed very recently, there are a number of directions for our work to go in, ranging from exploring basic properties to advanced computational techniques.

The first task to perform in exploring higher twisted K-theory is ensuring that it satisfies some expected properties analogous to those satisfied by topological and classical twisted K-theory. These include functoriality, cohomology properties and the existence of a graded module structure, as well as a formulation in terms of Fredholm operators.

The second major goal of this thesis was to investigate the higher twists themselves. While Pennig provides a geometric interpretation of the higher twists, it is often desirable to view twists of K-theory as cohomology classes where possible, for example in computations involving spectral sequences. In the case that twists can be identified with cohomology classes, however, there is no general method to construct an explicit bundle associated to a cohomology class, nor is there a method to do the reverse.

Of course, when studying something computable such as higher twisted K-theory, performing computations for a variety of spaces should be one of the main aims. Therefore the final and most major aim was to develop and apply tools for computation. These range from straightforward tools such as the Mayer–Vietoris sequence to tools which are more difficult to apply but gain deeper results, including spectral sequences. Although Pennig and Dadarlat's results apply only to the torsion-free setting, we also perform computations for some spaces with torsion in their cohomology.

Results

We begin by addressing the first goal, to explore the basic properties of higher twisted K-theory. Indeed, in Proposition 2.3.7 we show that higher twisted K-theory is a functor from an appropriate category of topological spaces equipped with twists to the category of abelian groups, where a twist over X is geometrically interpreted as an algebra bundle over X with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ and a morphism $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ of spaces equipped with twists is a continuous proper map $f : X \to Y$ together with an isomorphism $\theta : f^*\mathcal{A}_Y \to \mathcal{A}_X$. We also show that higher twisted K-theory forms a generalised cohomology theory in Proposition 2.3.8, i.e. given a topological space X equipped with a twist and a closed subspace $A \subset X$, there is a notion of relative higher twisted K-theory and these groups satisfy the axioms for a generalised cohomology theory. Furthermore, to show that higher twisted K-theory truly is a generalisation of both topological and classical twisted K-theory, we prove that higher twisted K-theory reduces to topological K-theory when a trivial twist is taken in Proposition 2.3.10, and similarly show in Proposition 2.3.11 that it reduces to classical twisted K-theory in the appropriate setting.

While Pennig has proved a variety of basic results about higher twisted K-theory, he does this using mostly homotopy-theoretic notions and C^*-algebraic K-theory, and so we aim to explore these results and more using an alternative, more topological viewpoint. To do so, we generalise a result of Rosenberg [Ros89] in Theorem 2.5.1 and prove that the higher twisted K-theory groups can be identified with

$$K^0(X, \delta) = [\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})};$$
$$K^1(X, \delta) = [\mathcal{E}_\delta, \Omega\, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})};$$

where $\mathcal{E}_\delta$ is the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X representing the abstract twist δ, $\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ denotes the Fredholm operators on the standard Hilbert $(\mathcal{O}_\infty \otimes \mathcal{K})$-module, Ω denotes the based loop space and $[-,-]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} = \pi_0(C(-,-)^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})})$ denotes unbased homotopy classes of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-equivariant maps. This formulation allows for greater insight into the structure of the K-theory groups, and will allow us to equip the higher twisted K-theory functor with additional properties.

Using this topological characterisation, we investigate the algebraic properties of higher twisted K-theory. We construct a general product of the form

$$K^m(X, \delta_X) \times K^n(Y, \delta_Y) \to K^{m+n}(X \times Y, {p_X}^*\delta_X + {p_Y}^*\delta_Y)$$

where p_X and p_Y denote projection from $X \times Y$, and explore its properties in Proposition 4.1.6. Fixing $X = Y$ and pulling this external product back along the diagonal map $X \to X \times X$ provides a product map

$$K^m(X, \delta) \times K^n(X, \delta') \to K^{m+n}(X, \delta + \delta'),$$

but for a general twist δ we do not obtain a graded ring structure on the higher twisted K-theory groups $K^*(X, \delta)$. Restricting this map further to the case that one of the twists is trivial we see in Proposition 4.1.8 that $K^*(X, \delta)$ forms a graded module over the graded ring $K^*(X)$ via the map

$$K^m(X) \times K^n(X, \delta) \to K^{m+n}(X, \delta).$$

To address the second goal of this thesis, we work with special cases. In the simplest case, when the topological space is an odd-dimensional sphere S^{2n+1}, we note that the clutching construction can be used to construct principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over S^{2n+1} by specifying a gluing map $S^{2n} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. These maps are classified up to homotopy by an integer, and using Pennig and Dadarlat's link between twists and cohomology classes [DP16] we see that the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over S^{2n+1} are also classified by an integer, providing an explicit geometric way of describing the twist over S^{2n+1} associated with a particular cohomology class in Section 3.1. Another case that we consider involves removing the assumption on the space but restricting to a particular type of cohomology class. For a general CW-complex X with torsion-free cohomology, we take a 5-class δ given by the cup product of a 2-class α and a 3-class β. By associating a principal $U(1)$-bundle to α and a principal PU-bundle to β and constructing an effective action of $U(1) \times PU$ on $\mathcal{O}_\infty \otimes \mathcal{K}$, we are able to construct a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X corresponding to the cohomology class δ, and obtain a more general result in Theorem 3.2.1.

Moving towards our final goal of performing computations, we highlight the Mayer–Vietoris sequence in higher twisted K-theory developed by Pennig in Proposition 2.3.9. This proves useful in some computations, but most spaces require more powerful machinery in the form of spectral sequences. Hence in order to be better equipped to perform these computations, we have developed an analogue of the twisted Atiyah–Hirzebruch spectral sequence for higher twisted K-theory. We show in Theorem 4.2.3 that when X is a CW complex, there is an analogue of the Atiyah–Hirzebruch spectral sequence with E_2-term $E_2^{p,q} = H^p(X, K^q(pt))$ and which strongly converges to the higher twisted K-theory $K^*(X, \delta)$. In particular, when the abstract twist δ can be identified with a cohomology class $\delta \in H^{2n+1}(X, \mathbb{Z})$ we see in Theorem 4.2.4 that the d_{2n+1} differential will be of the form $d_{2n+1}(x) = d'_{2n+1}(x) + \delta \cup x$ where d'_{2n+1} is the differential in the ordinary Atiyah–Hirzebruch spectral sequence in topological K-theory, which is in particular an operator whose image is torsion. There is, in fact, a more general Segal spectral sequence that can be applied in this setting and we generalise a result of Rosenberg [Ros17] to obtain this sequence. Letting $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ be a fibre bundle of CW complexes with δ a twist over E, we prove in Theorem 4.2.5 that there is a spectral sequence with E_2-term $E_2^{p,q} = H^p(B, K^q(F, \iota^*\delta))$ which strongly converges to the higher twisted K-theory $K^*(E, \delta)$. We also obtain more explicit information about the differentials of this sequence in Theorem 4.2.6, which is useful in proving general results about the higher twisted K-theory of Lie groups.

To conclude the thesis, we use these spectral sequences and other techniques to perform computations, generalising a wide variety of results of [BCM+02] from the classical case. Beginning with the simplest case, we compute the higher twisted K-theory of the odd-dimensional spheres in Proposition 5.1.1 using both the Mayer–Vietoris sequence and the spectral sequence, and obtain the same results as in the classical setting for S^3. Due to the simplicity of this computation, we are able to explicitly determine the generator of $K^1(S^{2n+1}, \delta) \cong \mathbb{Z}_N$ where the twist $\delta \in H^{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ is $N \neq 0$ times a generator, and present this in Proposition 5.1.3. We also compute the higher twisted K-theory of various products of spheres in Propositions 5.1.4 and 5.1.5. Another important class of spaces with torsion-free cohomology are the compact, connected, simply connected Lie groups. We use the spectral sequence to obtain information about the higher twisted K-theory groups of $SU(n)$ for twists of specific degree in Theorem 5.1.8, and follow techniques of Rosenberg [Ros17] to draw more general conclusions about the higher twisted K-theory of $SU(n)$ in Theorem 5.1.9.

Although the major result of Pennig and Dadarlat linking twists of K-theory to cohomology classes only applies when the space X has torsion-free cohomology, we show that this condition can be somewhat relaxed in order to allow for a wider class of spaces to be considered. This allows computations to be performed for real projective space and for Lens spaces in Propositions 5.2.1 and 5.2.2 respectively. A class of examples which may be of greater physical interest are $SU(2)$-bundles over 4-manifolds. These spaces belong in the setting of spherical T-duality in M-theory, and Bouwknegt, Evslin and Mathai prove that the spherical T-duality transformation induces a degree-shifting isomorphism on the 7-twisted K-theory groups of these bundles [BEM15a, BEM15b, BEM18]. They do not, however, consider the 5-twisted K-theory of these $SU(2)$-bundles, and this is what we compute in Subsection 5.2.3. We place restrictions on the base space M in order to ensure that the 5-twists of the bundle correspond exactly to the integral 5-classes of the bundle, and then use the spectral sequence to compute the 5-twisted K-theory. These groups are heavily dependent on the ring structure of the cohomology of M, and so several specific base 4-manifolds M are chosen to obtain complete computations.

Outline of this thesis

The first chapter of this thesis contains preliminaries on topological K-theory and operator algebraic K-theory, including background on vector bundles and Fredholm operators on both Hilbert spaces and Hilbert C^*-modules. This allows for a motivated introduction to higher twisted K-theory to be presented in the second chapter using Pennig's original formulation, after which various basic properties of higher twisted K-theory are explored and an alternative topological characterisation of the higher twisted K-theory groups is provided. In the third chapter, we describe methods of producing explicit geometric representatives for twists of K-theory described by cohomology classes using both the clutching construction and by considering decomposable cohomology classes. The fourth chapter describes the external product and graded module structure on higher twisted K-theory, and develops spectral sequences for computations. Finally, the fifth chapter contains explicit computations of higher twisted K-theory for various spaces. This chapter is separated into two sections; in the first we consider spaces with torsion-free cohomology while in the second we consider spaces where torsion must be taken into account.

Chapter 1

Preliminaries

This chapter serves to explore the notion of topological K-theory from a variety of viewpoints which will be relevant in the exploration of higher twisted K-theory, and also to introduce operator algebraic K-theory through which higher twisted K-theory was originally formulated. We present only the basic definitions and results required to obtain insight into these areas, and the reader who is familiar with this content can proceed directly to Chapter 2.

1.1 Topological K-theory

The idea of topological K-theory goes back to 1959, at which time Atiyah and Hirzebruch modified Grothendieck's recently defined algebraic K-theory to the topological setting [AH59]. At its core, complex topological K-theory is a generalised multiplicative cohomology theory whose zero-dimensional piece classifies complex vector bundles over topological spaces up to isomorphism.

We begin our exposition with a brief survey of topological vector bundles as could be found in [Ati67, Par08, Hat17] for instance. Using this, we follow the standard approach of these same references in defining topological K-theory, and provide results to show that it forms a generalised cohomology theory. We then define topological K-theory from a more analytic perspective using Fredholm operators on an infinite-dimensional Hilbert space, and conclude by presenting another useful way to view K-theory groups using classifying spaces.

1.1.1 Vector bundles

Definition 1.1.1. A *complex vector bundle* of rank k over a topological space X is a topological space $\mathcal{E}$ and a continuous surjective map $\pi : \mathcal{E} \to X$ such that

- $\pi^{-1}(p)$ is a complex vector space of dimension k for all $p \in X$;

- there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of X such that for every $\alpha \in A$ there exists a homeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ making the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times \mathbb{C}^k \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle \pi_{U_\alpha}} \\ & & U_\alpha, \end{array}$$

  and Φ_α restricts to a linear isomorphism $\pi^{-1}(\{x\}) \cong \mathbb{C}^k$ for each $x \in U_\alpha$.

More generally, a complex vector bundle over X need not have constant rank if X is not connected, but the rank is locally constant.

This shows that a complex vector bundle consists of data $(\mathcal{E}, \pi, X)$ where $\mathcal{E}$ is referred to as the total space, X is the base space, π is projection and $\mathcal{E}_p = \pi^{-1}(p)$ is the fibre over $p \in X$. Although there is a notion of real vector bundle which leads to real topological K-theory, this will not concern us. There exists a far more extensive literature in the case of complex K-theory, and so by vector bundle we will always mean complex vector bundle. We will often denote a vector bundle simply by the total space $\mathcal{E}$ where the base space is understood, and the projection will be denoted $\pi_{\mathcal{E}}$ where there are multiple vector bundles over the same space.

A *morphism* of vector bundles over X is a continuous map $\varphi : \mathcal{E} \to \mathcal{F}$ such that $\pi_{\mathcal{E}} = \pi_{\mathcal{F}} \circ \varphi$ and which restricts to a linear map $\varphi_x : \mathcal{E}_x \to \mathcal{F}_x$ for all $x \in X$. An *isomorphism* of vector bundles is a morphism with an inverse which is also a morphism, and if there exists an isomorphism between $\mathcal{E}$ and $\mathcal{F}$ then we say that these vector bundles are *isomorphic*. The set $\mathrm{Vect}(X)$ is defined to be the set of isomorphism classes of vector bundles over X.

Definition 1.1.2. A *section* of a complex vector bundle $\mathcal{E}$ over X is a continuous map $s : X \to \mathcal{E}$ such that $\pi \circ s = \mathrm{id}_X$. The *space of sections* of a vector bundle is denoted $C(X, \mathcal{E})$.

Example 1.1.1. The product vector bundle $X \times \mathbb{C}^k$ over X with π given by projection onto the first factor is called the *trivial bundle* of rank k over X. Any vector bundle which is isomorphic to a trivial bundle is said to be *trivial.* For this reason, the second point in Definition 1.1.1 is often referred to as local triviality, and the elements of $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$ are referred to as local trivialisations. A section of the trivial bundle of rank k over X is simply a continuous map $s : X \to \mathbb{C}^k$, and therefore $C(X, X \times \mathbb{C}^k)$ is equal to the set of continuous functions $X \to \mathbb{C}^k$, denoted $C(X, \mathbb{C}^k)$.

Example 1.1.2. If M is a manifold, then the suggestively named tangent bundle TM and cotangent bundle T^*M are vector bundles over M (see, for instance, [Lee03]). A section of the tangent bundle is a vector field and a section of the cotangent bundle is a 1-form, thus $C(M, TM)$ is the space of all vector fields on M and $C(M, T^*M) = \wedge^1 M$.

Example 1.1.3. Consider the complex projective space $\mathbb{C}P^n$ of lines through the origin in $\mathbb{C}^{n+1}$, with the equivalence class of the point $z = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $z \sim \lambda z$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ denoted by $[z_1 : \cdots : z_{n+1}] \in \mathbb{C}P^n$. There is a *canonical line bundle* over $\mathbb{C}P^n$ whose total space is the subspace $\mathcal{E}$ of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ consisting of pairs (l, z) such that $z \in l$, and which is equipped with the obvious projection map $\pi : \mathcal{E} \to \mathbb{C}P^n$. We will show that this is indeed a vector bundle over $\mathbb{C}P^n$. In order to construct local trivialisations, we define an open cover $\{U_i\}_{i\in\{1,\cdots,n+1\}}$ of $\mathbb{C}P^n$ by $U_i = \{[z_1 : \cdots : z_{n+1}] : z_i \neq 0\}$, and observe that the map $U_i \times \mathbb{C} \to \pi^{-1}(U_i)$ defined by $([z_1, \cdots, z_{i-1}, 1, z_{i+1}, \cdots, z_{n+1}], \lambda) \mapsto ([z_1, \cdots, z_{i-1}, 1, z_{i+1}, \cdots, z_{n+1}], (\lambda z_1, \cdots, \lambda z_{n+1}))$ is a homeomorphism. Thus $\mathcal{E}$ is a line bundle over $\mathbb{C}P^n$.

We now move towards placing a richer structure on our set $\mathrm{Vect}(X)$ by introducting ways of constructing new vector bundles from old.

Definition 1.1.3.

- Given a vector bundle $\pi : \mathcal{E} \to Y$ and a continuous map $\phi : X \to Y$, we define the *pullback bundle* of $\mathcal{E}$ over X to have total space

$$\phi^*\mathcal{E} = \{(x, v) \in X \times \mathcal{E} : f(x) = \pi(v)\}$$

  and the natural projection onto X.

- Given two vector bundles $\pi_\mathcal{E} : \mathcal{E} \to X$ and $\pi_\mathcal{F} : \mathcal{F} \to X$, we define the sum $\mathcal{E} \oplus \mathcal{F}$ to be the fibred product of $\mathcal{E}$ and $\mathcal{F}$, i.e.

$$\mathcal{E} \oplus \mathcal{F} = \{(e, f) \in \mathcal{E} \times \mathcal{F} : \pi_\mathcal{E}(e) = \pi_\mathcal{F}(f)\}$$

  with the natural projection map, and then $(\mathcal{E} \oplus \mathcal{F})_x \cong \mathcal{E}_x \oplus \mathcal{F}_x$ for all $x \in X$.

- Given two vector bundles $\mathcal{E}, \mathcal{F}$ over X, the tensor product bundle may be defined in a similar manner such that $(\mathcal{E} \otimes \mathcal{F})_x \cong \mathcal{E}_x \otimes \mathcal{F}_x$ for all $x \in X$.

The pullback operation interacts nicely with direct sum and tensor product as follows.

Lemma 1.1.4. *Let X, Y and Z be topological spaces, with $\mathcal{E}_1$ and $\mathcal{E}_2$ vector bundles over Y and $\mathcal{F}$ a vector bundle over Z. Furthermore, let $\phi : X \to Y$ and $\psi : Y \to Z$ be continuous maps. Then*

(i) $(\psi \circ \phi)^*\mathcal{F} \cong \phi^*(\psi^*\mathcal{F})$;

(ii) $id_Z{}^*\mathcal{F} \cong \mathcal{F}$;

(iii) $\phi^*(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \phi^*\mathcal{E}_1 \oplus \phi^*\mathcal{E}_2$;

(iv) $\phi^*(\mathcal{E}_1 \otimes \mathcal{E}_2) \cong \phi^*\mathcal{E}_1 \otimes \phi^*\mathcal{E}_2$.

As mentioned at the beginning of the chapter, we are interested in classifying vector bundles up to isomorphism, and so we must show that the constructions given in Definition 1.1.3 are well-defined on $\mathrm{Vect}(X)$ to place additional structure on this set.

Proposition 1.1.5. *Let $\phi : X \to Y$ be a continuous map between topological spaces with $\mathcal{E}_1 \cong \mathcal{E}_1'$ and $\mathcal{E}_2 \cong \mathcal{E}_2'$ vector bundles over X and $\mathcal{F} \cong \mathcal{F}'$ vector bundles over Y. Then*

(i) $\phi^*\mathcal{F} \cong \phi^*\mathcal{F}'$;

(ii) $\mathcal{E}_1 \oplus \mathcal{E}_2 \cong \mathcal{E}_1' \oplus \mathcal{E}_2'$;

(iii) $\mathcal{E}_1 \otimes \mathcal{E}_2 \cong \mathcal{E}_1' \otimes \mathcal{E}_2'$.

Therefore the operations $\phi^*[\mathcal{F}] = [\phi^*\mathcal{F}]$, $[\mathcal{E}_1] \oplus [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2]$ and $[\mathcal{E}_1] \otimes [\mathcal{E}_2] = [\mathcal{E}_1 \otimes \mathcal{E}_2]$ are well-defined on $\mathrm{Vect}(X)$.

These results allow us to place an algebraic structure on $\mathrm{Vect}(X)$. In particular, (ii) shows that $\mathrm{Vect}(X)$ forms a monoid under direct sum, where the identity element is the zero bundle (X, X, id_X). Furthermore, it can be shown that $\mathcal{E}_1 \oplus \mathcal{E}_2 \cong \mathcal{E}_2 \oplus \mathcal{E}_1$ and so $\mathrm{Vect}(X)$ actually forms an abelian monoid. In fact, (iii) along with the simple observation that taking the tensor product with the zero bundle gives the zero bundle implies that $\mathrm{Vect}(X)$ is a commutative semiring, which can informally be viewed as a commutative ring in which the underlying "group" is actually a monoid. Now, we would like K-theory to have the structure of a ring and so the final hurdle in defining $K^0(X)$ for a topological space X is constructing an abelian group from our abelian monoid $\mathrm{Vect}(X)$, which becomes a ring when equipped with the induced multiplication from $\mathrm{Vect}(X)$. This is done through a construction known as the Grothendieck group, which we outline in the following. Informally, we add "negative" elements into the abelian monoid in order to give each element an inverse – this is recognisable as the process by which the commutative semiring of natural numbers $\mathbb{N}$ is used to form the commutative ring of integers $\mathbb{Z}$. The formal construction and its basic properties are outlined below.

Definition 1.1.6. Let M be an abelian monoid. Define an equivalence relation on $M \times M$ by $(m_1, m_2) \sim (n_1, n_2)$ if and only if there is an $l \in M$ such that $m_1 + n_2 + l = m_2 + n_1 + l$. We define the *Grothendieck group* of M to be the set of equivalence classes of this relation in $M \times M$ with the operation $[(m_1, m_2)] + [(n_1, n_2)] = [(m_1 + n_1, m_2 + n_2)]$. An element $[(m, n)]$ is commonly denoted $m - n \in G(M)$. If M is a commutative semiring, then $G(M)$ becomes a commutative ring with multiplication defined by

$$(m_1 - m_2)(n_1 - n_2) = m_1 n_1 + m_2 n_2 - m_1 n_2 - m_2 n_1.$$

Lemma 1.1.7. *Let M be a commutative semiring. Then $G(M)$ is a well-defined commutative ring, i.e. the relation defined is truly an equivalence relation, addition is well-defined, commutative and associative; the class of $(0, 0)$ is the identity; the class of (n, m) is an inverse for the class of (m, n); and multiplication is well-defined, associative, commutative and distributive.*

We have now laid out enough groundwork to present the first definition in topological K-theory.

Definition 1.1.8. For X a compact Hausdorff space, we define the K-theory $K^0(X)$ to be the Grothendieck group of the commutative semiring $\mathrm{Vect}(X)$ of isomorphism classes of complex vector bundles over X.

Unraveling this definition, we see that the elements of $K^0(X)$ are formal differences of isomorphism classes of vector bundles over X, i.e. a general element of $K^0(X)$ is of the form $[\mathcal{E}] - [\mathcal{F}]$ for $\mathcal{E}, \mathcal{F}$ vector bundles over X. Note that $K^0(X)$ inherits the structure of a commutative ring, since $\mathrm{Vect}(X)$ forms a commutative semiring.

Example 1.1.4. While the definition is relatively simple to state, K-theory can be quite difficult to compute in general, at least until we build up some machinery to use in computations. At this stage, we can compute the K-theory of the simplest non-empty topological space: a single point. It is clear from the definition that a vector bundle over a point consists of a single vector space, and therefore these vector bundles will be classified up to isomorphism by a natural number. Hence $\mathrm{Vect}(pt) \cong \mathbb{N}$ and thus $K^0(pt) \cong \mathbb{Z}$.

To each compact Hausdorff space X we have associated a commutative ring $K^0(X)$. We now wish to slightly broaden our viewpoint to look at a wider class of spaces – locally compact Hausdorff spaces – and to turn K-theory into a graded ring by introducing higher groups K^n for $n \in \mathbb{Z}$. To the reader who is well-versed in algebraic topology and cohomology theories, K-theory ends up defining a generalised cohomology theory and so many of the techniques available for cohomology theories may be applied to K-theory. In particular, there is a notion of relative and reduced K-theory that we will briefly introduce. This will allow us to explore the fundamental result which separates K-theory from other cohomology theories: Bott periodicity. This result will imply that our K-theory groups are 2-periodic, making calculations immensely easier, but the proof is technical and we will not present the details here. For now, we explore the functorial properties of K-theory, allowing the notion of reduced K-theory to be introduced.

Proposition 1.1.9. *The assignment $X \mapsto K^0(X)$ defines a contravariant functor from the category of compact Hausdorff spaces with continuous maps to the category of commutative rings with unital ring homomorphisms.*

We use this functoriality to define the reduced K-theory ring.

Definition 1.1.10. Let (X, x_0) be a pointed space such that X is compact Hausdorff, and denote inclusion by $j : \{x_0\} \hookrightarrow X$. The *reduced* K-theory ring $\widetilde{K}^0(X)$ of (X, x_0) is the kernel of the induced homomorphism $j^* : K^0(X) \to K^0(\{x_0\}) = \mathbb{Z}$.

As in Proposition 1.1.9, this is a functorial assignment. Of particular interest allowing reduced K-theory to be related to ordinary K-theory is the following result.

Proposition 1.1.11. *For any pointed space* (X, x_0) *with* X *compact Hausdorff, we have* $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$.

This allows us to pass to the more general category of locally compact Hausdorff spaces equipped with proper continuous maps.

Definition 1.1.12.

(i) A Hausdorff space X is *locally compact* if for every $x \in X$ there is an open neighbourhood $U \ni x$ such that U has compact closure.

(ii) A continuous function $f : X \to Y$ is *proper* if $f^{-1}(C)$ is a compact subset of X for every compact subset $C \subset Y$.

(iii) The *one-point compactification* of X, denoted X^+, is the disjoint union $X \amalg \{\infty\}$ obtained by adding a point ∞ to X.

It can be verified that X^+ truly is a compact Hausdorff space if X is a locally compact Hausdorff space, so this allows us to associate a compact Hausdorff space to each locally compact Hausdorff space and in this way we can extend our definition of K-theory to these spaces.

Definition 1.1.13. For X a locally compact Hausdorff space, we define $K^0(X) = \widetilde{K}^0(X^+)$ taking ∞ as the basepoint of X^+.

It can be easily verified that if X is a compact Hausdorff space then this agrees with the original definition of $K^0(X)$, and also that this definition of K^0 provides a contravariant functor from the category of locally compact Hausdorff spaces with proper maps to the category of commutative rings with unital ring homomorphisms. As discussed earlier, there is also a notion of relative K-theory which we define here for completeness, and to introduce the notion of a generalised cohomology theory.

Definition 1.1.14. A compact pair (X, A) is a compact Hausdorff space X together with a closed subspace $A \subset X$. The *relative* K-theory ring of (X, A) is defined to be the reduced K-theory of the quotient space X/A, i.e. $K^0(X, A) = \widetilde{K}^0(X/A)$ where we take the basepoint of X/A to be the single point to which A is identified.

From this definition, it can be seen that the relative K-theory of the compact pair (X, x_0) is simply the reduced K-theory of (X, x_0), and that the relative K-theory of $(X, \emptyset)$ is the standard K-theory $K^0(X)$. Now, all that remains in order to express K-theory as a generalised cohomology theory is to define the higher order groups $K^n(X)$, which can be done in a number of equivalent ways. We give two equivalent definitions here which are both useful in differing circumstances, firstly introducing some necessary terminology from algebraic topology. In particular we define K^{-n} for $n \in \mathbb{N}$ and later extend this to $n \in \mathbb{Z}$.

Definition 1.1.15. Let (X, x_0) and (Y, y_0) be pointed spaces.

(i) The *wedge sum* of these spaces $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ by the discrete subspace $\{x_0, y_0\}$, whose basepoint is the image of $\{x_0, y_0\}$ under the quotient map. The wedge sum can also be identified as $(X \times Y) \setminus (X \setminus \{x_0\} \times Y \setminus \{y_0\})$.

(ii) The *smash product* of these spaces $X \wedge Y$ is the quotient of the product $X \times Y$ by the wedge sum $X \vee Y$, whose basepoint is the single point of $X \wedge Y$ to which $X \vee Y$ is identified.

(iii) The *reduced suspension* ΣX of X is defined to be $S^1 \wedge X$, where any point of S^1 is chosen as the basepoint. This can be iterated, i.e. we define $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$ for $n \geq 2$, and we have $\Sigma^n X \cong S^n \wedge X$.

Definition 1.1.16. Let X be a locally compact Hausdorff space and (Y, A) a compact pair. For $n \in \mathbb{N}$ we define $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$, and using functoriality this allows us to define $\widetilde{K}^{-n}(X)$ as in Definition 1.1.10 and $K^{-n}(Y, A)$ as in Definition 1.1.14. Alternatively, we define $\widetilde{K}^{-n}(X) = K^0(\Sigma^n X)$ and then $K^{-n}(X) = \widetilde{K}^{-n}(X^+)$ and $K^{-n}(Y, A) = \widetilde{K}^0(\Sigma^n(Y/A))$.

Theorem 1.1.17. *The definitions given in Definition 1.1.16 are equivalent.*

We have finally developed a sequence of contravariant functors from the category of locally compact Hausdorff spaces with proper maps (and also compact pairs with a suitable notion of morphism) to the category of commutative rings with unital ring homomorphisms. In order to summarise some of the key properties of K-theory by stating that K-theory forms a multiplicative generalised cohomology theory, we must now define the remaining functors K^n for $n \geq 1$. We do this using Bott periodicity – a result initially proved about the homotopy groups of the stable unitary group by Bott in 1957 [Bot59] and which is commonly restated as a periodicity theorem for the K-theory functors that we have defined. There exist a wide range of different proofs of this result; Bott's original proof made use of techniques in Morse theory, while many more recent proofs have used techniques from analysis including Fourier series. For a fairly standard proof, see [Ati67], [Par08] or [Hat17].

Theorem 1.1.18 (Bott Periodicity)**.** *For X a locally compact Hausdorff space, there is a natural isomorphism $K^0(X) \cong K^{-2}(X)$.*

We then define the remaining functors for $n \geq 1$ to be

$$K^n(X) = \begin{cases} K^0(X) & \text{if } n \text{ is even;} \\ K^{-1}(X) & \text{if } n \text{ is odd.} \end{cases}$$

We are now able to state the result that K-theory forms a generalised cohomology theory.

Definition 1.1.19. A *cohomology theory* is a sequence $\{H^p\}_{p\in\mathbb{Z}}$ of contravariant functors from the category of compact pairs to the category of abelian groups such that the *Eilenberg–Steenrod axioms* hold:

(i) Homotopy. If $\{f_t\}_{t\in[0,1]}$ is a homotopy of morphisms of compact pairs, then the induced maps $H^p(f_0)$ and $H^p(f_1)$ are equal for all p.

(ii) Long exact sequence. For any compact pair (X, A), the inclusion map $j : A \hookrightarrow X$ induces a long exact sequence of the form

$$\cdots \to H^p(X,A) \to H^p(X) \xrightarrow{j^*} H^p(A) \xrightarrow{\partial} H^{p+1}(X,A) \to H^{p+1}(X) \xrightarrow{j^*} H^{p+1}(A) \to \cdots$$

for some map $\partial : H^p(A) \to H^{p+1}(X, A)$, for all p.

(iii) Excision. For any compact pair (X, A) with U an open subset of X such that the closure of U is contained in the interior of A, the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism $H^p(X, A) \cong H^p(X \setminus U, A \setminus U)$ for all p.

(iv) Dimension. If X consists of a single point then $H^p(X) = \mathbb{Z}$ if $p = 0$ and is trivial otherwise.

A *generalised cohomology theory* satisfies all of the above axioms except (iv).

Theorem 1.1.20. *K-theory forms a generalised cohomology theory, i.e. it satisfies the first three axioms of Definition 1.1.19.*

Remark 1.1.1. It is straightforward to prove that there is also a long exact sequence in reduced K-theory, where the relative K-groups in the sequence remain the same and the remaining groups are replaced by their reduced counterparts. This is because reduced K-theory is isomorphic to regular K-theory in odd degree, and in the even case the map $K^n(X) \to K^n(A)$ is a map $\widetilde{K}^n(X) \oplus \mathbb{Z} \to \widetilde{K}^n(A) \oplus \mathbb{Z}$ which is an isomorphism between the $\mathbb{Z}$-factors of the unreduced groups.

Note that the sequence of functors in Definition 1.1.19 needs only land in the category of abelian groups, and so K-theory actually satisfies a stronger condition since it lands in the category of commutative rings. This gives the cohomology theory a multiplicative structure, but the term “multiplicative cohomology theory” requires a more general product than what has been developed so far. It is possible to develop a more general tensor product of vector bundles which induces a map $K^{-m}(X) \times K^{-n}(Y) \to K^{-n-m}(X \times Y)$, and this is precisely the form of product which makes K-theory into a multiplicative cohomology theory. We will not present the details here, but the interested reader may discover more about the product in [Hat17] or [Ati67].

Letting $X = Y$ in this product map described above, we may use the diagonal map $X \to X \times X$ to induce a map $K^{-m}(X) \times K^{-n}(X) \to K^{-m-n}(X)$, and in doing so we equip the K-theory ring $K^*(X) = K^0(X) \oplus K^{-1}(X)$ with the structure of a graded ring.

Definition 1.1.21. A ring R is *graded* by a group G if there exists a family of subgroups $\{R_g\}_{g\in G}$ of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g \cdot R_h \subset R_{gh}$ for all $g, h \in G$.

Then it is evident that $K^*(X)$ forms a $\mathbb{Z}_2$-graded ring, since Bott periodicity ensures that the multiplication $K^{-1}(X) \times K^{-1}(X) \to K^{-2}(X) \cong K^0(X)$ lands in the correct group.

So far we have provided a detailed presentation of the structure of K-theory, both as a multiplicative cohomology theory and as a graded ring. We now change focus to computation, presenting an important tool which will be critical in the computation of K-theory groups.

Theorem 1.1.22 (Six-term exact sequence)**.** *For (X, A) a compact pair with inclusion $j : A \hookrightarrow X$ there are natural maps $\partial_0 : K^0(A) \to K^{-1}(X, A)$ and $\partial_1 : K^{-1}(A) \to K^0(X, A)$ such that the cyclic sequence*

$$\begin{array}{ccccc} K^0(X,A) & \longrightarrow & K^0(X) & \xrightarrow{j^*} & K^0(A) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_0} \\ K^{-1}(A) & \xleftarrow[j^*]{} & K^{-1}(X) & \longleftarrow & K^{-1}(X,A) \end{array}$$

is exact. There is an analogous sequence in reduced K-theory.

An important corollary to the six-term exact sequence is the Mayer–Vietoris sequence in K-theory. There is an analogous long exact sequence for any generalised cohomology theory, but Bott periodicity once again makes the sequence particularly useful for computations of K-theory.

Corollary 1.1.23 (Mayer–Vietoris sequence)**.** *Let $X = U_1 \cup U_2$ where U_1 and U_2 are closed subsets of the locally compact space X. Then there is a cyclic exact sequence*

$$\begin{array}{ccccc} K^0(X) & \longrightarrow & K^0(U_1) \oplus K^0(U_2) & \longrightarrow & K^0(U_1 \cap U_2) \\ \uparrow & & & & \downarrow \\ K^{-1}(U_1 \cap U_2) & \longleftarrow & K^{-1}(U_1) \oplus K^{-1}(U_2) & \longleftarrow & K^{-1}(X). \end{array}$$

We finish this section by presenting some computations using the tools that have been developed thus far.

Example 1.1.5.

(i) Since $\mathbb{R}^m \times \{p\}$ is homeomorphic to $\mathbb{R}^m$, then by Definition 1.1.16 we have

$$K^0(\mathbb{R}^n) = K^{-n}(\{p\}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd;} \end{cases}$$

and

$$K^{-1}(\mathbb{R}^n) = K^{-n-1}(\{p\}) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) To compute the K-theory of S^n, we note that S^n is the one-point compactification of $\mathbb{R}^n$. So by Definitions 1.1.13 and 1.1.16 we have $\widetilde{K}^{-m}(S^n) = K^{-m}(\mathbb{R}^n)$ and thus

$$K^0(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \text{ is even;} \\ \mathbb{Z} & \text{if } n \text{ is odd;} \end{cases}$$

and

$$K^{-1}(S^n) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

(iii) The first factor of $\mathbb{Z}$ in $K^0(S^n)$ for n even is generated by the trivial bundle, but the second factor is more interesting. It can be shown e.g. as in [Hat17] that for S^2 this is generated by $[H] - 1$ where H denotes the canonical line bundle over $\mathbb{C}P^1 \cong S^2$ introduced in Example 1.1.3.

(iv) The K-theory of real projective space may be computed using the six-term exact sequence and analysing the boundary maps between the K^0 and K^1 groups, which is quite involved [Par08]. The end result is that $K^0(\mathbb{R}P^n) \cong \mathbb{Z} \oplus \mathbb{Z}_n$ and $K^1(\mathbb{R}P^n) = 0$, which reveals an interesting property of $\mathbb{R}P^n$ – there exists a non-trivial vector bundle over $\mathbb{R}P^n$ which becomes trivial upon taking its sum with itself n times.

These examples shed light on methods of calculation used in algebraic topology in general, and in particular methods which we will apply later in the more general setting of higher twisted K-theory.

1.1.2 Fredholm operators

An alternative definition of topological K-theory arises through the study of a special class of operators on Hilbert spaces known as Fredholm operators. We introduce Fredholm operators and reformulate topological K-theory from this point of view, following the original work of [Ati67]. In doing so, we explore two compatible H-space structures on the space of Fredholm operators, providing operations which are analogous to direct sum and tensor product of vector bundles.

Henceforth $\mathcal{H}$ will denote an infinite-dimensional separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ will denote the bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ or simply $\mathcal{K}$ will denote the ideal of compact operators, i.e. the norm-closure of the set of finite-rank operators. Fredholm operators are a class of bounded linear operators that arose through the study of linear equations of the form $Tx = 0$. Loosely, the size of the kernel of the operator T is an indication of how far away T is from being injective, while the size of the cokernel measures the extent to which T fails to be surjective. By combining these two values into a single value known as the index of T, we have a rough description of the existence and uniqueness of solutions to the equation $Tx = 0$. The Fredholm operators are those for which this type of analysis is meaningful, i.e. the operators that have a well-defined notion of index.

Definition 1.1.24. A *Fredholm operator* on $\mathcal{H}$ is a bounded linear operator with finite-dimensional kernel and finite-dimensional cokernel. The *index* of a Fredholm operator is $\operatorname{ind} T = \dim\ker T - \dim\operatorname{coker} T$. The space of all Fredholm operators on $\mathcal{H}$ is denoted by $\mathrm{Fred}_{\mathcal{H}}$ or simply Fred.

There is an important alternative definition of Fredholm operators given by Atkinson's theorem.

Theorem 1.1.25 (Atkinson. Thm 14.1.1 [WO93])**.** *An operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if and only if it is invertible modulo compact operators, i.e. there is an $S \in \mathcal{B}(\mathcal{H})$ such that $id_{\mathcal{H}} - ST$ and $id_{\mathcal{H}} - TS$ are compact operators on $\mathcal{H}$.*

Letting $\mathcal{Q}(\mathcal{H})$ denote the *Calkin algebra* of $\mathcal{H}$, i.e. the quotient space $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, we obtain a short exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \to 0$$

and thus see that the Fredholm operators can be viewed as the inverse image of the units in the Calkin algebra under π.

We now provide a collection of important results regarding the index map.

Lemma 1.1.26 (14.1.6 to 14.1.9 [WO93])**.** *Let S and T be Fredholm operators and Q a compact operator.*

(i) The composition ST is a Fredholm operator and $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

(ii) The adjoint operator T^* is a Fredholm operator and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

(iii) The set of Fredholm operators on $\mathcal{H}$ is an open subset of $\mathcal{B}(\mathcal{H})$.

(iv) The index map is continuous and thus locally constant.

(v) The index is unaffected by compact peturbations, i.e. $\operatorname{ind}(T + Q) = \operatorname{ind}(T)$.

(vi) Letting Fred_n denote the Fredholm operators of index $n \in \mathbb{Z}$, the connected components of Fred are precisely Fred_n and hence $\pi_0(\mathrm{Fred}) \cong \mathbb{Z}$.

In order to define a commutative ring which is isomorphic to topological K-theory using the Fredholm operators, we aim to define two compatible binary operations on Fred in order to obtain a ring-like structure on Fred. One of the natural operations to consider would be composition of Fredholm operators, but composition is in general not commutative and Fredholm operators are not all invertible. Using Lemma 1.1.26, however, it appears that considering composition up to homotopy may yield the desired results, because the indices of ST and TS are equal and thus these operators are homotopic through Fredholm operators. This leads to the notion of an H-space structure on a topological space.

Definition 1.1.27. An H-space is a topological space X with a fixed basepoint $e \in X$ and a continuous map $\mu : X \times X \to X$ such that $\mu(x, e) = x = \mu(e, x)$ for all $x \in X$.

We can see that Fred forms an H-space with identity element given by the identity operator I and with binary operation given by composition of operators. In fact, Fred has additional structure as alluded to above. Composition is associative, and while there do not exist inverses a priori, there are inverses up to homotopy: for $S \in \mathrm{Fred}$ we see that SS^* and S^*S both have index 0 and thus are homotopic to the identity. A space with these properties is sometimes referred to as an H-group, as for any space X the set $[X, \mathrm{Fred}]$ will inherit a group structure through the obvious definition. In fact, by introducing a second operation on Fred defined up to homotopy which satisfies certain distributivity axioms, we will turn Fred into what is sometimes referred to as an H-ring, for which $[X, \mathrm{Fred}]$ inherits a ring structure. Note that we are always using unbased homotopy classes of maps unless specified otherwise.

The second operation on Fred is a little harder to describe, and so we will stick to the simpler case of defining an operation on Fred_0. As argued in Section 4.1 of [MS03], given $S, T \in \mathrm{Fred}$ we may form the tensor product operator $S \otimes I + I \otimes T$, which is a Fredholm operator on $\mathcal{H} \otimes \mathcal{H}$ with index $\dim\ker(S)\dim\ker(T) - \dim\ker(S^*)\dim\ker(T^*)$. Hence if $S, T \in \mathrm{Fred}_0$, the tensor product will be a Fredholm operator of index 0. Then choosing an isometry $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$ we obtain a product map on Fred_0. It is straightforward to show that composition and tensor product are compatible, giving the following result.

Proposition 1.1.28. *The space* Fred_0 *equipped with composition and tensor product as described above is an H-ring, i.e.* $[X, \mathrm{Fred}_0]$ *inherits a ring structure.*

As one may expect, Fred also has an H-ring structure but the tensor product operation is more difficult to describe. It requires using a $\mathbb{Z}_2$-graded Hilbert space, and is explained in detail in [Jän65]. This is precisely the H-ring structure which allows us to describe topological K-theory using Fredholm operators.

Theorem 1.1.29 ([AH59, Jän65])**.** *For X a compact Hausdorff space, the ring $K^0(X)$ is isomorphic to* $[X, \mathrm{Fred}]$ *with operations as described above. The reduced K-theory ring* $\widetilde{K}^0(X)$ *is isomorphic to the ring* $[X, \mathrm{Fred}_0]$.

We may also define K-theory for locally compact spaces as in Definition 1.1.13, and define the relative and higher K-theory groups in the exact same way as in Definitions 1.1.14 and 1.1.16. From these definitions we are able to obtain an alternative characterisation of the higher K-theory groups, which requires the notion of based loop space.

Definition 1.1.30. Let (X, x_0) be a pointed space. The *based loop space* ΩX of X is the space of based loops in X, i.e. the set of continuous maps $f : S^1 \to X$ such that $f(1) = x_0$ equipped with the compact-open topology, and with binary operation given by concatenation of loops.

Proposition 1.1.31. *When X and Y are Hausdorff spaces, there is an isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$. When Y is an H-ring, ΩY inherits the H-ring structure and this becomes an isomorphism of rings. In particular, for compact Hausdorff X there is an isomorphism of rings $K^{-n}(X) \cong [X, \Omega^n \operatorname{Fred}]$.*

Thus Bott periodicity from this perspective states that there is a natural isomorphism $[X, \Omega^2 \operatorname{Fred}] \to [X, \operatorname{Fred}]$. With sufficient background on the operations turning Fred into an H-ring allowing topological K-theory to be described using Fredholm operators, we move on to our final characterisation of K-theory.

1.1.3 Classifying spaces

The final picture of topological K-theory that we will present is perhaps the most inspired by homotopy theory, as it will allow us to define the K-theory spectrum. This will require a basic understanding of classifying spaces and hence principal bundles, which will also be important when considering twists of K-theory, and so we briefly introduce these here.

To begin, we recall the definitions of fibre bundles and principal bundles, and discuss constructions allowing one to move between these two related notions. Fibre bundles are a generalisation of vector bundles in which the restriction that the fibres must be isomorphic to complex vector spaces is relaxed, and they are defined formally as follows.

Definition 1.1.32. Let F be a topological space. A *fibre bundle* with fibre F over a topological space X is a topological space $\mathcal{E}$ and a continuous surjective map $\pi : \mathcal{E} \to X$ such that

- $\pi^{-1}(p)$ is isomorphic to F for all $p \in X$;
- there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for every $\alpha \in A$ there exists a homeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ making the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times F \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi_{U_\alpha}} \\ & & U_\alpha, \end{array}$$

and Φ_α restricts to an isomorphism $\pi^{-1}(\{x\}) \cong F$ for each $x \in U_\alpha$.

A fibre bundle $\mathcal{E}$ over X with fibre F and projection π is often denoted by $F \to \mathcal{E} \xrightarrow{\pi} X$, and the elements of $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$ are called local trivialisations of the fibre bundle.

It is clear that a vector bundle over X is a fibre bundle with fibre $\mathbb{C}^k$, where we add the requirement that the Φ_α restrict to vector space isomorphisms on the fibres. There is an additional notion of the transition functions and structure group of a fibre bundle which roughly describe how the local trivialisations patch together to form the bundle.

Proposition 1.1.33 (Adapted from Lemma 10.5 [Lee03]). *Let $F \to \mathcal{E} \xrightarrow{\pi} X$ be a fibre bundle with local trivialisations $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$. Then there are continuous maps on the overlaps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Homeo(F)$ called transition functions such that the map*

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

is given by $(x, f) \mapsto (x, g_{\alpha\beta}(x)(f))$.

Definition 1.1.34. Let G be a topological group and suppose that G acts continuously and effectively on the topological space F. Equivalently, suppose that G is isomorphic to a subgroup of Homeo(F). Then the fibre bundle $F \to \mathcal{E} \xrightarrow{\pi} X$ has *structure group* G if there exist local trivialisations $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$ such that the transition functions $g_{\alpha\beta}$ defined in Proposition 1.1.33 land in G, i.e. $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ for all $\alpha, \beta \in A$. If there exists a subgroup $H < G$ satisfying the same condition, then we say that the structure group may be *reduced* to H.

For example, a vector bundle of rank k is a fibre bundle with structure group $GL(k, \mathbb{C})$, and under certain conditions this structure group may be reduced to $SL(k, \mathbb{C})$, $U(k)$ or $SU(k)$. We are now able to introduce principal bundles, which are a type of fibre bundle with some additional constraints.

Definition 1.1.35. Let $G \to P \xrightarrow{\pi} X$ be a fibre bundle with structure group G such that G has a free right action on P, and let G act on X via the trivial action. Given $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in A}$ local trivialisations, suppose that the map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is G-equivariant, i.e. $\Phi_\alpha(p \cdot g) = \Phi_\alpha(p) \cdot g$ for all $p \in \pi^{-1}(U_\alpha)$ and $g \in G$. Then this fibre bundle is a *principal G-bundle* over X, commonly denoted by

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow{\scriptstyle \pi} \\ & & X. \end{array}$$

As discussed, we wish to be able to pass between fibre bundles and principal bundles. This will allow us to find an alternative characterisation of K-theory by using principal G-bundles for some appropriate G to replace vector bundles. To construct a fibre bundle with fibre F from a principal G-bundle with G acting on F, we proceed as follows.

Proposition 1.1.36 (Proposition 2.1 [Wal04]). *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a topological space X, and G a topological group acting effectively on a topological space F. Suppose that there is a collection of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that $g_{\alpha\gamma}(p) = g_{\beta\gamma}(p)g_{\alpha\beta}(p)$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$ and $\alpha, \beta, \gamma \in A$. Then there exists a fibre bundle $F \to \mathcal{E} \xrightarrow{\pi} X$ with structure group G. Furthermore, if $F = G$ and G is acting on itself by left translation then this is a principal G-bundle.*

We have seen that a fibre bundle over X with fibre F and structure group G provides the collection of maps required in the proposition, and therefore we can construct a principal G-bundle over X from this data.

Conversely, suppose that we started with $\pi_P : P \to X$ a principal G-bundle, and F a topological space on which G acts effectively on the left. Define an equivalence relation on the product space $P \times F$ by $(p, f) \sim (pg, g^{-1}f)$ for all $p \in P$, $f \in F$ and $g \in G$. We denote the quotient of $P \times F$ under this relation by $P \times_G F$, and equip it with the projection map $\pi : P \times_G F \to X$ given by $\pi([p, f]) = \pi_P(p)$.

Theorem 1.1.37. *Given $\pi_P : P \to X$ a principal G-bundle and F a topological space on which G acts effectively on the left, then $F \to P \times_G F \xrightarrow{\pi} X$ as defined above is indeed a fibre bundle with structure group G. Moreover, applying the construction given in Proposition 1.1.36 to this fibre bundle returns the original principal G-bundle. The converse is also true; using Proposition 1.1.36 to construct a principal G-bundle from a fibre bundle over X with fibre F and structure group G, and then forming the associated bundle as above, yields the original fibre bundle.*

Thus we can conclude that for a fixed F, there is a bijective correspondence between fibre bundles over X with fibre F and structure group G, and principal G-bundles over X, where G acts effectively on F. In particular, there is a bijective correspondence between vector bundles of rank k over X and principal $GL(k, \mathbb{C})$-bundles over X. Furthermore, by noting that the principal $GL(k, \mathbb{C})$-bundle associated to a vector bundle is the frame bundle and that all complex vector bundles admit a hermitian structure e.g. as in Proposition 4.1.4 of [Huy05], we see that the structure group of the frame bundle can be reduced to $U(k)$ and hence the correspondence is actually with principal $U(k)$-bundles. Thus we want to be able to classify all principal $U(k)$-bundles over a topological space.

Given this motivation, we now move to defining the classifying space of a topological group G. Given a continuous map $\phi : X \to Y$ between topological spaces and a principal G-bundle P over Y, we can form the pullback bundle ϕ^*P over X in the same way as in Definition 1.1.3, where G acts on $(x, p) \in X \times P$ via $(x, p) \cdot g = (x, p \cdot g)$. We also have a notion of isomorphism of principal G-bundles, where two G-bundles are isomorphic if and only if there is a G-equivariant homeomorphism between them. We denote the set of isomorphism classes of principal G-bundles over X by P_GX.

Definition 1.1.38. A *universal principal G-bundle* is a principal G-bundle

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array}$$

with contractible total space EG such that for every paracompact Hausdorff space X, the map $[X, BG] \to P_G X$ sending the class of $f : X \to BG$ to f^*EG is an isomorphism. The space BG is known as a *classifying space* for G.

Observe that in particular, every CW-complex is a paracompact Hausdorff space and so this definition will hold for all of the spaces that we will be concerned with. This definition motivates the terminology "classifying space", since maps from X into BG up to homotopy classify the principal G-bundles over X up to isomorphism. Observe that the classifying space BG is only defined up to homotopy equivalence, and so a particular choice of space will be referred to as a model for BG. Of course, we want to ensure that such a classifying space exists for any topological group, and this is shown by Milnor in his original paper.

Theorem 1.1.39 (Theorem 5.2 [Mil56])**.** *Let G be a topological group. Then there exists a classifying space for G.*

So we are particularly interested in the space $BU(k)$ which can be used to classify all complex vector bundles of rank k over a space, and whether we can form some large space enveloping $BU(k)$ for all $k \in \mathbb{N}$ which will allow us to classify all complex vector bundles over a space. Section I.7 of [Kar05] shows that there is a natural embedding $BU(k) \to BU(k+1)$ for all k, and thus we may define the space BU to be the inductive limit of the directed system $BU(1) \to \cdots \to BU(k) \to \cdots$. For more detail on direct limits, see [Wei94]. It will suffice for our purposes to view the direct limit of a directed system of objects to be a large enveloping object containing all of the smaller objects embedded inside it.

As expected, the space BU is related to topological K-theory and forms a classifying space for the even-degree groups with the following slight modification. For more details on this, see Section II.1 of [Kar05].

Theorem 1.1.40. *For every compact Hausdorff space X there is a natural isomorphism $K^0(X) \cong [X, \mathbb{Z} \times BU]$.*

Thus it is said that $\mathbb{Z} \times BU$ is a classifying space for $K^0(X)$. A natural question that arises when viewing K-theory as a generalised cohomology theory is whether there exists a sequence of topological spaces $\{K_n\}_{n\in\mathbb{N}}$ such that $K^{-n}(X) = [X, K_n]$ for all topological spaces X. Such spaces could be viewed as classifying spaces for K-theory, and the sequence $\{K_n\}_{n\in\mathbb{N}}$ is known as the spectrum of a cohomology theory.

Using this theorem, we are able to obtain an alternative characterisation for the first-order K-theory group and hence obtain a model for the K-theory spectrum. Let U denote the inductive limit of the directed system $U(1) \to \cdots \to U(k) \to \cdots$ with maps $U(k) \hookrightarrow U(k+1)$ given by $A \mapsto \operatorname{diag}(A, 1)$.

Theorem 1.1.41. *For every compact Hausdorff space X there is a natural isomorphism $K^{-1}(X) \cong [X, U]$.*

This result follows because $\Omega BG \cong G$, as shown e.g. in Proposition 4.66 of [Hat00]. This provides an affirmative answer to our previous question: the 2-periodic sequence $\mathbb{Z} \times BU, U, \cdots$ forms the K-theory spectrum.

Finally, since these characterisations of K-theory are well-defined for more general spaces than compact Hausdorff X, this allows our previous notion of K-theory to be extended to what is known as representable K-theory for more general topological spaces.

Definition 1.1.42. Let X be any topological space. We define the *representable K-theory* of X to be $RK^0(X) = [X, \mathbb{Z} \times BU]$ and $RK^{-1}(X) = [X, U]$.

Given these three approaches to defining topological K-theory, each of which will be useful in different contexts, we are prepared to move on to a related variant of K-theory.

1.2 Operator algebraic K-theory

Operator algebraic K-theory is interesting in its own right as a variant of algebraic K-theory which can be viewed as a noncommutative generalisation of topological K-theory, but in particular we can use it to introduce higher twisted K-theory in an intuitive geometrical fashion. Here we briefly introduce the notion of a C^*-algebra and discuss the K-theory of these C^*-algebras from two differing but useful viewpoints. A more detailed account of operator algebraic K-theory may be found in a standard reference such as [WO93] or [Bla86], and in particular the setting that we will use in Subsection 1.2.3 can be found specifically in Part III of [WO93].

1.2.1 C^*-algebras

Firstly, we recall the definition of an algebra.

Definition 1.2.1. An *algebra* over a field K is a vector space A over K with a multiplication operation $A \times A \to A$ satisfying left and right distributivity and compatibility with scalar multiplication, i.e. $(ax) \cdot (by) = (ab)(x \cdot y)$ for all $a, b \in K$ and $x, y \in A$.

Example 1.2.1. As a motivating example, we let X be a locally compact Hausdorff space and denote by $C_0(X)$ the space of continuous complex-valued functions on X which vanish at infinity. Here, we say that a function vanishes at infinity if and only if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $|f(x)| < \epsilon$ for all $x \in X \setminus K_\epsilon$. It is clear that $C_0(X)$ forms a complex algebra, because functions vanishing at infinity can be added together, multiplied together and multiplied by scalars to return a function vanishing at infinity. We wish to define a specific class of complex algebra of which $C_0(X)$ is a member, and so we collect some of its important properties here. Firstly, $C_0(X)$ can be equipped with the supremum norm and it is complete with respect to this submultiplicative norm. An algebra satisfying these properties is called a Banach algebra, similar to a Banach space being a vector space equipped with a norm with respect to which it is compete. Secondly, $C_0(X)$ comes with a natural antilinear involution map, which is the operation of conjugation $^* : C_0(X) \to C_0(X)$. Finally, combining the involution with the norm we see that $\|f^*f\| = \|f\|^2$ for all $f \in C_0(X)$. This identity is suggestively termed the C^*-identity.

Definition 1.2.2. A C^*-algebra is a complex Banach algebra A with an antilinear involution $^* : A \to A$ satisfying the C^*-identity.

It is clear that $C_0(X)$ forms a C^*-algebra from our motivating example. Similarly, the space of bounded linear operators on a Hilbert space is easily verified to be a C^*-algebra. A C^*-algebra is called unital when it contains an identity element with respect to multiplication. In the case that X is compact, it can be seen that the condition of vanishing at infinity is trivially true for all functions – take $K = X$ to be the compact set outside of which the function vanishes. Thus $C_0(X)$ is equal to $C(X)$ and is unital if and only if X is compact. Unital C^*-algebras turn out to be easier to work with than non-unital C^*-algebras in the same way that compact spaces are nicer than locally compact spaces, and so we present a way of embedding a non-unital C^*-algebra in a unital one analogous to the compactification of a locally compact space.

Definition 1.2.3. Let A be a non-unital C^*-algebra. The *unitisation* A^+ of A is the set $A \times \mathbb{C}$ equipped with pointwise sum and involution, and with multiplication defined by $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$.

It is routine to show that the operator norm $\|(a, \lambda)\| = \|a + \lambda\|_{\mathcal{B}(A)}$ turns A^+ into a C^*-algebra, where $a + \lambda$ is viewed as an operator on the Banach space A by left multiplication. This is also a functorial construction, where a morphism $\varphi : A \to B$ induces a unital morphism $\varphi^+ : A^+ \to B^+$ sending (a, λ) to $(\varphi(a), \lambda)$. We henceforth use the notation A^+ to denote the unitisation as defined above if A is a non-unital algebra, or simply to refer to the direct sum C^*-algebra $A \oplus \mathbb{C}$ with pointwise operations and the maximum norm when A has a unit.

The unitisation defined in Definition 1.2.3 is the smallest unitisation of A in that any other unital C^*-algebra in which A may be embedded as an essential ideal must contain A^+. There is also a largest unitisation of A containing A as an essential ideal called the multiplier algebra $\mathcal{M}(A)$ which we will not define (see 2.2 of [WO93]), but which will be important to us later. The quotient $\mathcal{M}(A)/A$ is known as the corona algebra of A.

We finish this brief introduction to C^*-algebras with a classical result of Gelfand and Naimark, which shows that the examples that we have considered essentially account for all C^*-algebras.

Theorem 1.2.4 (Gelfand–Naimark. Theorem II.2.2.4 and Corollary II.6.4.10 [Bla06])**.** *Every commutative C^*-algebra is isometrically *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. Moreover, X is homeomorphic to Y if and only if $C_0(X)$ is *-isomorphic to $C_0(Y)$. More generally, every C^*-algebra is isometrically *-isomorphic to a norm-closed subalgebra of the space of bounded operators on a separable Hilbert space.*

This result explains why C^*-algebras may be viewed as a noncommutative generalisation of topological spaces, as we have an equivalence of categories between locally compact Hausdorff spaces and commutative C^*-algebras but more generally there exist noncommutative C^*-algebras which do not correspond to topological spaces.

1.2.2 Projections

Based on the link between commutative C^*-algebras and locally compact Hausdorff spaces provided by Gelfand and Naimark's theorem, we aim to define a notion of K-theory for C^*-algebras such that the topological K-theory of a space X is equal to the operator algebraic K-theory of $C_0(X)$. One such approach to defining operator algebraic K-theory is similar to the way in which topological K-theory is defined using vector bundles: a monoid structure can be defined using projection matrices over a C^*-algebra, and the Grothendieck group can be used to define the K_0-group.

Definition 1.2.5. A *projection* in a C^*-algebra A is an element $p \in A$ which satisfies $p^2 = p^* = p$. An element $v \in A$ is a *partial isometry* if v^*v is a projection. We say that two projections $p, q \in A$ are *equivalent* when there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$; *unitarily equivalent* when $p = u^*qu$ for a unitary element $u \in A$ when A is unital or $u \in A^+$ when A is non-unital; or *homotopic* when p and q are connected by a continuous path of projections in A.

These are all sensible notions of equivalence of projections, but unfortunately they are not all equivalent in general. We can, however, pass from the C^*-algebra A to the infinite matrix algebra over A, in which case these notions do coincide. We let $M_n(A)$ denote the C^*-algebra of $n \times n$ matrices over A equipped with pointwise addition and matrix multiplication along with the operator norm, and then let $M_\infty(A)$ denote the inductive limit of this directed system where we embed $M_n(A)$ into $M_{n+1}(A)$ via $T \mapsto \mathrm{diag}(T, 0)$. Note that $M_\infty(A)$ forms only a pre C^*-algebra, i.e. it satisfies all of the conditions in Definition 1.2.2 except norm-completeness, but the notions of equivalent projections in Definition 1.2.5 carry over to this setting.

Lemma 1.2.6 (Section 5.2 [WO93])**.** *The three notions of equivalence of projections introduced in Definition 1.2.5 are equivalent in $M_\infty(A)$.*

This provides us with a monoid structure with which we can introduce K-theory.

Proposition 1.2.7. *The set of equivalence classes of projections in $M_\infty(A)$, denoted $P(A)$, forms an abelian monoid when equipped with the operation sending the sum of the classes of p and q to the class of the diagonal matrix $diag(p, q)$. Given a morphism $\varphi : A \to B$, there is an induced map $\varphi_* : P(A) \to P(B)$ defined by $\varphi_*([a_{ij}]) = [\varphi(a_{ij})]$.*

This shows that we have introduced a covariant functor from the category of C^*-algebras to the category of abelian monoids. We now complete this to the K-theory group.

Definition 1.2.8. The K-theory of a unital C^*-algebra A, denoted $K_0(A)$, is defined to be the Grothendieck group of the abelian monoid $P(A)$.

Remark 1.2.1. The subscript notation K_0 is used for operator algebraic K-theory since it forms a covariant functor whereas topological K-theory formed a contravariant functor.

In order to define K-theory for C^*-algebras in general, we must use the unitisation in the same way that compactification is used to define topological K-theory for locally compact spaces. We also need the following basic example.

Example 1.2.2. One of the simplest C^*-algebras is the algebra of complex numbers. Any element in $P(\mathbb{C})$ can be represented by some finite-dimensional complex projection matrix, and these projections are equivalent if and only if their ranges have the same dimension. Thus $P(\mathbb{C})$ is isomorphic to $\mathbb{N}$ as an abelian monoid, and so we may conclude that $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Definition 1.2.9. Let A be a C^*-algebra and $\pi : A^+ \to \mathbb{C}$ projection. The K-theory $K_0(A)$ is defined to be the kernel of the induced map $\pi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$.

If A is unital then this definition agrees with the previous definition of $K_0(A)$, so we have consistency. We must also define the higher operator algebraic K-theory groups. There is a way to do this using invertible or unitary elements in the infinite matrix algebra over A, but we will not need this construction. The interested reader may see Chapter 7 of [WO93]. It will suffice for our purposes to define the higher groups as follows.

Definition 1.2.10. The higher K-theory groups of any C^*-algebra A are defined via $K_n(A) = K_0(S^nA)$, where

$$SA = \{f : S^1 \to A \text{ continuous} : f(1) = 0\}$$

denotes the *suspension* of A equipped with the supremum norm and $S^nA = S^{n-1}SA$.

Finally, these definitions unify topological K-theory with operator algebraic K-theory as motivated at the beginning of the section. The following more general theorem is due to Serre and Swan in the case $n = 0$ [Swa62].

Theorem 1.2.11. *Let X be a locally compact Hausdorff space. There are isomorphisms $K^n(X) \cong K_n(C_0(X))$ for all n.*

We finish this section with some important results that we will require later, beginning with the stability of K-theory.

Proposition 1.2.12 (Corollaries 6.2.11 and 7.1.9 [WO93])**.** *For any C^*-algebra A, there are isomorphisms $K_n(A) \cong K_n(A \otimes \mathcal{K})$ for all n where $\mathcal{K}$ denotes the compact operators on an infinite-dimensional seperable Hilbert space.*

We also have a six-term exact sequence in operator algebraic K-theory, which will be useful for computing higher twisted K-theory groups.

Theorem 1.2.13 (Theorem 9.3.2 [WO93])**.** *Let*

$$0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$$

be a short exact sequence of C^*-algebras. Then there are group homomorphisms known as boundary maps $\partial_0 : K_0(A/J) \to K_1(J)$ and $\partial_1 : K_1(A/J) \to K_0(J)$ making the following sequence exact:

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/J) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_0} \\ K_1(A/J) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(J). \end{array}$$

1.2.3 Hilbert C^*-modules

There is a less conventional approach to defining C^*-algebraic K-theory, which is closer to the definition of topological K-theory using Fredholm operators presented in Subsection 1.1.2. This will be equally useful for our purposes, as it will lead to an alternative formulation of higher twisted K-theory. In order to take this approach, we must introduce the notion of a Hilbert C^*-module. These provide a generalisation of the standard notion of Hilbert space, where the complex-valued inner product on the Hilbert space is replaced by one which takes values in some C^*-algebra. We have seen that Fredholm operators on Hilbert spaces play a crucial role in a more analytical formulation of topological K-theory, and so we introduce the analogous notion of Fredholm operators on Hilbert C^*-modules, and these turn out to be relevant in a more topological formulation of higher twisted K-theory. We begin by formalising the definition of Hilbert C^*-module.

Definition 1.2.14. Let A be a C^*-algebra. A right A-module $\mathcal{H}$ is said to be a *pre Hilbert A-module* if there exists a map $\langle - \,|{-}\rangle : \mathcal{H} \times \mathcal{H} \to A$ that is sesquilinear, positive definite and respects the module action, i.e.

(i) $\langle x \,| y_1 + y_2 \rangle = \langle x \,| y_1 \rangle + \langle x \,| y_2 \rangle$ for $x, y_1, y_2 \in \mathcal{H}$;

(ii) $\langle x \,| ya \rangle = \langle x \,| y \rangle \, a$ for $x, y \in \mathcal{H}$ and $a \in A$;

(iii) $\langle x \,| zy \rangle = z \langle x \,| y \rangle$ for $x, y \in \mathcal{H}$ and $z \in \mathbb{C}$;

(iv) $\langle x \,| y \rangle = \langle y \,| x \rangle^*$ for $x, y \in \mathcal{H}$;

(v) $\langle x \,| x \rangle \geq 0$ for $x \in \mathcal{H}$ and $\langle x \,| x \rangle = 0 \iff x = 0$.

Then $\|x\| = \sqrt{\|\langle x \,| x \rangle\|_A}$ for $x \in \mathcal{H}$ defines a norm on $\mathcal{H}$. If a pre Hilbert A-module is complete with respect to this norm then it is said to be a *Hilbert A-module.*

There is one specific Hilbert C^*-module which will be of interest to us.

Example 1.2.3. One of the most important Hilbert spaces is ℓ^2, so we seek a generalisation of this to the Hilbert C^*-module setting. For any C^*-algebra A we define the *standard Hilbert A-module* to be

$$\mathcal{H}_A = \left\{ (a_i) \in \prod_{i=1}^{\infty} A : \sum_i a_i^* a_i \text{ converges in norm in } A \right\}.$$

The A-valued inner product is defined by

$$\langle (a_i) \,| (b_j) \rangle = \sum_{i=1}^{\infty} a_i^* b_i$$

and hence the norm is defined by

$$\|(a_i)\| = \sqrt{\left\| \sum_{i=1}^{\infty} a_i^* a_i \right\|_A}.$$

Then it is not difficult to see that conditions (i) to (v) in Definition 1.2.14 are satisfied and that $\mathcal{H}_A$ is complete with respect to this norm (see the section on $\mathcal{H}_A$ in Examples 15.1.7 of [WO93] for details). Note also that $\mathcal{H}_{\mathbb{C}} \cong \ell_2$, and so this example does generalise ℓ_2.

In order to define a notion of Fredholm operator on a Hilbert C^*-module, we require analogues of the bounded and compact operators. It turns out that linearity and boundedness are not the natural assumptions to place when dealing with Hilbert modules, and we instead consider adjointability.

Definition 1.2.15. Let $\mathcal{H}$ be a Hilbert A-module. A map $T : \mathcal{H} \to \mathcal{H}$ is said to be *adjointable* if there exists a map $T^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle x \,|Ty\rangle = \langle T^*x \,|y\rangle$$

for all $x, y \in \mathcal{H}$. We denote the set of all adjointable maps on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Unlike when dealing with the bounded linear operators on a Hilbert space, adjointability is not guaranteed in this setting and so this notion does form an interesting class of operators. We also define a subclass of compact adjointable operators in a similar vein to defining the compact operators on a Hilbert space to be the norm-closure of the finite-rank operators.

Definition 1.2.16. For $x, y \in \mathcal{H}$, let $\theta_{x,y} : \mathcal{H} \to \mathcal{H}$ be defined by $\theta_{x,y}(z) = x \langle y \,|z\rangle$, and let $\Theta = \{\theta_{x,y} : x, y \in \mathcal{H}\}$. The set of *compact adjointable operators* on $\mathcal{H}$ is the closed subspace of $\mathcal{B}(\mathcal{H})$ generated by the $\theta_{x,y}$, i.e. $\mathcal{K}(\mathcal{H}) = \overline{\mathrm{Span}\,\Theta}$.

As is true for Hilbert spaces, we are able to quotient $\mathcal{B}(\mathcal{H})$ by $\mathcal{K}(\mathcal{H})$ as a result of the following lemma.

Lemma 1.2.17 (Proposition 15.2.4 and Corollary 15.2.10 [WO93])**.** *Both the adjointable and the compact adjointable operators form C^*-algebras, and $\mathcal{K}(\mathcal{H})$ is an essential ideal in $\mathcal{B}(\mathcal{H})$.*

In the case of the standard Hilbert A-module introduced in Example 1.2.3, it is shown in Examples 15.2.11 of [WO93] that $\mathcal{B}(\mathcal{H}_A) \cong \mathcal{M}(A \otimes \mathcal{K})$ and $\mathcal{K}(\mathcal{H}_A) \cong A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the compact operators on some infinite-dimensional separable Hilbert space. Additional background on the properties of these operators may be obtained from Chapter 15 of [WO93] but we are now in a position to define Fredholm operators.

Definition 1.2.18. An adjointable operator F on a Hilbert A-module $\mathcal{H}$ is said to be a *Fredholm operator* if $\pi(F)$ is invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ where π is projection to the quotient. In the case that $\mathcal{H}_A$ is the standard Hilbert A-module, the set of all Fredholm operators on $\mathcal{H}_A$ is denoted Fred_A.

There is an equivalent Atkinson-style definition for these Fredholm operators in terms of kernels and cokernels, but this will not be relevant for us. We are now able to reformulate the definition of K-theory for unital C^*-algebras.

Theorem 1.2.19 (Theorem 17.3.11 [WO93])**.** *Let A be a unital C^*-algebra. Then there is an isomorphism $K_0(A) \cong \pi_0(\mathrm{Fred}_A)$.*

This theorem is the subject of Chapter 17 of [WO93], and it is proved by constructing an index map which generalises the index $[pt, \mathrm{Fred}] \to K^0(pt)$ introduced earlier to a map

$[pt, \mathrm{Fred}_A] \to K_0(A)$. Recalling the result of Serre and Swan in Theorem 1.2.11, we see that $K^0(pt) = K_0(C(pt)) = K_0(\mathbb{C})$. We also know that $\mathcal{H}_{\mathbb{C}} \cong \ell^2$, meaning that $\mathrm{Fred}_{\mathbb{C}}$ can be identified with the ordinary Fredholm operators on a Hilbert space, and hence the index map $[pt, \mathrm{Fred}] \to K^0(pt)$ can be viewed as a map $[pt, \mathrm{Fred}_{\mathbb{C}}] \to K_0(\mathbb{C})$.

Finally, while [WO93] and the original reference [Min87] that he cites only present this result for unital C^*-algebras, it is possible to generalise this definition to non-unital C^*-algebras as in [Bla86].

Theorem 1.2.20 (Corollary 12.2.3 and 12.2.4 [Bla86])**.** *For any C^*-algebra A there is an isomorphism $K_0(A) \cong \pi_0(\mathrm{Fred}_A)$.*

Since the C^*-algebras that we will be dealing with are non-unital, we will need to rely on this version of the theorem.

This concludes our introduction to topological and operator algebraic K-theory, with which the reader should be well-equipped to understand the remainder of this thesis.

Chapter 2

Higher twisted K-theory

To every multiplicative generalised cohomology theory there is an associated homotopy-theoretic notion of twist and a corresponding twisted cohomology theory. In this way, higher twisted K-theory is the twisted cohomology theory of topological K-theory. This definition, however, is far too abstract to work with for the purposes of computation and potential applications to physics. For this reason, a limited set of twists for which geometric representatives were known were studied extensively for a great deal of time. Many great mathematicians acknowledged the existence of more general twists, but since no geometric interpretation was known at the time, nobody was able to do any work in the more general setting. This finally changed with several papers by Marius Dadarlat and Ulrich Pennig [DP16, DP15a, DP15b], culminating in work by Pennig in defining higher twisted K-theory in [Pen15]. In this chapter, we will introduce the abstract notion of twist, and give a brief history of twisted K-theory. We will then provide background on the Cuntz algebra $\mathcal{O}_\infty$ which is central to the development of higher twisted K-theory, formulate higher twisted K-theory in the same way as Pennig in [Pen15] and provide some of his fundamental results. To conclude, we will provide an alternative, more topological formulation of higher twisted K-theory à la Rosenberg [Ros89], and discuss the potential for applications of this area to physics.

2.1 Introduction

We begin with a brief non-technical introduction to twists of cohomology theories. The definitions presented here will not be motivated, as the motivation comes from deep within homotopy theory, but the knowledgeable reader may see a standard reference such as [MS06b, Dou05] for details.

As discussed at the end of Subsection 1.1.3, there is a notion of spectrum for a cohomology theory, which is a sequence of topological spaces $\{E_n\}_{n\in\mathbb{N}}$ satisfying particular properties such that $h^n(X) = [X, E_n]$ for a cohomology theory $h^\bullet$. In the case of topological K-theory, this is the 2-periodic sequence $\mathbb{Z} \times BU, U, \cdots$. To each spectrum there is an associated unit spectrum consisting of the unital elements in each space, which we will denote $\{GL_1(E_n)\}_{n\in\mathbb{N}}$, and from this another cohomology theory denoted $gl_1(h)^\bullet$ may be defined via $gl_1(h)^n(X) = [X, GL_1(E_n)]$. The notation $gl_1(h)$ here reflects that this cohomology theory is in some sense constructed out of the unital elements of the cohomology theory $h^\bullet$. Then the twists of $h^\bullet$ over some space X are classified by the first group of this cohomology theory, i.e. $gl_1(h)^1(X)$. As $GL_1(E_0)$ denotes the units in E_0, we see that the twists are classified by $[X, BGL_1(E_0)]$. To be slightly more technical, a twisting of a cohomology theory over a space X is defined to be a bundle of spectra over X with fibre given by the spectrum R of the cohomology theory. Then letting $GL_1(R)$ denote the automorphism group of the spectrum R, these bundles of spectra are classified by $[X, BGL_1(R)]$. One may then define the groups of the twisted cohomology theory, but we will not present this level of detail in generality.

While these notions are all very general, we are only interested in applying them to topological K-theory. Recalling the vector bundle formulation of K-theory, we see that the invertible elements of the ring $K^0(X)$ are represented by virtual line bundles. Unifying this with the spectrum picture, these classes correspond to homotopy classes $[X, \mathbb{Z}_2 \times BU]$, and so in the notation of the previous paragraph we have $GL_1(\mathbb{Z} \times BU) = \mathbb{Z}_2 \times BU$. Thus the twists of topological K-theory over X are classified by

$$gl_1(KU)^1(X) = [X, B(\mathbb{Z}_2 \times BU)].$$

Taking this slightly further, it has been shown that BU is homotopy equivalent to $K(\mathbb{Z}, 2) \times BSU$ [MST77] and hence $B(\mathbb{Z}_2 \times BU) \simeq K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}, 3) \times BBSU$. This means that twists of K-theory are classified by homotopy classes of maps

$$X \to K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}, 3) \times BBSU,$$

and therefore for a compact space X the twists of K-theory correspond to elements of $H^1(X, \mathbb{Z}_2)$, $H^3(X, \mathbb{Z})$ and $[X, BBSU]$. The third of these groups is not well-understood, which led to the lack of understanding of this class of twists for K-theory. Furthermore, while this may be used to define what a twist of K-theory is, it does not give any clear picture of how to view a twist. In particular, this approach via stable homotopy theory does not provide any geometric information about twists, rendering any results and computations specific to twisted K-theory very difficult to prove and perform. In spite of this, while this will not be a focus of this thesis, further investigation into the topology of $BBSU$ – a topic which is interesting in its own right – may prove useful in the study of higher twisted K-theory.

Rather than extracting geometric information from the homotopy theory, however, our approach will be to formulate an appropriate geometric object which could be shown to classify the twists of K-theory. To be clear, the type of geometric object that we wish to associate to a twist of K-theory over a space X is some variety of bundle over X. By finding an appropriate fibre for a bundle over X, we may make the classification of twists and the definition of twisted K-theory much simpler.

At this point, we will give a brief overview of the history of twisted K-theory. Before the stable homotopy approach was well-known, mathematicians had already considered natural geometric objects with which topological K-theory could be modified or "twisted". What has classically been called twisted K-theory is a generalisation of topological K-theory which gradually emerged over the course of the 1960s after Atiyah and Hirzebruch's initial work in topological K-theory. Interest in the area was sparked when Atiyah, Bott and Shapiro investigated the relationship between topological K-theory and Clifford algebras, providing an isomorphism between the K-theory ring of a point and a space defined in terms of Clifford modules [ABS64]. This provided a new perspective from which K-theory could be viewed, which was studied in depth in Karoubi's doctoral thesis [Kar68] using Clifford bundles associated to vector bundles. Collaboration between Karoubi and Donovan in 1970 then extended this work, proving that algebra bundles could be used in place of Clifford bundles [DK70]. This resulted in the original definition of what was then called "K-theory with local coefficients" using graded Brauer groups, in which the local coefficient systems over X were classified by $H^1(X, \mathbb{Z}/2)$ and the torsion elements of $H^3(X, \mathbb{Z})$, corresponding to finite-dimensional complex algebra bundles over X whose fibres were isomorphic to complex matrix algebras. This was the first notion of twisted K-theory which was defined, and it was done so geometrically using these algebra bundles to represent twists.

The next major development in the field was by Rosenberg, who presented results in 1988 that this definition could be extended to any class in $H^3(X, \mathbb{Z})$ rather than specifically considering torsion classes [Ros89], corresponding to using infinite-dimensional algebra bundles over X. In particular, the twists of K-theory being considered were shown to be represented by algebra bundles with fibres isomorphic to the algebra of compact operators on an infinite-dimensional, separable complex Hilbert space. After Rosenberg's fundamental work, the next significant contributions came from the Adelaide school, beginning with a paper by Bouwknegt, Carey, Mathai, Murray and Stevenson in which bundle gerbe K-theory was developed, various computations were performed and a twisted Chern character was defined in the even case [BCM$^+$02]. This was followed by two papers by Mathai and Stevenson, the first of which introduces the twisted Chern character in the odd case [MS03], and the second of which studied the Connes–Chern character for twisted K-theory and showed that it agrees with the twisted Chern character [MS06a].

Further developments were later made by Atiyah and Segal in formulating twisted K-theory using Fredholm modules and exploring the differentials in an Atiyah–Hirzebruch spectral sequence for twisted K-theory [AS04, AS06]. It was in the first of these papers that the authors acknowledge the existence of more general twists of K-theory which had not found a geometric realisation, and due to this there was no known way to incorporate them into the existing theory. The authors also show that the geometric twists which are being considered, represented by elements of $H^3(X,\mathbb{Z})$, do form a subset of the full set of twists in the homotopy-theoretic picture. We should also mention that while K-theory can be twisted by elements of $H^1(X,\mathbb{Z}/2)$ as in Donovan and Karoubi's work, these twists were studied comprehensively in [AH04] and are often neglected in more recent work.

While twisted K-theory with twists corresponding to 3-classes continued to flourish, with links drawn to T-duality in string theory by Bouwknegt, Evslin and Mathai [BEM04a, BEM04b] and an important result tying equivariant twisted K-theory to representation theory in a series of papers by Freed, Hopkins and Teleman [FHT11a, FHT13, FHT11b], it was not until a series of papers was published by Dadarlat and Pennig in 2015 and 2016 that a geometric view of the more general twists of K-theory emerged. We will dedicate the next section to introducing the algebra which is central in Dadarlat and Pennig's work, which will culminate in Pennig's formulation of higher twisted K-theory.

2.2 The Cuntz algebra $\mathcal{O}_\infty$

One of the factors that led to the compact operators being useful in representing twists of K-theory was the extensive Dixmier–Douady theory introduced in [DD63] and developed by many authors. In 2005, Toms and Winter introduced a special class of C^*-algebras described as strongly self-absorbing [TW07], and ten years later Dadarlat and Pennig began developing a parallel Dixmier–Douady theory for these algebras [DP16, DP15a, DP15b]. As mentioned, this work culminated in a paper by Pennig introducing the most general class of K-theory twists from a bundle-theoretic point of view, something that had eluded mathematicians for many years. Of particular importance in Pennig's work is the Cuntz algebra $\mathcal{O}_\infty$, which is the main focus of our discussion.

2.2.1 Definitions

We will firstly introduce Toms and Winter's class of strongly self-absorbing C^*-algebras, for which the higher Dixmier–Douady theory was developed and with which higher twisted K-theory can be defined. Apart from having this application to K-theory, this class of algebras is interesting in its own right as it has proved useful in the quest of Elliott to classify all simple nuclear C^*-algebras. Firstly, we remark that a C^*-algebra is referred to as self-absorbing if it is isomorphic to its tensor product with itself. We will be concerned only with nuclear C^*-algebras, i.e. those for which all possible tensor products are equivalent, and so we need not be concerned with choosing a particular tensor product. Based on this definition, an algebra should be strongly self-absorbing if it satisfies a stronger condition than simply being isomorphic to its tensor product with itself. Here we present a slightly modified but equivalent definition posed by Pennig and Dadarlat, which is more applicable to topological problems. For the original definition, see [TW07].

Definition 2.2.1. A seperable and unital C^*-algebra D is called *strongly self-absorbing* if there exists a *-isomorphism $\psi : D \to D \otimes D$ and a path of unitaries $u : [0,1) \to U(D \otimes D)$ such that, for all $d \in D$, $\lim_{t\to 1} \|\psi(d) - u_t(d \otimes 1)u_t^*\| = 0$.

Rather than studying the abstract theory of strongly self-absorbing C^*-algebras, we are more concerned with specific examples of these algebras. The most relevant example for us is the Cuntz algebra $\mathcal{O}_\infty$, first introduced by Cuntz in [Cun77].

Definition 2.2.2. The *Cuntz algebra* $\mathcal{O}_n$ with n generators for $n = 1, 2, \cdots$ is defined to be the C^*-algebra generated by a set of isometries $\{S_i\}_{i=1}^n$ acting on a separable Hilbert space satisfying $S_i^* S_j = \delta_{ij} I$ for $i, j = 1, \cdots, n$ and

$$\sum_{i=1}^{n} S_i S_i^* = I.$$

Similarly, the Cuntz algebra $\mathcal{O}_\infty$ with infinitely many generators is defined in an analogous way for an infinite sequence $\{S_i\}_{i\in\mathbb{N}}$ satisfying $S_i^* S_j = \delta_{ij} I$ for $i, j \in \mathbb{N}$ and

$$\sum_{i=1}^{k} S_i S_i^* \leq I$$

for all $k \in \mathbb{N}$.

Note that it is proved in the original reference that this definition is independent of the choice of Hilbert space and of isometries. For the sake of completeness, we will list various other examples of strongly self-absorbing C^*-algebras without definitions. These algebras are shown to satisfy the definition given above in Toms and Winter's original paper [TW07].

Example 2.2.1.

(1) The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are strongly self-absorbing.

(2) The Jiang-Su algebra $\mathcal{Z}$ introduced by Jiang and Su in [JS99] is strongly-self absorbing.

(3) Uniformly hyperfinite (UHF) algebras (defined in III.5.1 of [Dav91]) of infinite type are strongly-self absorbing.

(4) The tensor product of a UHF algebra of infinite type with $\mathcal{O}_\infty$ is strongly-self absorbing. Note that this is the only way to form a new class of algebras out of the previous examples, as $\mathcal{O}_2$ absorbs UHF algebras of infinite type, all of the examples absorb $\mathcal{Z}$ and $\mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$.

Although all of these algebras can be used to formulate a notion of twisted K-theory as we will see, it is the Cuntz algebra $\mathcal{O}_\infty$ which can be used to realise the most general notion of twist. This is thus the algebra whose properties we will explore in more detail.

2.2.2 Properties

We will firstly give a survey of Dadarlat and Pennig's main results about $\mathcal{O}_\infty$ from [DP16], regarding the homotopy type of its automorphism group and the automorphism group of its stabilisation. We will then explore its automorphisms more explicitly, presenting an action of the stable unitary group on $\mathcal{O}_\infty$ by outer automorphisms.

A great deal of Pennig and Dadarlat's work in [DP16] is about determining the homotopy type of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, and this turns out to be relevant in higher twisted K-theory. It is sensible to firstly consider automorphisms of $\mathcal{O}_\infty$ itself, but the authors show that the automorphism group does not have an interesting homotopy type.

Theorem 2.2.3 (Theorem 2.3 [DP16])**.** *Let D be a strongly self-absorbing C^*-algebra. Then the space* $\mathrm{Aut}(D)$ *is contractible.*

Upon stabilisation, the homotopy type of the automorphism group becomes much more interesting.

Theorem 2.2.4 (Theorem 2.18 [DP16])**.** *There are isomorphisms of groups*

$$\pi_i(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})) \cong \begin{cases} K_0(\mathcal{O}_\infty)_+^\times & \text{if } i = 0; \\ K_i(\mathcal{O}_\infty) & \text{if } i \geq 1; \end{cases}$$
$$= \begin{cases} \mathbb{Z}_2 & \text{if } i = 0; \\ \mathbb{Z} & \text{if } i > 0 \text{ even}; \\ 0 & \text{if } i \text{ odd}. \end{cases}$$

This theorem also gives greater insight into the structure of the automorphism group of $\mathcal{O}_\infty \otimes \mathcal{K}$ through the following corollary.

Corollary 2.2.5 (Corollary 2.19 [DP16])**.** *There is an exact sequence of topological groups*

$$0 \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathcal{K}) \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \to \mathbb{Z}_2 \to 0$$

where $\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathcal{K})$ *denotes the connected component of the identity.*

Furthermore, it is shown in [BKP03] that there is an automorphism α of $\mathcal{O}_\infty \otimes \mathcal{K}$ which has order two and is such that $\alpha_* = -1$ on $K_0(\mathcal{O}_\infty)$, which implies that this sequence is split. Thus we may conclude that

$$\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \cong \mathbb{Z}_2 \times \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathcal{K}).$$

To gain further insight into the automorphisms of $\mathcal{O}_\infty$, in particular the outer automorphisms, we investigate an action of $U(\infty)$ on $\mathcal{O}_\infty$. Note that this is one of the significant differences between the Cuntz algebra and the algebra of compact operators on a Hilbert space which is used in the formulation of classical twisted K-theory – all automorphisms of $\mathcal{K}$ are inner but this is not the case for $\mathcal{O}_\infty$.

We follow the work of Enomoto, Fujii, Takehana and Watatani in describing an action of $U(\infty)$ on $\mathcal{O}_n$. We reproduce their work in detail in order to fill in some details of the proof for $\mathcal{O}_\infty$ and because their paper [EFTW79] is somewhat difficult to locate.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices over $\mathbb{C}$. For any $u = (u_{ij}) \in M_n(\mathbb{C})$, we define a map $\alpha_u : \mathcal{O}_\infty \to \mathcal{O}_\infty$ to act on the generators $S_1, S_2, \cdots$ of $\mathcal{O}_\infty$ by

$$\alpha_u(S_j) = \begin{cases} \sum_{i=1}^{n} u_{ij} S_i & \text{for } j = 1, \cdots, n; \\ S_j & \text{for } j > n; \end{cases}$$

and then extend this map to $\mathcal{O}_\infty$ such that it is a homomorphism. It is clear that $\alpha_u \circ \alpha_{u'} = \alpha_{uu'}$, but for general u this map does not give an automorphism of $\mathcal{O}_\infty$. For example, taking u to be the zero matrix we certainly do not obtain an automorphism, and so we seek a class of matrices for which α_u is an automorphism.

Lemma 2.2.6. *For $u = (u_{ij}) \in M_n(\mathbb{C})$, the map α_u defines an automorphism of $\mathcal{O}_\infty$ if and only if $u \in U(n)$.*

Proof. Let $u \in U(n)$. In order to show that α_u is an automorphism of $\mathcal{O}_\infty$, we must show that the elements $T_i = \alpha_u(S_i)$ are generators of $\mathcal{O}_\infty$, i.e. they are isometries which satisfy the Cuntz relations. For $i > n$ the T_i are clearly isometries, and in the case that $1 \leq i \leq n$ we see that

$$\begin{aligned} T_i^* T_i &= \left(\sum_{k=1}^{n} u_{ki} S_k \right)^* \left(\sum_{l=1}^{n} u_{li} S_l \right) \\ &= \sum_{k,l=1}^{n} u_{ki}^* u_{li} \delta_{kl} \\ &= \sum_{k=1}^{n} u_{ki}^* u_{ki} \\ &= (u^* u)_{ii} \\ &= 1, \end{aligned}$$

and hence T_i is an isometry. A similar calculation shows that the T_i are pairwise orthogonal. Now, in order to prove that the T_i satisfy the Cuntz relations, we need only show that

$$\sum_{i=1}^{n} T_i T_i^* = \sum_{i=1}^{n} S_i S_i^*$$

and the result will follow since the S_i satisfy the Cuntz relations. To see this, we observe that

$$\begin{aligned}
\sum_{i=1}^{n} T_i T_i^* &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} u_{ji} S_j \right) \left(\sum_{k=1}^{n} u_{ki} S_k \right)^* \\
&= \sum_{i,j,k=1}^{n} u_{ji} u_{ki}^* S_j S_k^* \\
&= \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} u_{ji} u_{ki}^* \right) S_j S_k^* \\
&= \sum_{j,k=1}^{n} \delta_{jk} S_j S_k^* \\
&= \sum_{j=1}^{n} S_j S_j^*
\end{aligned}$$

as required. Thus $\{T_1, T_2, \cdots\}$ is a generating set for $\mathcal{O}_\infty$, and so α_u provides an automorphism of $\mathcal{O}_\infty$. To prove the converse, suppose that α_u does define an automorphism on $\mathcal{O}_\infty$. Then

$$\begin{aligned}
\delta_{ij} &= \alpha_u(S_i^* S_j) \\
&= \alpha_u(S_i)^* \alpha_u(S_j) \\
&= \sum_{k,l=1}^{n} u_{ki}^* u_{lj} S_k^* S_l \\
&= \sum_{k=1}^{n} u_{ki}^* u_{kj} \\
&= (u^* u)_{ij}
\end{aligned}$$

and a similar computation shows that $(uu^*)_{ij} = \delta_{ij}$, which implies that $u \in U(n)$ as required. $\square$

This provides an action of $U(n)$ on $\mathcal{O}_\infty$ for all $n = 1, 2, \cdots$ which we will see extends to an action of $U(\infty)$ on $\mathcal{O}_\infty$. Here we are taking $U(\infty)$ to be the algebraic direct limit of $U(n) \overset{\iota}{\hookrightarrow} U(n+1)$ with $\iota(A) = \operatorname{diag}(A, 1)$. Taking $u \in U(\infty)$, there exists a finite representative $\hat{u} \in U(n)$ for some n, and every representative of u will be of the form $\operatorname{diag}(\hat{u}, 1, \cdots)$. Therefore all representatives of u define the same action on $\mathcal{O}_\infty$, so we define the action of u on $\mathcal{O}_\infty$ to be that of its finite representatives. This provides a map $U(\infty) \to \operatorname{Aut}(\mathcal{O}_\infty)$ which satisfies some desirable properties as we will see. In particular, the map α lands in the group $\operatorname{Out}(\mathcal{O}_\infty)$ of outer automorphisms of $\mathcal{O}_\infty$.

Theorem 2.2.7. *The map* $\alpha : U(\infty) \to \mathrm{Out}(\mathcal{O}_\infty)$ *is continuous and injective.*

Proof. We must show that the map $\alpha : U(\infty) \to \mathrm{Aut}(\mathcal{O}_\infty)$ is a continuous, injective group homomorphism such that the only element of $U(\infty)$ whose image is an inner automorphism is the identity, i.e. $(\pi \circ \alpha)(u)$ is not equal to the identity for any $u \in U(\infty)$ except the identity where $\pi : \mathrm{Aut}(\mathcal{O}_\infty) \to \mathrm{Aut}(\mathcal{O}_\infty)/\,\mathrm{Inn}(\mathcal{O}_\infty)$ denotes the projection map. Firstly, taking $u, u' \in U(\infty)$ there exist representatives $\hat{u} \in U(n), \widehat{u'} \in U(n')$ for some n, n' as discussed earlier. Then assuming $n > n'$, we can view $\widehat{u'}$ as an element of $U(n)$ by adding ones down the diagonal, obtaining another representative $\widetilde{u'} \in U(n)$ for u'. We then take the product of $\hat{u}$ and $\widetilde{u'}$ in $U(n)$, and define the class of this product in $U(\infty)$ to be the product of u and u'. Based on this product, we see that α is a group homomorphism. Secondly, we will show that α is injective. Suppose that $\alpha_u = 1$ for some $u \in U(\infty)$ represented by $\hat{u} \in U(n)$. Then for $1 \leq j \leq n$ we have

$$S_j = \alpha_u(S_j) = \sum_{k=1}^{n} \widehat{u}_{kj} S_k,$$

and

$$\delta_{ij} = S_i^* S_j = S_i^* \alpha_u(S_j) = \sum_{k=1}^{n} \widehat{u}_{kj} S_i^* S_k = \widehat{u}_{ij},$$

meaning that $\widehat{u} = 1$ and thus $u = 1$. Next, we will show that α is continuous. Let $\{u(k)\}_{k\in\mathbb{N}}$ be a sequence in $U(\infty)$ converging to $u \in U(\infty)$, and let $\widehat{u(k)} \in U(n_k)$ and $\widehat{u} \in U(n)$ be finite representatives for $u(k)$ and u respectively. In general, the sequence $\{n_k\}_{k\in\mathbb{N}}$ may not have an upper bound. If it does have an upper bound m, we view each $\widehat{u(k)}$ and $\widehat{u}$ as being an element of $U(l)$ where $l = \max\{m, n\}$, and then it is clear that the sequences of complex numbers $\{(\widehat{u(k)})_{ij}\}_{k\in\mathbb{N}}$ converge to $(\widehat{u})_{ij}$. Then for $1 \leq j \leq l$ we have

$$\begin{aligned}\left\|\alpha_{u(k)}(S_j) - \alpha_u(S_j)\right\| &= \left\|\sum_{i=1}^{l} (\widehat{u(k)}_{ij} - \widehat{u}_{ij}) S_i\right\| \\ &\leq \sum_{i=1}^{l} |\widehat{u(k)}_{ij} - \widehat{u}_{ij}|\end{aligned}$$

which tends to 0 as $k \to \infty$. In the case $j > l$ we have that $\alpha_u(S_j) = S_j = \alpha_{u(k)}(S_j)$, and so this limit is true for all j. If $\{n_k\}_{k\in\mathbb{N}}$ does not have an upper bound, however, then for any $n_k > n$ and $1 \leq j \leq n$ we have

$$\begin{aligned}\left\|\alpha_{u(k)}(S_j) - \alpha_u(S_j)\right\| &= \left\|\sum_{i=1}^{n_k} \widehat{u(k)}_{ij} S_i - \sum_{i=1}^{n} \widehat{u}_{ij} S_i\right\| \\ &\leq \sum_{i=n+1}^{n_k} |\widehat{u(k)}_{ij}| + \sum_{i=1}^{n} |\widehat{u(k)}_{ij} - \widehat{u}_{ij}|,\end{aligned}$$

where the second term can be made arbitrarily small as above. For the first term, we need to look more carefully at what it means for the sequence $\{u(k)\}_{k\in\mathbb{N}}$ to converge to u. This convergence means that we can make the sum

$$\sum_{i,j=1}^{\max(n,n_k)} |\widehat{u(k)}_{ij} - \widehat{u}_{ij}|$$

arbitrarily small as k grows, where we embed $\widehat{u(k)}$ into $U(n)$ if $n > n_k$ or embed $\widehat{u}$ into $U(n_k)$ if $n_k > n$. In this case, we have $n_k > n$ and so restricting this sum to the terms of interest we see that

$$\sum_{i=n+1}^{n_k} |\widehat{u(k)}_{ij}|$$

can be made arbitrarily small as k grows as required. Thus the original expression $\left\|\alpha_{u(k)}(S_j) - \alpha_u(S_j)\right\|$ tends to 0 as k tends to ∞. The same argument applies if $j > n$, but the terms in the sum will be slightly different and we will have $u(k)_{ij} \to \delta_{ij}$. Hence the limit $\left\|\alpha_{u(k)}(S_j) - \alpha_u(S_j)\right\| \to 0$ as $k \to \infty$ is true in general.

Now, let $X = S_i S_j$ for some i and j. Then

$$\begin{aligned}\|\alpha_{u(k)}(X) - \alpha_u(X)\| &\leq \|\alpha_{u(k)}(S_i)\alpha_{u(k)}(S_j) - \alpha_{u(k)}(S_i)\alpha_u(S_j)\| \\ &\quad + \|\alpha_{u(k)}(S_i)\alpha_u(S_j) - \alpha_u(S_i)\alpha_u(S_j)\| \\ &\leq \|\alpha_{u(k)}(S_i)\|\|\alpha_{u(k)}(S_j) - \alpha_u(S_j)\| + \|\alpha_u(S_j)\|\|\alpha_{u(k)}(S_i) - \alpha_u(S_i)\|,\end{aligned}$$

and since $\|\alpha_v\| = 1$ for all $v \in U(\infty)$ we may conclude that this tends to 0 as $k \to \infty$. The same argument can be used to show that the limit holds for any X in the *-algebra generated by $\{S_1, S_2, \cdots\}$. The Cuntz algebra $\mathcal{O}_\infty$ is the norm-closure of this *-algebra, so letting $X \in \mathcal{O}_\infty$ we take a sequence $\{X_m\}_{m\in\mathbb{N}}$ in the *-algebra generated by $\{S_1, S_2, \cdots\}$ converging to X. Then

$$\begin{aligned}\|\alpha_{u(k)}(X) - \alpha_u(X)\| &\leq \|\alpha_{u(k)}(X) - \alpha_{u(k)}(X_m)\| \\ &\quad + \|\alpha_{u(k)}(X_m) - \alpha_u(X_m)\| + \|\alpha_u(X_m) - \alpha_u(X)\| \\ &\leq \|\alpha_{u(k)}\|\|X - X_m\| + \|\alpha_{u(k)}(X_m) - \alpha_u(X_m)\| + \|\alpha_u\|\|X_m - X\|,\end{aligned}$$

and since $\|\alpha_v\| = 1$ for all $v \in U(\infty)$ then all of these terms can be made arbitrarily small as k, m tend to infinity. Thus we may conclude that $\|\alpha_{u(k)}(X) - \alpha_u(X)\| \to 0$ as $k \to \infty$ for all $X \in \mathcal{O}_\infty$. Then given a sequence $\{(u(k), X_k)\}_{k\in\mathbb{N}}$ in $U(\infty) \times \mathcal{O}_\infty$ converging to (u, X), we see that

$$\begin{aligned}\|\alpha_{u(k)}(X_k) - \alpha_u(X)\| &\leq \|\alpha_{u(k)}(X_k) - \alpha_{u(k)}(X)\| + \|\alpha_{u(k)}(X) - \alpha_u(X)\| \\ &\leq \|\alpha_{u(k)}\|\|X_k - X\| + \|\alpha_{u(k)}(X) - \alpha_u(X)\|,\end{aligned}$$

and both of these terms tend to 0 as $k \to \infty$. So α viewed as a map $U(\infty) \times \mathcal{O}_\infty \to \mathcal{O}_\infty$ is continuous, and hence the representation α is continuous. Finally, we must show that α_u is an outer automorphism for all $u \in U(\infty)$. Letting $u \in U(n)$, there exist $v, w \in U(n)$ such that $u = vwv^{-1}$ and w is a diagonal matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Following a similar argument with representatives, any matrix $u \in U(\infty)$ can be diagonalised to $u = vwv^{-1}$ where w is a diagonal matrix with only finitely many entries not equal to one. Then the automorphism α_u is outer if and only if α_w is outer, since u and w differ by conjugation by v, which is an inner automorphism. So it is sufficient to show that α_u is outer for a diagonal matrix $u \in U(\infty)$ with eigenvalues $\lambda_1, \lambda_2, \cdots, 1, 1, \cdots$ and where we may further assume without loss of generality that $\lambda_1 \neq 1$. Suppose that α_u is inner, so $\alpha_u = \operatorname{ad} V$ for some unitary $V \in \mathcal{O}_\infty$. Let $\mathscr{H}$ be a separable Hilbert space with countable orthonormal basis $\{e_k\}$. We realise S_i in $\mathcal{B}(\mathscr{H})$ by setting $S_i e_k = e_h$ where $h = 2^{i-1}(2k-1)$, and then $S_1 e_1 = e_1$. We also know that $Ve_1 = \sum_{k=1}^{\infty} x_k e_k$ for some $x_k \in \mathbb{C}$ not all equal to zero, and hence

$$\alpha_u(S_1)(Ve_1) = (VS_1V^*)(Ve_1) = VS_1e_1 = Ve_1 = \sum_{k=1}^{\infty} x_k e_k.$$

We also have $\alpha_u(S_1)(Ve_1) = \sum_{k=1}^{\infty} \lambda_1 x_k e_{2k-1}$, and since $\lambda_1 \neq 1$ we may conclude that $x_k = 0$ for all k and thus $Ve_1 = 0$. This is a contradiction, so α_u must be an outer automorphism as required. □

This action provides further insight into $\mathcal{O}_\infty$ – there are non-trivial outer automorphisms of this algebra, and there must be a large number of them since a space as large as $U(\infty)$ can act effectively via these automorphisms.

Given this understanding of the Cuntz algebra $\mathcal{O}_\infty$, we are now in a position to present the formulation of higher twisted K-theory by Pennig.

2.3 Formulation

As alluded to at the beginning of the chapter, we wish to classify the twists of K-theory in a geometric fashion by using a bundle with an appropriate fibre. In the classical case, this was done using algebra bundles and so firstly we will recall the definition of algebra bundle. Since we will only be concerned with C^*-algebras, we restrict our attention to these algebras and this will give us an induced topology on the bundle via the norm.

Definition 2.3.1. An *algebra bundle* $A \to \mathcal{A} \xrightarrow{\pi} X$ over a topological space X is a fibre bundle such that the fibre A is a C^*-algebra (possibly infinite-dimensional) and the trivialisation maps Φ_α restrict to algebra isomorphisms on each fibre.

Since the fibres of an algebra bundle carry a multiplicative structure, this means that multiplication of sections of an algebra bundle is possible. In particular, since we are working with C^*-algebras we see that the space of continuous sections of an algebra bundle over a compact Hausdorff space X itself forms a C^*-algebra equipped with the induced norm and involution from the fibres. Furthermore, if X is only a locally compact Hausdorff space then there is a sensible notion of a continuous section of an algebra bundle $\mathcal{A}$ over X vanishing at infinity, defined in much the same way as $C_0(X)$ and denoted $C_0(X, \mathcal{A})$. So to each locally compact Hausdorff space X and each algebra bundle $A \to \mathcal{A} \xrightarrow{\pi} X$ we are able to associate a C^*-algebra $C_0(X, \mathcal{A})$. It is by taking the operator algebraic K-theory of this C^*-algebra that we wish to define higher twisted K-theory. Indeed, when A is isomorphic to the algebra of compact operators, this is how classical twisted K-theory is defined

We no longer wish to use the algebra of compact operators as in the classical case; we wish to find an appropriate algebra to replace $\mathcal{K}$ as the fibre in order to classify all twists of K-theory. As such, one might expect that using a fibre isomorphic to a strongly self-absorbing C^*-algebra gives the desired construction. This is not quite the case; in fact, $\mathrm{Aut}(D)$ is contractible for all strongly self-absorbing D as stated in Theorem 2.2.3 and therefore $B\,\mathrm{Aut}(D)$ is also contractible. This means that there are no non-trivial algebra bundles with fibre D over a space X. Instead, we take the stabilisation of the strongly self-absorbing C^*-algebra, the automorphism group of which has a far more interesting homotopy type as mentioned in Theorem 2.2.4 for $\mathcal{O}_\infty$ specifically. This culminates in one of the main theorems of Pennig and Dadarlat's paper.

Theorem 2.3.2 (Theorem 3.8 (a), (b) [DP16])**.** *Let X be a compact metrisable space and let D be a strongly self-absorbing C^*-algebra. The set $Bun_X(D \otimes \mathcal{K})$ of isomorphism classes of algebra bundles over X with fibre $D \otimes \mathcal{K}$ becomes an abelian group under the operation of tensor product. Furthermore, $B\,\mathrm{Aut}(D \otimes \mathcal{K})$ is the first space in a spectrum defining a cohomology theory $E_D{}^\bullet$.*

While this result is interesting from a homotopy-theoretic point of view, it does not yet tell us that we can obtain the twists of K-theory using this construction. What we want is for the cohomology theory $E_D{}^\bullet$ to be $gl_1(KU)^\bullet$, so that the twists of K-theory may be identified with algebra bundles with fibre $D \otimes \mathcal{K}$. The cohomology theory obtained, however, depends on the choice of strongly self-absorbing C^*-algebra. In fact, in the introduction of [DP15b] the authors claim that using $\mathcal{Z}$ yields a subset of twists of K-theory where $\mathbb{Z}_2 \times BU$ is replaced by $\{1\} \times BU$, and using a tensor product of a UHF algebra of infinite type with $\mathcal{O}_\infty$ yields twists for localisations of KU. The full set of twists is the subject of the main theorem of this paper.

Theorem 2.3.3 (Adapted from Theorem 1.1 [DP15b])**.** *The twists of K-theory over X are classified by algebra bundles over X with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$. To be more precise, ${E_{\mathcal{O}_\infty}}^\bullet = gl_1(KU)^\bullet$ and hence $\mathrm{Bun}_X(\mathcal{O}_\infty \otimes \mathcal{K}) \cong gl_1(KU)^1(X)$.*

The significance of this theorem should not be overlooked. The proof requires heavy machinery from stable homotopy theory, much of which is built up over the series of three papers by the authors. It is only through this deep understanding of the abstract notion of twist that the authors were able to determine an appropriate model for the twists of K-theory using geometry and operator algebras.

Given this geometric notion of twist, we are now able to define the higher twisted K-theory groups.

Definition 2.3.4. The higher twisted K-theory of the locally compact Hausdorff space X with twist δ represented by the algebra bundle $\mathcal{O}_\infty \otimes \mathcal{K} \to \mathcal{A}_\delta \xrightarrow{\pi} X$ is defined to be $K^n(X,\delta) = K_n(C_0(X,\mathcal{A}_\delta))$.

Remark 2.3.1. This is actually not the definition originally given by Pennig – he follows the homotopy-theoretic approach of using bundles of spectra and defines the higher twisted K-theory groups to be colimits of certain homotopy groups. The equivalent characterisation that we present above is given in his Theorem 2.7(c) directly after the definition.

Remark 2.3.2. We also note that it is possible to define various other versions of twisted K-theory by replacing $\mathcal{O}_\infty$ with other strongly self-absorbing C^*-algebras. This will modify the set of twists under consideration. We have chosen to focus on $\mathcal{O}_\infty$ as this corresponds to the full set of twists of K-theory, but interesting results may be obtained by using different algebras. For example, Evans and Pennig use infinite UHF-algebras corresponding to twists of localisations of K-theory in a recent paper [EP19].

For the sake of completeness, we also include the definitions of the relative higher twisted K-theory groups, which requires some minor groundwork. Letting X be a compact Hausdorff space with $A \subset X$ a closed subspace, we define $c(X,A) = (X \amalg (A \times [0,1)))/\sim$ where $a \in A \subset X$ is identified with $(a,0) \in A \times [0,1)$. Then given a twist δ over X represented by $\mathcal{A}_\delta$, this algebra bundle extends canonically to a bundle $c\mathcal{A}_\delta$ over $c(X,A)$ defined defined in the same way, i.e. as a disjoint union under an appropriate equivalence relation.

Definition 2.3.5. In the setting above, the relative higher twisted K-theory group of the pair (X,A) is defined to be $K^n(X,A;\delta) = K_n(C_0(c(X,A), c\mathcal{A}_\delta))$.

As expected, taking $A = \emptyset$ gives $K^n(X, A; \delta) = K^n(X, \delta)$, and taking X to be a locally compact Hausdorff space with $X^+ = X \cup \{\infty\}$ the one-point compactification described in Definition 1.1.12 and $A = \{\infty\}$, we may define the higher twisted K-theory of X to be $K^n(X, A, \delta)$, agreeing with the previous definition. We will restrict our attention to the ordinary higher twisted K-theory groups for the remainder of the thesis, only using the relative groups to show that higher twisted K-theory does indeed form an extraordinary cohomology theory.

As one might expect, there is also a notion of higher twisted K-homology which is introduced by Pennig. While we will be focusing on the twisted cohomology version of K-theory, the twisted homology version will be important in some computations as we will see and so we also include this definition here.

Definition 2.3.6. The higher twisted K-homology of the locally compact Hausdorff space X with twist δ represented by the algebra bundle $\mathcal{O}_\infty \otimes \mathcal{K} \to \mathcal{A}_\delta \xrightarrow{\pi} X$ is defined to be $K_n(X, \delta) = KK_n(C_0(X, \mathcal{A}_\delta), \mathcal{O}_\infty)$.

Again, this is not the way that Pennig initially defines the higher twisted K-homology – he introduces the topological definition via ∞-categories – but the version using KK-theory is shown to be an equivalent definition in Corollary 3.5 using a Poincaré duality homomorphism [Pen15].

2.3.1 Basic Properties

Pennig also proves several important results about higher twisted K-theory, including functoriality and the existence of a six-term exact sequence as well as a module structure. We are equipped to prove most of these facts with only the theory that we have developed so far, and so we provide proofs of these basic properties using less homotopy-theoretic methods. We will defer the existence of the module structure to a later section, as this will require the development of a product on higher twisted K-theory.

While higher twisted K-theory is, of course, a contravariant functor, we must determine the appropriate category on which this functor acts. Since we have defined the higher twisted K-groups for locally compact Hausdorff spaces X with a fixed twist δ represented by an algebra bundle with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ over X, the objects in our category must consist of locally compact Hausdorff spaces equipped with these algebra bundles. In topological K-theory the natural class of maps to consider are proper maps, but we need a stronger condition to ensure compatability between the algebra bundles representing the twists. In particular, we specify a morphism $(X, \mathcal{A}_{\delta_X}) \to (Y, \mathcal{A}_{\delta_Y})$ to be a proper map $f : X \to Y$ together with an algebra isomorphism $\theta : f^*\mathcal{A}_{\delta_Y} \to \mathcal{A}_{\delta_X}$, to ensure a relationship between the twist on X and the twist on Y.

Proposition 2.3.7. *Higher twisted K-theory forms a contravariant functor from the category of locally compact Hausdorff spaces with twists to the category of abelian groups.*

Proof. We view the assignment $(X, \mathcal{A}_\delta) \mapsto K^*(X, \delta)$ as the composition of the assignments $(X, \mathcal{A}_\delta) \mapsto C_0(X, \mathcal{A}_\delta) \mapsto K_*(C_0(X, \mathcal{A}_\delta))$. Then proving functoriality boils down to showing that the first of these assignments is a contravariant functor and the second is a covariant functor. These are standard results, but we will briefly spell out the details. The pullback operation for algebra bundles is defined in an analogous way as it is for vector bundles in Definition 1.1.3, and satisfies the same properties as presented in Lemma 1.1.4. Therefore composing the map $f^* : C_0(Y, \mathcal{A}_{\delta_Y}) \to C_0(X, f^*\mathcal{A}_{\delta_Y})$ with the map $C_0(X, f^*\mathcal{A}_{\delta_Y}) \to C_0(X, \mathcal{A}_{\delta_X})$ induced by θ we obtain the desired map $C_0(Y, \mathcal{A}_{\delta_Y}) \to C_0(X, \mathcal{A}_{\delta_X})$. Then combining the first and second properties from Lemma 1.1.4 and using the fact that θ is an isomorphism shows that $(X, \mathcal{A}_\delta) \mapsto C_0(X, \mathcal{A}_\delta)$ does indeed form a contravariant functor. Similarly, we show in Proposition 1.2.7 that K_0 is a covariant functor and claim that it is straightforward to show that suspension of a C^*-algebra is as well, thus $C_0(X, \mathcal{A}_\delta) \mapsto K_*(C_0(X, \mathcal{A}_\delta))$ is a covariant functor as required. □

We will also show that higher twisted K-theory forms an extraordinary cohomology theory, i.e. it satisfies axioms (i), (ii) and (iii) stated in Definition 2.4.1, and in particular that it is an additive cohomology theory, meaning that the higher twisted K-theory of a finite disjoint union is the direct sum of the higher twisted K-theory groups of the pieces.

Proposition 2.3.8. *Higher twisted K-theory forms an extraordinary cohomology theory, i.e.*

(i) if $f_0 : (X, A) \to (Y, B)$ and $f_1 : (X, A) \to (Y, B)$ are homotopic then the maps induced on higher twisted K-theory are the same;

(ii) each pair (X, A) induces a long exact sequence in higher twisted K-theory via the inclusions $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$;

(iii) if (X, A) is a pair and U is an open subset of X whose closure is contained in the interior of A then the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism on higher twisted K-theory;

(iv) the higher twisted K-theory of a finite disjoint union of X_α is the direct sum of the higher twisted K-theory groups of the X_α.

Note that the long exact sequence in (ii) will reduce to a six-term exact sequence by Bott periodicity.

A key ingredient in the proof will be determining the map induced on higher twisted K-theory by a morphism of pairs. The appropriate class of maps $f : (X, A) \to (Y, B)$ to consider here are proper maps $f : X \to Y$ such that $f(A) \subset B$ together with an isomorphism $\theta : f^*\mathcal{A}_Y \to \mathcal{A}_X$. In terms of the abstract twists δ_X and δ_Y corresponding to these algebra bundles, we will write $f^*\delta_Y = \delta_X$ to represent the isomorphism θ. Letting f be such a map, we see that this induces $cf : c(X, A) \to c(Y, B)$ via $cf(x) = f(x)$

if $x \in X$ and $cf(a,t) = f(a)$ if $(a,t) \in A \times [0,1)$. This is clearly well-defined and lands in $c(Y,B)$ since $f(A) \subset B$. Then cf induces a pullback map between the spaces of sections of these bundles, i.e. $cf^* : C_0(c(Y,B), c\mathcal{A}_Y) \to C_0(c(X,A), f^*c\mathcal{A}_X)$, in the natural way. Furthermore, it can be observed that θ induces an isomorphism $f^*c\mathcal{A}_Y \cong c\mathcal{A}_X$, and composing the map induced on the spaces of sections of these bundles by this isomorphism with cf^* we obtain a map $C_0(c(Y,B), c\mathcal{A}_Y) \to C_0(c(X,A), c\mathcal{A}_X)$ as desired. Finally, this induces a map on C^*-algebraic K-theory which is the desired map on higher twisted K-theory, which we will denote by $f^* : K^*(Y,B;\delta_Y) \to K^*(X,A;\delta_X)$. With this map defined, we are equipped to prove the proposition.

Proof.

1. Suppose that $H : X \times [0,1] \to Y$ is a homotopy between $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. Then we can define $cH : c(X,A) \to c(Y,B)$ in the same way that cf is defined above, and it is clear that cH will be a homotopy between cf_0 and cf_1. Then it is standard that homotopic maps induce the same map $cf_0^* = cf_1^*$ between spaces of sections, and thus the maps induced on higher twisted K-theory is the same.

2. We will use the six-term exact sequence in C^*-algebraic K-theory to obtain the six-term exact sequence in higher twisted K-theory. To do so, we will use the short exact sequence of C^*-algebras

$$0 \to SC(A, \mathcal{A}|_A) \xrightarrow{f} C_0(c(X,A), c\mathcal{A}) \xrightarrow{g} C(X, \mathcal{A}) \to 0, \tag{2.3.1}$$

where S denotes the suspension and thus the first non-trivial algebra can be viewed as sections $C_0(A \times [0,1), c\mathcal{A}|_{A\times[0,1)})$ which are trivial over $[0,1)$. Then f can simply be viewed as inclusion where the section is extended trivially to the rest of X, while g takes the section over X and forgets the rest of the data. This is a short exact sequence of C^*-algebras by the definition of the suspension, since a map $f \in SC(A, \mathcal{A}|_A)$ takes the value of the trivial section at $1 \in S^1$. Applying the six-term exact sequence in K-theory yields the exact sequence

$$\begin{array}{ccccc} K^1(A,\delta) & \xrightarrow{f^*} & K^0(X,A,\delta) & \xrightarrow{g^*} & K^0(X,\delta) \\ \partial\uparrow & & & & \downarrow\partial \\ K^1(X,\delta) & \xleftarrow[g^*]{} & K^1(X,A,\delta) & \xleftarrow[f^*]{} & K^2(A,\delta). \end{array}$$

This sequence is of the desired form, but we need to check that the maps in the sequence are induced by the correct maps. Firstly, the inclusion $j : (X,\emptyset) \to (X,A)$ induces the map $cj : X \to c(X,A)$ as described above, and this induces the map $cj^* = g$ on spaces of sections. Hence g^* in the sequence really is the map induced on higher twisted K-theory by j. Then all that remains to check is that the map $K^0(X,\delta) \to K^2(A,\delta) \cong K^0(A,\delta)$ agrees with the map induced by the inclusion map $i : A \to X$, where the isomorphism shown is given by Bott periodicity. In order to prove this, we restrict the short exact sequence (2.3.1) to the following:

$$0 \to SC(A, \mathcal{A}|_A) \xrightarrow{f} C_0(A \times [0,1), c\mathcal{A}|_{A\times[0,1)}) \xrightarrow{g} C(A, \mathcal{A}|_A) \to 0. \tag{2.3.2}$$

Naturality of the boundary map for C^*-algebraic K-theory then gives the commutative diagram

$$\begin{array}{ccc} K_0(C(X,\mathcal{A})) & \longrightarrow & K_1(SC(A,\mathcal{A}|_A)) \\ \downarrow & & \| \\ K_0(C(A,\mathcal{A}|_A)) & \xrightarrow{\cong} & K_1(SC(A,\mathcal{A}|_A)). \end{array}$$

Here, the downwards map is that induced by inclusion and the horizontal maps are those given by Bott periodicity. Thus the boundary map in the six-term exact sequence does agree with the map induced by inclusion and so the six-term exact sequence in higher twisted K-theory is of the correct form with the right maps.

3. The excision property follows from applying the six-term exact sequence in C^*-algebraic K-theory to the short exact sequence

$$0 \to C_0(U \times [0,1), c\mathcal{A}|_{U\times[0,1)}) \to C(c(X,A), c\mathcal{A}) \to C(c(X \setminus U, A \setminus U), c\mathcal{A}|_{c(X\setminus U, A\setminus U)}) \to 0.$$

   Note that the C^*-algebra $C_0(U \times [0,1), c\mathcal{A}|_{U\times[0,1)})$ has trivial K-theory, which is implied by the short exact sequence (2.3.1) since the K-theory groups of the first and last non-trivial algebras in this sequence are isomorphic. Thus the higher twisted K-theory of the pair (X,A) is equal to that of $(X \setminus U, A \setminus U)$ with the isomorphism induced by inclusion as required.

4. If $X = \amalg X_\alpha$, then an algebra bundle $\mathcal{A}$ over X will be a disjoint union of algebra bundles over each X_α, i.e. $\mathcal{A} = \amalg \mathcal{A}_\alpha$. Hence the space of sections of $\mathcal{A}$ will be the direct sum of the spaces of sections of each $\mathcal{A}_\alpha$, and thus by the additivity of C^*-algebraic K-theory the higher twisted K-theory of X will be the direct sum of the higher twisted K-theories of X_α.

□

Hence we can see that higher twisted K-theory really does form an extraordinary cohomology theory, as expected.

We are also in a position to prove the Mayer–Vietoris sequence, which is a feature of any generalised cohomology theory and which will be of critical importance for computations.

Proposition 2.3.9. *Let $X = U_1 \cup U_2$ for closed subsets U_k such that their interiors still cover X. Let $i_k : U_k \to X$ and $j_k : U_1 \cap U_2 \to U_k$ denote inclusion, and $\delta|_{U_k} = i_k^*\delta$ and $\delta|_{U_2 \cap U_2} = (i_1 \circ j_1)^*\delta$ denote restriction of the twist to the corresponding subspaces. Then there is a six-term Mayer–Vietoris sequence as follows:*

$$\begin{array}{ccccc} K^0(X,\delta) & \xrightarrow{(i_1^*, i_2^*)} & K^0(U_1, \delta|_{U_1}) \oplus K^0(U_2, \delta|_{U_2}) & \xrightarrow{j_1^* - j_2^*} & K^0(U_1 \cap U_2, \delta|_{U_1 \cap U_2}) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K^1(U_1 \cap U_2, \delta|_{U_1 \cap U_2}) & \xleftarrow[j_1^* - j_2^*]{} & K^1(U_1, \delta|_{U_1}) \oplus K^1(U_2, \delta|_{U_2}) & \xleftarrow[(i_1^*, i_2^*)]{} & K^1(X, \delta). \end{array}$$

Proof. Observe that we have the pullback diagram of C^*-algebras

$$\begin{array}{ccc} C(X, \mathcal{A}) & \longrightarrow & C(U_1, \mathcal{A}|_{U_1})) \\ \downarrow & & \downarrow \\ C(U_2, \mathcal{A}|_{U_2})) & \longrightarrow & C(U_1 \cap U_2, \mathcal{A}|_{U_1 \cap U_2})). \end{array}$$

Then by Theorem 21.2.3 of [Bla86], the existence of the Mayer–Vietoris sequence is established. □

To conclude our discussion of higher twisted K-theory, we will show that taking the trivial twist over X yields the standard topological K-theory of X, and as such higher twisted K-theory encodes all of the information of topological K-theory and a great deal more.

Proposition 2.3.10. *For any locally compact Hausdorff space X, taking the trivial twist δ represented by the trivial algebra bundle $X \times (\mathcal{O}_\infty \otimes \mathcal{K})$ yields topological K-theory; $K^n(X, \delta) = K^n(X)$.*

Proof. The higher twisted K-theory of X is defined to be the K-theory of the space of continuous sections vanishing at infinity of the algebra bundle, but of course a continuous section of the trivial bundle vanishing at infinity is simply a continuous map from X into the fibre vanishing at infinity, i.e. $K^n(X, \delta) = K_n(C_0(X, \mathcal{O}_\infty \otimes \mathcal{K}))$. It is standard that $C_0(X, A) \cong C_0(X) \otimes A$ for any C^*-algebra A, for instance by Theorem II.9.4.4 of [Bla06], and so this combined with the stability of K-theory given in Proposition 1.2.12 gives $K^n(X, \delta) = K_n(C_0(X) \otimes \mathcal{O}_\infty)$. Here we must use a Künneth theorem – the Künneth theorem presented in V.1.5.10 of [Bla06] states that if A and B are C^*-algebras such that A satisfies a technical property which is satisfied by $\mathcal{O}_\infty$ and either $K_*(A)$ or $K_*(B)$ is torsion-free, then $K_*(A \otimes B) \cong K_*(A) \otimes K_*(B)$. By a result of Cuntz [Cun81], the K-theory of $\mathcal{O}_\infty$ is $\mathbb{Z}$ in even degree and trivial in odd degree, and hence we may conclude that

$$\begin{aligned} K^n(X,\delta) &= K_n(C_0(X)\otimes \mathcal{O}_\infty) \\ &\cong (K_n(C_0(X))\otimes K_0(\mathcal{O}_\infty)) \oplus (K_{n+1}(C_0(X))\otimes K_1(\mathcal{O}_\infty)) \\ &\cong (K^n(X)\otimes \mathbb{Z}) \oplus (K^{n+1}(X)\otimes 0) \\ &= K^n(X) \end{aligned}$$

as required. □

We will also prove that higher twisted K-theory reduces to classical twisted K-theory when the twists can be represented by algebra bundles with fibres isomorphic to $\mathcal{K}$.

Proposition 2.3.11. *Let $\mathcal{K} \to \mathcal{A} \to X$ be an algebra bundle with fibre $\mathcal{K}$ over X representing a classical twist of K-theory. Then the classical notion of twisted K-theory agrees with our notion of higher twisted K-theory.*

Proof. Since the algebra bundle $\mathcal{A}$ is being used to represent a twist of K-theory, this twist must also have a geometric representative in terms of bundles with fibre isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$. In order to obtain a suitable bundle, we may take the tensor product of $\mathcal{A}$ with the trivial algebra bundle with fibre $\mathcal{O}_\infty$ over X. The tensor product of algebra bundles can be defined in much the same way as the tensor product of vector bundles in Definition 1.1.3, but since we need to determine the relationship between sections of $\mathcal{A}$ and of the tensor product bundle we will need a definition in terms of transition functions which can be found in the section "Tensor Products" in 1.1 of [Hat17].

Suppose that $\{U_\alpha\}_{\alpha\in A}$ is a trivialising open cover for $\mathcal{A}$ over X with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathcal{K})$. The trivial bundle $X \times \mathcal{O}_\infty$ will then have transition functions all equal to the identity automorphism of $\mathcal{O}_\infty$ over the same open cover. So the tensor product bundle will have transition functions $(I \otimes g_{\alpha\beta})(x) = I \otimes g_{\alpha\beta}(x) \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ for all $x \in U_\alpha \cap U_\beta$. A section of the tensor product bundle can then be viewed as a collection of maps $\{s_\alpha\}_{\alpha\in A}$ with $s_\alpha : U_\alpha \to \mathcal{O}_\infty \otimes \mathcal{K}$ such that

$$s_\beta(x) = s_\alpha(x) \cdot (I \otimes g_{\alpha\beta})(x),$$

or viewing this as $s_\alpha = s'_\alpha \otimes s''_\alpha$ for $s'_\alpha : U_\alpha \to \mathcal{O}_\infty$ and $s''_\alpha : U_\alpha \to \mathcal{K}$ we see that

$$s'_\beta(x) \otimes s''_\beta(x) = (s'_\alpha(x)) \otimes (s''_\alpha(x) \cdot s_{\alpha\beta}(x)).$$

But this is simply the tensor product of a section of $\mathcal{A}$ with a constant section of the trivial $\mathcal{O}_\infty$ bundle over X, i.e. a constant element of $\mathcal{O}_\infty$. Hence we conclude that the higher twisted K-theory of X may be expressed as

$$\begin{aligned} K^n(X,\delta) &= K_n(C_0(X, \mathcal{A} \otimes (X \times \mathcal{O}_\infty))) \\ &= K_n(C_0(X,\mathcal{A}) \otimes \mathcal{O}_\infty) \\ &= K_n(C_0(X,\mathcal{A})) \end{aligned}$$

which is equal to the classical twisted K-theory of X as required. □

As expected, this result shows that higher twisted K-theory contains all of the information of classical twisted K-theory, along with a great deal more.

2.4 Links to cohomology

In the classical case, twists of K-theory were not only visualised as algebra bundles but often this viewpoint was complemented using cohomology classes. This naturally raises the question as to whether there is any link between algebra bundles over a space X with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ and the cohomology of X. As briefly discussed in Section 2.1, classical twists were often viewed as elements of $H^3(X, \mathbb{Z})$, and it is precisely the Dixmier–Douady theory which provided a link between these cohomology classes and algebra bundles with fibre $\mathcal{K}$. As such, the higher Dixmier–Douady theory posed by Pennig and Dadarlat in relation to strongly self-absorbing C^*-algebras is the key to understanding the relationship between higher twists and cohomology classes. The following results are discussed in generality for all strongly self-absorbing C^*-algebras in Section 4 of [DP16], but we will specifically consider the use of $\mathcal{O}_\infty$ in order to work with all twists of K-theory.

What we desire is a way to interpret the twists of K-theory, i.e. the elements of the first group of some generalised cohomology theory $E^1_{\mathcal{O}_\infty}(X)$, in terms of the ordinary cohomology of X. This is precisely what a spectral sequence allows. While the reader unfamiliar with spectral sequences could simply read the results of this section, which do not rely on a knowledge of spectral sequences, a suitable background in spectral sequences and their use can be found in a standard reference in homology such as [CE99] or [BT82]. As with any generalised cohomology theory, there is an Atiyah–Hirzebruch spectral sequence converging to $E_{\mathcal{O}_\infty}{}^\bullet$, and the coefficients of this sequence are determined by Dadarlat and Pennig. They use this to argue that the E_2 term of the spectral sequence is as follows.

	0	1	2	3
0	$H^0(X, \mathbb{Z}_2)$	$H^1(X, \mathbb{Z}_2)$	$H^2(X, \mathbb{Z}_2)$	$H^3(X, \mathbb{Z}_2)$
-1	0	0	0	0
-2	$H^0(X, \mathbb{Z})$	$H^1(X, \mathbb{Z})$	$H^2(X, \mathbb{Z})$	$H^3(X, \mathbb{Z})$
-3	0	0	0	0
-4	$H^0(X, \mathbb{Z})$	$H^1(X, \mathbb{Z})$	$H^2(X, \mathbb{Z})$	$H^3(X, \mathbb{Z})$

At this stage there are complications. The differentials in this sequence are unknown, and even if they were known there may be non-trivial extension problems in determining $E^1_{\mathcal{O}_\infty}(X)$. At this point, Pennig and Dadarlat restrict to the setting in which X has torsion-free cohomology, as in this case the differentials of the sequence are necessarily zero as they are torsion operators, meaning that their image is torsion, as shown in Theorem 2.7 of [Arl92]. It is then clear that there will be no extension problems, as there are no non-trivial extensions of free groups and the only torsion will be in the final summand, $H^1(X, \mathbb{Z}_2)$. Thus we obtain the following, noting that to apply the spectral sequence we must be working with a finite connected CW complex.

Theorem 2.4.1 (Corollary 4.7(ii) [DP16])**.** *Let X be a finite connected CW complex such that the cohomology ring of X is torsion-free. Then*

$$E^1_{\mathcal{O}_\infty}(X) \cong \mathrm{Bun}_X(\mathcal{O}_\infty \otimes \mathcal{K}) \cong H^1(X, \mathbb{Z}_2) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z}).$$

This shows that there is a relationship between the twists of K-theory and cohomology, at least in the restrictive case when X is torsion-free. Even when the cohomology of X has torsion, the twists will correspond to some subset of these odd-degree cohomology groups depending on differentials and extension problems. This also confirms that, in this case, the classical twists contained in $H^1(X, \mathbb{Z}_2) \oplus H^3(X, \mathbb{Z})$ are indeed twists of K-theory, and provides insight into Pennig's chosen name – "higher" twisted K-theory. The twists introduced by Pennig are higher in the sense that they can be represented by higher degree cohomology classes, as opposed to the classical degree 1 and 3 twists.

Note that henceforth, when we are in a setting in which Theorem 2.4.1 applies, we will identify the twists of K-theory over X with the odd-degree integral cohomology classes of X. Given a twist we may view this as a cohomology class, and given a cohomology class this will represent a twist. This will be particularly important in the development of spectral sequences and in computations.

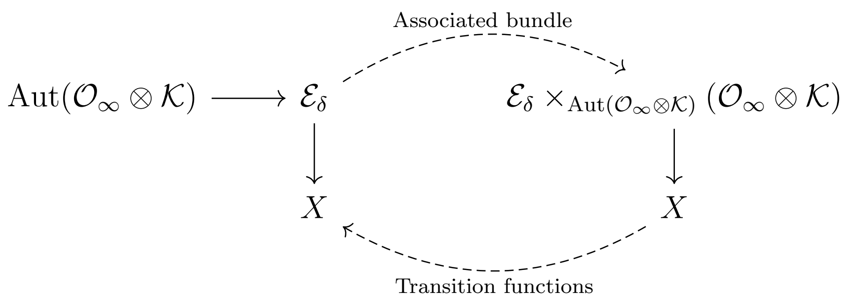
Although not considered by Pennig and Dadarlat, there are some slightly more general statements that can be made even if the cohomology of the base space has torsion. This will be the case if the torsion does not have any effect on the previous argument, i.e. if the relevant differentials are necessarily zero and there are no extension problems. Since we are only interested in the degree 1 group of this cohomology theory, only the groups whose row and column index sum to 1 are relevant, and so we only need to worry about the differentials entering and leaving these groups. If, for instance, only the odd cohomology groups of X are torsion-free, the differentials between these relevant groups will all necessarily be zero, and we will be able to reach the same conclusion that twists correspond to odd-degree cohomology classes. Even these slightly relaxed assumptions allow for a wider class of spaces to be considered, including real projective space and some variety of Lens spaces.

This link between the twists of K-theory and cohomology provides a wide variety of directions to explore. We may use the cohomology picture to develop explicit constructions of twists which correspond to specific cohomology classes, to obtain information about differentials in a spectral sequence for calculating higher twisted K-theory groups, and to aid in computations. These are all important tasks, and as such will all be explored in later chapters.

2.5 Topological characterisation

While Pennig's original formulation of higher twisted K-theory proves to be useful in computations, it is difficult to explicitly describe the elements of these groups using this definition. We follow an argument of Rosenberg presented in [Ros89] about classical twisted K-theory in order to adapt this to a more topological definition. This alternative characterisation will allow greater insight into elements of the higher twisted K-groups, and will also be more useful in motivating and defining a product structure on higher twisted K-theory.

To do this, we will need to slightly shift our perspective from that of algebra bundles to that of principal bundles. Recall from Theorem 1.1.37 that there is a correspondence between certain fibre bundles and certain principal bundles. In this setting, we may use this to obtain a correspondence between algebra bundles over X with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ and principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over X as follows, because $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ acts effectively on $\mathcal{O}_\infty \otimes \mathcal{K}$ by automorphisms.



This is a bijective correspondence; moving from one perspective to the other and back again yields an isomorphic bundle, and therefore we may view either of the objects above as twists of K-theory over X. Viewing a twist δ as a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle as opposed to an algebra bundle, we define the K-theory of X twisted by δ to be the K-theory of the continuous sections vanishing at infinity of the associated algebra bundle, to agree with our previous definition.

We are now able to state the main result of this section.

Theorem 2.5.1. *Let X be a compact Hausdorff space and $\mathcal{E}_\delta$ a principal* $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$*-bundle over X representing a twist δ. There are natural identifications*

$$K^0(X,\delta) \cong [\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})}$$

and

$$K^1(X,\delta) \cong [\mathcal{E}_\delta, \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})}$$

where $[\ ,\ ]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})}$ denotes the unbased homotopy classes of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$*-equivariant maps, i.e. $\pi_0(C(\ ,\ )^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})})$, and $\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ is the based loop space of $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$, i.e. the space of continuous maps $S^1 \to \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ with $1 \in S^1$ mapped to $I \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$.*

Note that $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ acts on $\mathcal{E}_\delta$ as the structure group of the principal bundle and acts on $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ via conjugation in the same way that PU acts on Fred, meaning that

$$F \cdot T = {T^{-1}}_{\mathcal{H}} F T_{\mathcal{H}} \tag{2.5.1}$$

where $T \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, $F \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ and we denote the map induced on $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ by applying T pointwise by $T_{\mathcal{H}}$. The action on $\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ is defined to be this action at every point in the loop.

Before proceeding with the proof, we need a standard lemma about principal bundles and associated bundles.

Lemma 2.5.2. *Let $\mathcal{E}$ be a principal G-bundle over a topological space X with projection map $\pi : \mathcal{E} \to X$, and suppose that G has a continuous and effective left action on a topological space F. Then the space of sections of the associated fibre bundle $\mathcal{E} \times_G F$ over X can be identified with the space of G-equivariant maps $\mathcal{E} \to F$.*

Proof. A section $s : X \to \mathcal{E} \times_G F$ will be of the form $x \mapsto [(e, f)]$, but a choice of $e \in \pi^{-1}(x)$ will uniquely determine an $f \in F$. This means that we may write this section as a continuous map $h : \mathcal{E} \to F$, and in particular the equivalence relation in the definition of the associated bundle $[(e, h(e))] = [(e \cdot g, g^{-1} \cdot h(e))]$ implies that $g^{-1} \cdot h(e) = h(e \cdot g)$. Hence h is G-equivariant. Conversely, given a G-equivariant map $h : \mathcal{E} \to F$, we define $s : X \to \mathcal{E} \times_G F$ by $s(x) = [(e, h(e))]$ for a choice of $e \in \pi^{-1}(x)$. Suppose $e_1, e_2 \in \pi^{-1}(x)$, then there is an element $g \in G$ such that $e_1 \cdot g = e_2$. Hence

$$[(e_1, h(e_1))] = [(e_1 \cdot g, g^{-1} \cdot h(e_1))] = [(e_2, h(e_1 \cdot g))] = [(e_2, h(e_2))],$$

and so s is well-defined. Furthermore, it is clear that s defines a section and that these two constructions are inverses of each other, so the lemma is proved. □

We are now equipped to prove Theorem 2.5.1.

Proof. From Theorem 1.2.20 we have

$$\begin{aligned} K^0(X,\delta) &= K_0(C(X, \mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}))) \\ &= \pi_0(\mathrm{Fred}_{C(X,\mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}))}), \end{aligned}$$

and applying Lemma 2.5.2 allows us to replace $C(X, \mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}))$ with $C(\mathcal{E}_\delta, \mathcal{O}_\infty \otimes \mathcal{K})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$. Then we see from the definition that

$$\mathcal{H}_{C(\mathcal{E}_\delta, \mathcal{O}_\infty \otimes \mathcal{K})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}} = C(\mathcal{E}_\delta, \mathcal{H}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})},$$

which allows us to conclude that

$$\mathrm{Fred}_{C(X,\mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}))} = C(\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$$

as in the proof of Proposition 2.1 of [Ros89]. Then

$$\begin{aligned} K^0(X,\delta) &= \pi_0(C(\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}) \\ &= [\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \end{aligned}$$

as required. In order to obtain the result for K^1, we recall from Definition 1.2.10 that $K_1(A) = K_0(SA)$ for a C^*-algebra A where SA denotes the suspension. In this case, we are interested in the C^*-algebra $SC(X, \mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}))$, which can be viewed as

$$\{f : S^1 \to C(X, \mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K})) \text{ continuous} : f(1) = 0\}.$$

We will suppress the continuity of the function and the fact that $f(1) = 0$ for brevity, but the same conditions are required to hold in the following sets where 0 is taken to be the additive identity in each case. In the same way as above, we can view the Fredholm operators on the standard Hilbert C^*-module of this C^*-algebra as

$$\begin{aligned} &\{f : S^1 \to C(X, \mathcal{E}_\delta \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})\} \\ =\ &\{f : S^1 \to C(\mathcal{E}_\delta, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}\} \\ =\ &C(\mathcal{E}_\delta \times S^1, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \\ =\ &C(\mathcal{E}_\delta, C(S^1, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}))^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \\ =\ &C(\mathcal{E}_\delta, \Omega\, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}. \end{aligned}$$

Thus we may conclude that

$$K^1(X,\delta) = \pi_0(C(\mathcal{E}_\delta, \Omega\, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}) = [\mathcal{E}_\delta, \Omega\, \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$$

as required. ∎

In the case that principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over a space can be explicitly described, this provides a way of expressing elements in the higher twisted K-theory groups of that space. This will be useful when performing computations for spheres and also in exploring the graded module structure on higher twisted K-theory.

One advantage in this formulation is that it can simplify proofs, for instance we can develop the functoriality of higher twisted K-theory in a more straightforward and explicit manner as follows.

Proof of Proposition 2.3.7. Given a morphism $f : (X, \mathcal{A}_{\delta_X}) \to (Y, \mathcal{A}_{\delta_Y})$, i.e. a proper map $f : X \to Y$ together with an isomorphism $\theta : f^*\mathcal{A}_Y \to \mathcal{A}_X$, and letting $\mathcal{E}_{\delta_X}$ and $\mathcal{E}_{\delta_Y}$ denote the associated principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles, we define the induced map

$$\widetilde{f} : [\mathcal{E}_{\delta_Y}, \Omega^n \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \to [\mathcal{E}_{\delta_X}, \Omega^n \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$$

to send a map h to the composite $h \circ f^* \circ \widehat{\theta^{-1}}$, where $\widehat{\theta^{-1}} : \mathcal{E}_{\delta_X} \to f^*\mathcal{E}_{\delta_Y}$ is the isomorphism induced on the principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles by θ^{-1}. While we do not give an explicit form for $\widehat{\theta^{-1}}$ in terms of θ^{-1}, the fact that the algebra bundles $f^*\mathcal{A}_Y$ and $\mathcal{A}_X$ are isomorphic implies that the principal bundles $f^*\mathcal{E}_{\delta_Y}$ and $\mathcal{E}_{\delta_X}$ constructed using their transition functions are isomorphic. Then by the contravariant functoriality of the pullback construction and the fact that $\widehat{\theta^{-1}}$ is an isomorphism it is clear that the identity map induces the identity on higher twisted K-theory and if $f : (X, \mathcal{A}_{\delta_X}) \to (Y, \mathcal{A}_{\delta_Y})$ and $g : (Y, \mathcal{A}_{\delta_Y}) \to (Z, \mathcal{A}_{\delta_Z})$ then $\widetilde{(f \circ g)} = \widetilde{g} \circ \widetilde{f}$ as required. □

This formulation can also be extended to obtain expressions for the higher twisted K-theory groups of higher degree, where we see that the method used in the proof can be used to show that

$$K^n(X, \delta) = [\mathcal{E}_\delta, \Omega^n \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}.$$

Although this may be a useful expression, the statement of the theorem covers the important cases since Bott periodicity implies that everything will reduce to these two groups. Furthermore, we can obtain a topological characterisation of the reduced higher twisted K-theory groups where we replace the Fredholm operators with the connected component of $\operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ containing the identity, denoted $(\operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})_0$, i.e.

$$\widetilde{K}^n(X, \delta) = [X, \Omega^n((\operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})_0)]^{\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}.$$

This follows in the same way as in topological K-theory where $\widetilde{K}^i(X) = [X, \Omega^i(\operatorname{Fred}_0)]$.

2.6 Physical applications

Both topological and classical twisted K-theory are of great importance in mathematical physics, particularly in string theory, and so it is expected that higher twisted K-theory will prove to possess even greater applications.

D-branes are one of the fundamental objects in string theory, and there exists a vast literature detailing the relationship between these objects and K-theory. The link was first popularised by work of Witten, in which he shows that the charges of D-branes on a spacetime can be naturally identified with elements of the K-theory of the spacetime manifold [Wit00]. This also led to a greater understanding of tachyon condensation. It was later shown by Bouwknegt and Mathai that in the presence of a B-field, which is topologically classified by an element of $H^3(X, \mathbb{Z})$, D-brane charges take values in the twisted K-theory of X [BM00]. For a more detailed expository view on the links between twisted K-theory and D-branes, see [Moo04].

Another aspect of string theory in which twisted K-theory has found application is T-duality, introduced by Bouwknegt, Evslin and Mathai in [BEM04a] and [BEM04b] from a physical perspective. The authors show that the T-duality transformation induces an isomorphism on twisted K-theory, which was an interesting and unexpected result as the T-duality transformation often leads to significant differences between the topologies of the circle bundles in question. A number of generalisations came out of this, most notably for our purposes being a series of papers by the same authors on spherical T-duality [BEM15a, BEM15b, BEM18] in which the relevant cohomology class is of degree 7. This led to the result that the spherical T-duality transform induces an isomorphism on higher twisted K-theory. In the first of this series of papers, the authors provide some insight into how higher twisted K-theory fits in with the D-brane picture. In the setting of Type IIB string theory, the data in 10-dimensional supergravity includes a 10-manifold Y which is commonly diffeomorphic to $\mathbb{R} \times X$ for an appropriate 9-manifold X. They explain that the K-theory of X twisted by a 7-class corresponds to the set of conserved charges of a certain subset of branes. This implies that there is a richer relationship between D-branes and higher twisted K-theory than what currently exists in the literature, and it may be possible to gain greater insight into the behaviour of D-branes by studying higher twisted K-theory.

Relevance to physics also appears in the study of strongly self-absorbing C^*-algebras, as some common algebras in physics such as the canonical anticommutation relations (CAR) algebra are strongly self-absorbing. This algebra is not only relevant in quantum physics, but also in the study of Clifford algebras. By exploring the twisted K-theory whose twists arise from the CAR algebra, a deeper understanding of this algebra and its applications may be gained.

A final application of twisted K-theory of critical importance in mathematical physics is the work of Freed, Hopkins and Teleman in proving that the Verlinde ring of positive energy representations of loop groups is isomorphic to the equivariant twisted K-theory of a compact Lie group in the series of papers [FHT11a, FHT13, FHT11b]. The Verlinde ring is an object which arises in conformal field theory, first introduced by Verlinde in [Ver88], and on which there exists no simple expression for the product in general. In the setting of Freed, Hopkins and Teleman, the equivariant twisted K-theory group can be equipped with a product which essentially comes from multiplication in the Lie group, and using this the product on the Verlinde ring can be simplified greatly; a task which is of relevance in conformal field theory. In spite of this, the isomorphism between the two rings is also very complicated, which presents difficulties transferring the product over to the Verlinde ring. It is likely that there exists an analogous result in the higher twisted setting, where it is as of yet unknown what can replace the Verlinde ring to be isomorphic to higher twisted K-theory, but another object from physics may arise in this case. Pennig and Evans [EP19] hint at possible approaches.

Chapter 3

Explicit geometric construction of twists

Whilst knowing that the twists of K-theory over a space X may be identified with algebra bundles over X with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ is useful in its own right, this does not provide us with an explicit construction of a bundle to represent each homotopy-theoretic twist. In particular, since the definition of higher twisted K-theory involves the algebra of sections of such an algebra bundle, it is easier to compute higher twisted K-theory groups when there is an explicit bundle to work with. In the general case, even classifying the $\mathcal{O}_\infty \otimes \mathcal{K}$ bundles over X is a difficult task. In the case that twists can be identified with cohomology classes, however, by associating an explicit bundle to each cohomology class this will allow for simpler methods of computation.

In this chapter we explicitly construct algebra bundles with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ over topological spaces with torsion-free cohomology. While Pennig and Dadarlat proved that these bundles are classified up to isomorphism by odd-degree cohomology classes when the base space is torsion-free, they provide no explicit constructions and so we aim to bridge this gap in the literature. In the case of twisted K-theory and Lie groups, these geometric constructions are well-understood and can be obtained through loop groups and transgression of cohomology classes represented by differential forms [MW16].

We begin by restricting our attention to a limited class of spaces – those over which all principal bundles can be constructed via the clutching construction. We prove that by specifying a gluing map we are able to construct algebra bundles with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ represented by any cohomology class over the odd-dimensional spheres, and mention generalisations of these methods to other spaces. We then move to more general topological spaces, and simplify the twisting class. We consider the simplest non-trivial case, in which we take a decomposable class $\alpha \cup \beta \in H^5(X, \mathbb{Z})$ with $\alpha \in H^2(X, \mathbb{Z})$ and $\beta \in H^3(X, \mathbb{Z})$ and construct the desired bundle associated to this cohomology class, loosely based upon the work done in [MM17].

3.1 The clutching construction

There is a well-known construction which builds fibre bundles over topological spaces viewed as the union of two closed subsets, ideally whose intersection has a simple form. This construction is most commonly applied to the n-sphere S^n viewed as the union of its upper and lower hemispheres D^n_+ and D^n_-, which intersect along the equatorial $(n-1)$-sphere. We will limit our discussion to spheres for the the most part, but note that more general constructions apply which we will mention briefly. Loosely, the construction takes a fibre bundle over each of the hemispheres and glues them together using a gluing function, which tells the bundles how to interact on the intersection. Therefore the gluing map should be a function defined on the intersection, and in order to sensibly tell the bundles how to interact it should land in the structure group of the bundle. For example, to construct a principal G-bundle it should be a map into G, or in the simple case of constructing a vector bundle it should map into $GL(n, \mathbb{C})$. The construction is particularly useful for the spheres because the gluing map can be viewed as an element of a homotopy group of the structure group, which provides a way of classifying these maps in cases that the homotopy type of the structure group is understood. It is also useful because the two closed subsets being considered are contractible, meaning that all fibre bundles over the hemispheres are necessarily canonically trivialised. In particular, a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over S^n may be constructed by specifying a map $f : S^{n-1} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ which will glue trivial bundles over the upper and lower hemispheres. More precisely, we make the following definition.

Definition 3.1.1. Let $f : S^{n-1} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ be a continuous map. The *clutching bundle* $\mathcal{E}_f$ over S^n associated to f is defined to be the quotient of the disjoint union $(D^n_+ \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})) \amalg (D^n_- \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ under $(x, T) \sim (x, f(x) \circ T)$ for all $x \in S^{n-1}$ and $T \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$.

Note that technically this equivalence is between points (x_+, T) and $(x_-, f(x) \circ T)$ where $x_+ \in D^n_+$ and $x_- \in D^n_-$ both represent the same point $x \in S^{n-1}$, but we will suppress these subscripts. It is straightforward to show that $\mathcal{E}_f$ equipped with the natural projection onto S^n is a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over S^n. Note that we could have equivalently constructed an algebra bundle with fibre isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ over S^n by replacing $T \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ with $v \in \mathcal{O}_\infty \otimes \mathcal{K}$, but the principal bundle construction will be more convenient.

We also mention that an added benefit of using the clutching construction is that there is a simple way to describe the sections of a clutching bundle, using sections of the trivial bundles over the two hemispheres. In particular, a section of a clutching bundle can be identified with sections of the trivial bundles which interact via the gluing map as follows. Note that we abbreviate $C(D^n_+, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})) \oplus C(D^n_-, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ as $C(D^n_+ \amalg D^n_-, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ for brevity.

Lemma 3.1.2. *Let $\mathcal{E}_f$ be the clutching bundle over S^n with gluing function given by $f : S^{n-1} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. The space of sections of $\mathcal{E}_f$ is of the form*

$$C(S^n, \mathcal{E}_f) = \{(g,h) \in C(D^n_+ \amalg D^n_-, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})) : g(x) = f(x) \cdot h(x) \textit{ for all } x \in S^{n-1}\}.$$

Proof. Obviously a pair of maps of this form defines a section of the clutching bundle, and conversely any section of the clucthing bundle will be built from sections of the trivial bundles over the upper and lower hemispheres which interact via the gluing map over the equator. □

This is particularly useful because the higher twisted K-theory groups are defined using the algebra of sections, and so having an explicit realisation of this algebra will allow computations to be performed more easily.

Now, we have an explicit construction of a bundle from a gluing map, but we want to be able to explicitly construct a bundle from a cohomology class of the sphere. To move towards this goal, we show that any principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over S^n can be constructed in this way, and furthermore that the isomorphism class of the bundle depends only on the homotopy class of the gluing map. This is a standard result in the case of vector bundles, but we provide the proof for completeness. We follow the approach in the proof of Theorem 2.7 in [Coh98].

Proposition 3.1.3. *There is a bijective correspondence between the set of isomorphism classes of principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over S^n and $\pi_{n-1}(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$.*

Proof. Let $\mathcal{E}$ be a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over S^n. Fixing a trivialisation of $\mathcal{E}$ over a chosen basepoint $x_0 \in S^{n-1}$, i.e. an identification of the fibre with $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, trivialisations of the bundle over D^n_+ and D^n_- can be defined which restrict to this chosen trivialisation over x_0 because the bundle is trivial over D^n_+ and D^n_-. This defines a map $f : S^{n-1} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ which sends the basepoint x_0 to the identity automorphism of $\mathcal{O}_\infty \otimes \mathcal{K}$, and hence $\mathcal{E}$ can be viewed as the clutching bundle defined by this gluing map f. This defines our correspondence from the set of isomorphism classes of principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over S^n to $\pi_{n-1}(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$.

Firstly, we must verify that this is well-defined by showing that isomorphic bundles correspond to homotopic clutching functions. Suppose that $\mathcal{E}_1 \cong \mathcal{E}_2$ via an isomorphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ which respects the trivialisation over the basepoint x_0. Restricting this isomorphism to the hemispheres defines maps $\varphi_\pm : D^n_\pm \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ such that $\varphi_\pm(x_0) = I$, and if f_1 and f_2 denote the clutching functions for $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively then $\varphi_+(x)f_1(x) = f_2(x)\varphi_-(x)$ for all $x \in S^{n-1}$. Now, identifying $0 \in D^n_+$ and $0 \in D^n_-$ with the north and south poles of the sphere respectively, note that the map $\varphi_+(tx)f_1(x)\varphi_-(tx)^{-1}$ for $t \in [0,1]$ is a homotopy from $f_2(x)$ to $\varphi_+(0)f_1(x)\varphi_-(0)^{-1}$. Furthermore, since $\varphi_\pm$ are defined on connected spaces, their images lie in the same connected component of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, and since their images on the basepoint x_0 are both the identity then there are paths $\alpha_\pm$ in S^n from $\varphi_\pm(0)$ to $I \in \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. So the map $\alpha_+(t)f_1(x)\alpha_-(t)^{-1}$ is a homotopy from $\varphi_+(0)f_1(x)\varphi_-(0)^{-1}$ to $f_1(x)$ and hence f_1 is homotopic to f_2 as required.

Secondly, we note that this correspondence is surjective since every element of the homotopy group $\pi_{n-1}(\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ defines a principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over S^n as in Definition 3.1.1.

Finally, we must show that homotopic gluing maps correspond to isomorphic principal bundles. Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are defined via homotopic maps $f_1, f_2 : S^{n-1} \to \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ respectively, and let $F : S^{n-1} \times [0,1] \to \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ be a homotopy. We may use the same clutching construction to define a principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $S^n \times [0,1]$ which restricts to $\mathcal{E}_1$ over $S^n \times \{0\}$ to $\mathcal{E}_2$ over $S^n \times \{1\}$. Finally, Theorem 2.1 in [Coh98] implies that these bundles are isomorphic. □

Note that the map defined by the bijective correspondence in Proposition 3.1.3 is an explicit realisation of the isomorphism induced by viewing S^n as the suspension ΣS^{n-1}:

$$\begin{aligned}
[S^n, B\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] &= [\Sigma S^{n-1}, B\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] \\
&= [S^{n-1}, \Omega B\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] \\
&\cong [S^{n-1}, \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})].
\end{aligned}$$

Now, since the cohomology groups of the spheres are torsion-free, we see via Theorem 2.4.1 that the twists of K-theory over the spheres are classified by their odd-degree cohomology groups, i.e. $H^{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ for $n \geq 1$. Finally, these correspondences

$$H^{2n+1}(S^{2n+1}, \mathbb{Z}) \cong [S^{2n+1}, B\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] \cong \pi_{2n}(\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$$

allow us to obtain explicit geometric representatives for twists over S^{2n+1} given in terms of cohomology classes. Letting $[\delta_0] \in H^{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ denote a generator and taking any $N \in \mathbb{Z}$, we see that the bundle representing the twist $N[\delta_0]$ is constructed via a degree N gluing map, i.e. N times the generator of $\pi_{2n}(\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ corresponding to $[\delta_0]$ under the above identification. Using this result, we are able to explicitly compute the higher twisted K-theory of the odd-dimensional spheres from the definition rather than using any higher-powered machinery such as spectral sequences, and crucially this method allows us to determine the generator of the higher twisted K-groups as we will see in Chapter 5.

We also note that the construction can only produce trivial principal $\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over even-dimensional spheres, which is expected from Theorem 2.4.1. This is because a bundle over S^{2n} would come from a gluing map $S^{2n-1} \to \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, but the homotopy group $\pi_{2n-1}(\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ is trivial as stated in Theorem 2.2.4 and so every gluing map is homotopic to a constant. This means that any gluing map will construct a bundle which is isomorphic to the trivial bundle, which agrees with the fact that the odd-dimensional cohomology of S^{2n} is trivial.

Now, we have provided a construction to obtain principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles as these are necessary in the topological formulation of higher twisted K-theory. In order to compute higher twisted K-theory directly from the definition, however, we must also be able to construct algebra bundles with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$. Of course, we can construct these in the same way by specifying a gluing map into the structure group $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, but it is non-trivial that carrying out the clutching construction commutes with taking the associated algebra bundle. We make this precise in the following lemma.

Lemma 3.1.4. *Let $f : S^{2n} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ be a gluing map, with $\mathcal{E}_f$ the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle constructed via the clutching construction. The associated algebra bundle to $\mathcal{E}_f$ with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ is isomorphic to the algebra bundle constructed directly via the clutching construction with gluing map f.*

Proof. For brevity, let $\mathcal{A}_f$ be the algebra bundle with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ constructed via the clutching construction. We aim to show that $\mathcal{E}_f \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}) \cong \mathcal{A}_f$. To do so, we let $\varphi : \mathcal{E}_f \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K}) \to \mathcal{A}_f$ be defined by $\varphi([[T, x], o]) = [T(o), x]$. If $x \in S^{2n}$ with $x_+ \in D_+^{2n+1}$ and $x_- \in D_-^{2n+1}$ both representing x then we see that $[[T, x_+], o] = [[f(x) \circ T, x_-], o]$ but

$$\varphi([[f(x) \circ T, x_-], o]) = [(f(x) \circ T)(o), x_-] = [T(o), x_+] = \varphi([[T, x_+], o]).$$

Furthermore, for any $T' \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ we have $[[T, x], o] = [[T \circ T', x], T'^{-1}(o)]$, but then $\varphi([[T \circ T', x], T'^{-1}(o)]) = [T(o), x]$ and hence φ is well-defined. Continuity follows by constructing a lift

$$\widetilde{\varphi} : \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \times X \times (\mathcal{O}_\infty \otimes \mathcal{K}) \to \mathcal{A}_f$$

which maps (T, x, o) to $(x, T(o))$ such that $\widetilde{\varphi} = \varphi \circ q$ where q denotes the quotient map. Since $\widetilde{\varphi}$ is clearly continuous then properties of the quotient topology imply that φ is continuous. Now, defining $\psi : \mathcal{A}_f \to \mathcal{E}_f \times_{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} (\mathcal{O}_\infty \otimes \mathcal{K})$ by $\psi([o, x]) = [[I, x], o]$ we see that for any $x \in S^{2n}$, $[o, x_+] = [f(x)(o), x_-]$ but

$$\psi([f(x)(o), x_-]) = [[I, x_-], f(x)(o)] = [[f(x), x_-], o] = [[I, x_+], o],$$

and so ψ is well-defined. Again it is clear that ψ is continuous. Then we have

$$\begin{aligned}
&(\varphi \circ \psi)([o, x]) = \varphi([[I, x], o]) = [o, x];\\
&(\psi \circ \varphi)([[T, x], o]) = \psi([T(o), x]) = [[I, x], T(o)] = [[T, x], o];
\end{aligned}$$

hence $\psi = \varphi^{-1}$ and so ψ is a homeomorphism. Finally, restricting to the fibre over $x \in S^{2n+1}$ gives $\psi_x : \mathcal{O}_\infty \otimes \mathcal{K} \to \mathcal{O}_\infty \otimes \mathcal{K}$ to be the identity and thus ψ is an isomorphism of algebra bundles. □

To finish this section, we briefly make some more general remarks regarding the clutching construction. As mentioned at the beginning of the section, this construction is more general than what we have presented here for spheres. In fact, given data consisting of any cover of a space X and a principal bundle over the disjoint union with certain isomorphism conditions imposed on points in the disjoint union which are identified to construct X, a principal bundle over X can be constructed. For the sake of simplicity, we restrict our attention to spaces which can be covered by two sets as was the case with the sphere. For example, by viewing complex projective space $\mathbb{C}P^n$ as the quotient of the disk D^{2n} by the equivalence relation identifying antipodal points on the boundary, $\mathbb{C}P^n$ can be covered by the image of a set containing a neighbourhood of the boundary of the disk under the projection map and a set which does not contain the boundary. The latter of these sets is contractible and the intersection is homeomorphic to S^{2n-1}, but the former is topologically more complicated, meaning that the principal bundles over this set would need to be better understood in order to construct any general principal bundle over $\mathbb{C}P^n$. This method could still be used to construct some principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over $\mathbb{C}P^n$, even if it is not possible to construct all in this way. Even more simply, this construction can be used to construct principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over products of spheres containing at least one odd-dimensional sphere where said sphere is split into two hemispheres as above, as we will see in Chapter 5.

3.2 Decomposable twists

Another approach to constructing geometric representatives for twists of K-theory is to consider general spaces X but to simplify the twisting cohomology class. One way in which to do this is to consider decomposable classes, because there already exist bundle-theoretic representatives for some low-dimensional cohomology classes.

To begin with, let X be a finite connected CW-complex with torsion-free cohomology so that we are in the setting of Theorem 2.4.1. Suppose that $\delta \in H^5(X, \mathbb{Z})$ decomposes as $\delta = \alpha \cup \beta$ with $\alpha \in H^2(X, \mathbb{Z})$ and $\beta \in H^3(X, \mathbb{Z})$. By the standard identification $H^n(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)]$ where $K(G, n)$ denotes an Eilenberg–Mac Lane space, i.e. a space whose only non-trivial homotopy group is G in degree n, and the fact that there exist simple geometric models for $K(\mathbb{Z}, n)$ in the case that $n = 1, 2, 3$, we identify δ with $H_5 : X \to K(\mathbb{Z}, 5)$ such that $H_5 = H_2 \wedge H_3$ with $H_2 : X \to BU(1)$ and $H_3 : X \to BPU$. Then H_2 determines a principal $U(1)$-bundle

$$\begin{array}{ccc} U(1) & \longrightarrow & P_{H_2} \\ & & \downarrow{\scriptstyle \pi_{H_2}} \\ & & X \end{array}$$

with Chern class α, and similarly H_3 determines a principal PU-bundle

$$\begin{array}{ccc} PU & \longrightarrow & Q_{H_3} \\ & & \big\downarrow \pi_{H_3} \\ & & X \end{array}$$

with Dixmier–Douady invariant β. We form the fibred product bundle over X, whose total space is $P_{H_2} \times_X Q_{H_3} = \{(p,q) \in P_{H_2} \times Q_{H_3} : \pi_{H_2}(p) = \pi_{H_3}(q)\}$, and this gives us a principal $U(1) \times PU$-bundle

$$\begin{array}{ccc} U(1) \times PU & \longrightarrow & P_{H_2} \times_X Q_{H_3} \\ & & \big\downarrow \pi \\ & & X. \end{array}$$

Such principal $U(1) \times PU$-bundles over X are classified by homotopy classes of maps from X into $B(U(1) \times PU) \simeq BU(1) \times BPU \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, i.e. elements of $H^2(X,\mathbb{Z}) \times H^3(X,\mathbb{Z})$. This shows not only that $\alpha \cup \beta$ is an invariant of the bundle $P_{H_2} \times_X Q_{H_3}$, but also that any principal $U(1) \times PU$-bundle arises in this way from a pair of cohomology classes.

Now, to this principal bundle we wish to associate a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X, to obtain a twist of K-theory. This is done by defining an injective group homomorphism from the structure group $U(1) \times PU$ into the automorphism group $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, or equivalently an effective action of the structure group $U(1) \times PU$ on the algebra $\mathcal{O}_\infty \otimes \mathcal{K}$. Since PU is isomorphic to the automorphism group of $\mathcal{K}$ by conjugation, we have the obvious action $PU \xrightarrow{\cong} \mathrm{Aut}(\mathcal{K})$, so we seek an effective action of $U(1)$ on $\mathcal{O}_\infty$.

As noted in Section 3 of [KK97], there is a one-parameter automorphism group of $\mathcal{O}_\infty$ obtained by scaling the generators as follows. Letting λ_k for $k = 1, 2, \cdots$ be a sequence of real constants, we obtain a map $\gamma : \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_\infty)$ defined by $\gamma_t(S_k) = e^{i\lambda_k t} S_k$ for $k = 1, 2, \cdots$ where the S_k are the generators in the definition of the Cuntz algebra $\mathcal{O}_\infty$. Then taking $\lambda_k = 2k\pi$ we see that γ is periodic in t with a period of 1. In fact, this is a special case of the action that we described in Theorem 2.2.7, and thus we can view it as a map $\gamma : U(1) \to \mathrm{Out}(\mathcal{O}_\infty)$, yielding the desired action.

Out of our maps $U(1) \to \mathrm{Aut}(\mathcal{O}_\infty)$ and $PU \to \mathrm{Aut}(\mathcal{K})$, we obtain the product map $U(1) \times PU \to \mathrm{Aut}(\mathcal{O}_\infty) \times \mathrm{Aut}(\mathcal{K})$. Then by Corollary T.5.19 of [WO93] we see that the tensor product of two automorphisms of C^*-algebras is an automorphism of the tensor product algebra, and thus $\mathrm{Aut}(\mathcal{O}_\infty) \times \mathrm{Aut}(\mathcal{K}) \subset \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. Finally, since the map $U(1) \to \mathrm{Aut}(\mathcal{O}_\infty)$ is given by scaling generators whereas $PU = U(\mathcal{H})/U(1)$ acts by conjugation on $\mathcal{K}$, there is no non-trivial action of the $U(1)$ factor on the $\mathcal{K}$ component or of the PU factor on the $\mathcal{O}_\infty$ component and hence the map $U(1) \times PU \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ that we have constructed is injective.

Thus we may form the associated bundle $(P_{H_2} \times_X Q_{H_3}) \times_{U(1)\times PU} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, which is a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X. As the 5-class $\alpha \cup \beta$ is an invariant of the principal $U(1) \times PU$-bundle, this is a prime candidate for the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X which corresponds to the class $\alpha \cup \beta$ under the isomorphism of Dadarlat and Pennig. Due to the inexplicit nature of the isomorphism, however, it is not immediate that this will indeed be the correct bundle. In order to get around this issue, we apply the techniques of Dadarlat and Pennig used to obtain the isomorphism between principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles and cohomology to the case of principal $U(1) \times PU$-bundles.

Section 3 of [DP16] is dedicated to proving that $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ is an infinite loop space, allowing the authors to define the generalised cohomology theory ${E_{\mathcal{O}_\infty}}^\bullet$ which we discuss in Section 2.3 and of which the first group classifies the twists of K-theory. In this case, we get this for free for our space $U(1) \times PU$ as this is a model for $K(\mathbb{Z},1) \times K(\mathbb{Z},2)$, and the Eilenberg–Mac Lane spaces are the infinite loop spaces defining ordinary cohomology. Hence we are able to define a cohomology theory ${E_{U(1)\times PU}}^n(X) = [X, B^n(U(1) \times PU)]$. In particular, we have that the first group of this cohomology theory classifies the principal $U(1) \times PU$-bundles over X. We will use the spectral sequence technique to show that this group is isomorphic to $H^2(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z})$, and conclude that the isomorphism between bundles and cohomology is what we expect it to be.

Using the homotopy groups of the Eilenberg–Mac Lane spaces $U(1)$ and PU and the fact that homotopy groups of products are products of homotopy groups, we see that

$$\pi_i(U(1) \times PU) \cong \begin{cases} \mathbb{Z} & \text{if } i = 2 \text{ or } 3; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore E_2-term of the Atiyah–Hirzebruch spectral sequence for the generalised cohomology theory ${E_{U(1)\times PU}}^\bullet$ is as follows.

	0	1	2	3	$\cdots$
0	0	0	0	0	$\cdots$
-1	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$\cdots$
-2	$H^0(X,\mathbb{Z})$	$H^1(X,\mathbb{Z})$	$H^2(X,\mathbb{Z})$	$H^3(X,\mathbb{Z})$	$\cdots$
-3	0	0	0	0	$\cdots$

Then since the differentials in the Atiyah–Hirzebruch spectral sequence are torsion operators by Theorem 2.7 of [Arl92], and assuming that X has torsion-free cohomology, we obtain an isomorphism

$$E_{U(1)\times PU}{}^{1}(X) \cong H^2(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z}).$$

But we already have an isomorphism between $H^2(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z})$ and the isomorphism classes of principal $U(1) \times PU$-bundles over X. Therefore we conclude that this map assigns to a principal bundle its invariant 2-class and 3-class in the sense described earlier. Furthermore, by taking the cup product of these classes we view this as a map

$$[X, B(U(1) \times PU)] \to H^5(X,\mathbb{Z})$$

induced by the isomorphism. Now, since this map was constructed via the same spectral sequence argument as the isomorphism of Dadarlat and Pennig, we conclude that the diagram

$$\begin{array}{ccc} [X, B(U(1)\times PU)] & \longrightarrow & H^5(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ [X, B\,\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] & \longrightarrow & H^1(X,\mathbb{Z}_2) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X,\mathbb{Z}) \end{array}$$

commutes, where the left vertical map takes a principal $U(1) \times PU$-bundle over X to the associated $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle via our injective group homomorphism. This allows us to conclude that the principal bundle $(P_{H_2} \times_X Q_{H_3}) \times_{U(1)\times PU} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ constructed earlier truly does correspond to the 5-class $\alpha \cup \beta$ under the isomorphism of Dadarlat and Pennig.

We now extend this argument to the case in which δ is a general element of the cup product of $H^2(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$, i.e. δ is given by a sum of N decomposable classes of the form considered above. We take $\alpha \in H^2(X,\mathbb{Z}^N)$ and $\beta \in H^3(X,\mathbb{Z}^N)$ such that $\delta = \langle \alpha \,|\beta\rangle$ where $\langle \cdot\,|\cdot\rangle$ is the pairing $H^2(X,\mathbb{Z}^N) \times H^3(X,\mathbb{Z}^N) \to H^5(X,\mathbb{Z})$ given by the cup product and the standard inner product $\mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}$. As above, we identify α with a map $H_2 : X \to BU(1)^N$ and β with a map $H_3 : X \to BPU^N$, and form the principal torus bundle

$$\begin{array}{ccc} U(1)^N & \longrightarrow & P_{H_2} \\ & & \big\downarrow{\scriptstyle \pi_{H_2}} \\ & & X \end{array}$$

with Chern class α and the principal PU^N-bundle

$$\begin{array}{ccc} PU^N & \longrightarrow & Q_{H_3} \\ & & \big\downarrow{\scriptstyle \pi_{H_3}} \\ & & X \end{array}$$

with Dixmier–Douady invariant β. Once again we take the fibred product to obtain the principal bundle

$$\begin{array}{ccc} U(1)^N \times PU^N & \longrightarrow & P_{H_2} \times_X Q_{H_3} \\ & & \big\downarrow \pi \\ & & X, \end{array}$$

and using $B(U(1)^N \times PU^N) \simeq BK(\mathbb{Z},2)^N \times BK(\mathbb{Z},3)^N$ we see that δ will be an invariant of this bundle. This time, we require an injective map $U(1)^N \times PU^N \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ to adapt the previous argument and construct the associated bundle.

Again by Corollary T.5.19 of [WO93] and the fact that $\mathcal{O}_\infty$ is nuclear, we see that $\mathrm{Aut}(\mathcal{O}_\infty)^N \subset \mathrm{Aut}(\mathcal{O}_\infty^{\otimes N}) = \mathrm{Aut}(\mathcal{O}_\infty)$. We may then combine two copies of our action $\gamma, \gamma' : U(1) \to \mathrm{Aut}(\mathcal{O}_\infty)$ described previously to form an injective map $U(1)^2 \to \mathrm{Aut}(\mathcal{O}_\infty^{\otimes 2})$ given by $(\gamma \otimes \gamma')_{(t,t')}(S_k \otimes S_{k'}) = e^{2\pi i(kt+k't')} S_k \otimes S_{k'}$. This argument can be extended to N maps $U(1) \to \mathrm{Aut}(\mathcal{O}_\infty)$ in the same way, and therefore the map $U(1)^N \to \mathrm{Aut}(\mathcal{O}_\infty)$ that we construct is injective. More simply, the map $PU^N \to \mathrm{Aut}(\mathcal{K})$ is injective because the automorphisms do not involve scaling by constants. Thus we obtain our desired injective group homomorphism $U(1)^N \times PU^N \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$.

We then construct the associated bundle $(P_{H_2} \times_X Q_{H_3}) \times_{U(1)^N \times PU^N} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ over X, at which point we apply the same argument as previously. We have that $U(1)^N \times PU^N$ is an infinite loop space, and using the same spectral sequence technique we obtain an isomorphism $[X, B(U(1)^N \times PU^N)] \cong H^2(X, \mathbb{Z}^N) \oplus H^3(X, \mathbb{Z}^N)$ which we view as a map $[X, B(U(1)^N \times PU^N)] \to H^5(X, \mathbb{Z})$. Then the diagram

$$\begin{array}{ccc} \big[X, B(U(1)^N \times PU^N)\big] & \longrightarrow & H^5(X,\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ [X, B\,\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})] & \longrightarrow & H^1(X,\mathbb{Z}_2) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X,\mathbb{Z}) \end{array}$$

commutes, and thus the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle constructed in this section does correspond to δ under the isomorphism.

We have proved the following.

Theorem 3.2.1. *Let X be a finite connected CW-complex with torsion-free cohomology, and take $\alpha \in H^2(X, \mathbb{Z}^N)$ and $\beta \in H^3(X, \mathbb{Z}^N)$ with $\delta = \langle \alpha \,|\beta \rangle$. Denote by P_α the total space of the principal $U(1)$-bundle with Chern class α and by Q_β the total space of the principal PU-bundle with Dixmier–Douady invariant β. Then $(P_\alpha \times_X Q_\beta) \times_{U(1)^N \times PU^N} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ is a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X which corresponds to δ under the isomorphism of Dadarlat and Pennig.*

Based on this result, the natural question to ask is whether decomposable classes in higher degrees may be represented in a similar way. For example, we could take

$\alpha_1 \cup \alpha_2 \cup \beta \in H^2(X,\mathbb{Z}) \cup H^2(X,\mathbb{Z}) \cup H^3(X,\mathbb{Z}) \subset H^7(X,\mathbb{Z})$ and aim to construct a bundle in much the same way. One could follow the same constructions as above in order to do so, and would obtain an extension of the result.

Example 3.2.1. We will outline the most straightforward application of this result; to the space $S^2 \times S^3$. This space has torsion-free cohomology, and the cup product of the generators of the second- and third-degree cohomology groups is the generator of $H^5(S^2 \times S^3, \mathbb{Z})$. Thus we may apply the constructions detailed above to obtain geometric representatives for all twists of degree 5 over this space.

Given $\delta \in H^5(S^2 \times S^3, \mathbb{Z})$, this will be of the form $N\alpha \cup \beta$ for $N \in \mathbb{Z}$ and α, β generators of the second- and third-degree integral cohomology groups respectively. Then we may construct the principal $U(1)$-bundle over $S^2 \times S^3$ with Chern class $N\alpha$, which is simply the $U(1)$-bundle over S^2 with Chern class $N\alpha$ extended trivially over the S^3 factor, and similarly construct the principal PU-bundle over $S^2 \times S^3$ with Dixmier–Douady invariant β which will be the corresponding PU-bundle over S^3 extended trivially over S^2. Taking the fibred product of these bundles then yields the principal $U(1) \times PU$-bundle over $S^2 \times S^3$ with invariant $(N\alpha, \beta)$, and lastly taking the associated bundle via our injective group homomorphism $U(1) \times PU \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ we obtain the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $S^2 \times S^3$ which is classified by δ.

Note that in this example we viewed the class $N\alpha \cup \beta$ as $(N\alpha) \cup \beta$, but we equivalently could have used $\alpha \cup (N\beta)$ or split N into two factors if possible. This is a limitation of our approach; while the bundles constructed in these slightly different ways are necessarily isomorphic, it is not obvious from the construction that they will be isomorphic.

More generally than the limitation above, it is even difficult to tell whether a bundle constructed in this way is trivial. For instance, taking a space such as $S^2 \times S^1$ which has non-trivial second and third degree integral cohomology, this construction can be used to form a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $S^1 \times S^2$ represented by an element of the fifth-degree cohomology, which is trivial. It is not obvious from the construction, however, that such a bundle is necessarily trivial. More generally, taking any assortment of 2- and 3-classes whose cup product is zero, it is not obvious that the bundle constructed in this way will be trivial.

We end this chapter with some discussion of directions for future research motivated by these results. Whilst Theorem 3.2.1 provides geometric representatives for a specific class of decomposable twists, the majority of twists cannot be decomposed into pieces as simple as these. Thus it would be desirable to obtain a more general theorem which is applicable to a wider class of decomposable twists, but it is difficult to obtain geometric representatives for higher degree cohomology classes. If models for higher Eilenberg–Mac Lane spaces could be obtained, and injective group homomorphisms from these into $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ could be determined, then the methods used would directly generalise to provide geometric representatives in higher degrees. Alternatively, if results in representing cohomology classes using maps into the stable unitary group could be obtained in some special cases, then the action by outer automorphisms explored in Subsection 2.2.2 could be used to extend the results presented here.

In general, the problem of associating geometric representatives to cohomology classes is very difficult. A large amount of work has been done on this for classical twists using the theory of loop groups and transgression of cohomology classes, but it is not apparent how this work can be carried over to the higher twisted setting. Further research in this area following these ideas may yield more general results.

Chapter 4

Product structure and spectral sequences

Besides the basic results such as functoriality and the existence of a Mayer–Vietoris sequence, there are additional properties and tools expected to generalise to higher twisted K-theory from the classical case. The first of these which we shall explore is the existence of a product on higher twisted K-theory, and while this does not equip higher twisted K-theory with the structure of a graded ring as such it does make it into a graded module over topological K-theory. The second is the existence of spectral sequences for higher twisted K-theory, which will be critical to our computations.

In order to work towards a graded module structure on higher twisted K-theory, we firstly develop a general external product between the higher twisted K-theory groups of spaces X and Y using Fredholm operators, which mirrors the construction in the setting of topological K-theory. Restricting this to the case $X = Y$ we obtain something which resembles a graded ring structure, but where the product of $K^*(X, \delta_1)$ and $K^*(X, \delta_2)$ lands in $K^*(X, \delta_1 + \delta_2)$ for a suitable notion of addition of twists. Taking one of these twists to be trivial, we obtain an explicit realisation of the graded module structure. We also briefly discuss the graded module structure using operator algebraic K-theory.

The second half of this chapter is dedicated to the development of spectral sequences, which are tools that will greatly improve our ability to perform computations in the final chapter. We show the existence of an Atiyah–Hirzebruch spectral sequence and a Segal spectral sequence in higher twisted K-theory, both of which are expected for any generalised cohomology theory, but in particular we obtain some information regarding the differentials in these sequences. While our results are limited in that we are only able to provide an expression for one differential in the Atiyah–Hirzebruch spectral sequence, this will still prove to be very useful when it comes to computations.

Note that we only outline some constructions and arguments in this chapter, particularly concerning the external product, and do not work out all details. The results on the product are not used in the remainder of the thesis.

4.1 External product and graded module structure

We firstly aim to develop an external product map allowing multiplication between higher twisted K-theory groups of different spaces. Let X and Y be topological spaces, and let $\mathcal{E}_{\delta_X}$ and $\mathcal{E}_{\delta_Y}$ denote principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over X and Y respectively corresponding to twists δ_X and δ_Y. Denote by p_X and p_Y projection from the product space $X \times Y$ to X and Y respectively. In order to define a product which lands in the higher twisted K-theory of $X \times Y$, we need a sensible notion of the sum of the twists δ_X over X and δ_Y over Y. Since the bundles are over different spaces, the first step towards this is pulling them both back to $X \times Y$ under the respective projection maps. We then need a suitable notion of the product of the bundles, which is provided in Section 3.2 of [DP16].

Pennig and Dadarlat develop a tensor product for principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles with respect to which the higher Dixmier–Douady classes are additive. We outline their construction here. Firstly, by the nuclearity of $\mathcal{O}_\infty \otimes \mathcal{K}$ we fix once and for all an isomorphism $\psi : \mathcal{O}_\infty \otimes \mathcal{K} \to (\mathcal{O}_\infty \otimes \mathcal{K}) \otimes (\mathcal{O}_\infty \otimes \mathcal{K})$, and induce a group homomorphism $\mathrm{Ad}_{\psi^{-1}} : \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ via $(\alpha, \beta) \mapsto \psi^{-1} \circ (\alpha \otimes \beta) \circ \psi$. Let P_1 and P_2 be principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over X. By taking the fibred product $P_1 \times_X P_2$ we obtain a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over X, and then we use ψ to define

$$P_1 \otimes_\psi P_2 = (P_1 \times_X P_2) \times_{\mathrm{Ad}_{\psi^{-1}}} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) = ((P_1 \times_X P_2) \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))/\sim \quad (4.1.1)$$

where $(p_1 \cdot \alpha, p_2 \cdot \beta, \gamma) \sim (p_1, p_2, \mathrm{Ad}_{\psi^{-1}}(\alpha, \beta)\gamma)$ for all $(p_1, p_2) \in P_1 \times_X P_2$ and for every $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. Pennig and Dadarlat show that different choices of ψ yield equivalent products in Lemma 3.4 of [DP16], so we denote the tensor product in (4.1.1) simply by $\otimes$. They also prove that the higher Dixmier–Douady invariants are additive with respect to the tensor product in Definition 4.6 [DP16].

Returning to the setting of interest, the twist $p_X^*\delta_X + p_Y^*\delta_Y$ over $X \times Y$ will be represented by the bundle $p_X^*\mathcal{E}_{\delta_X} \otimes p_Y^*\mathcal{E}_{\delta_Y}$. Thus we are seeking a map of the form

$$K^m(X, \delta_X) \times K^n(Y, \delta_Y) \to K^{m+n}(X \times Y, p_X^*\delta_X + p_Y^*\delta_Y), \quad (4.1.2)$$

and using our identification with Fredholm operators in Theorem 2.5.1 this is a map

$$\begin{aligned}[]
[\mathcal{E}_{\delta_X}, \Omega^m \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} &\times [\mathcal{E}_{\delta_Y}, \Omega^n \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} \\
&\to [p_X^*\mathcal{E}_{\delta_X} \otimes p_Y^*\mathcal{E}_{\delta_Y}, \Omega^{m+n} \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}.
\end{aligned} \quad (4.1.3)$$

The first ingredient that we will need is a tensor product operation on $\operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$, which will be central to the development of the product (4.1.3). To simplify this discussion, we will also need a notion of index for generalised Fredholm operators, which requires an understanding of the homotopy groups of the space of these operators.

Lemma 4.1.1. *The space of Fredholm operators on the standard Hilbert $(\mathcal{O}_\infty \otimes \mathcal{K})$-module $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ has homotopy groups*

$$\pi_i(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Using the reduced version of Theorem 2.5.1 we see, for δ a trivial twist represented by the trivial bundle $S^n \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, that

$$\begin{aligned} \widetilde{K}^i(S^n, \delta) &= [S^n \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}), \Omega^i((\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0)]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})} \\ &= [S^n, \Omega^i((\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0)] \\ &= [S^{n+i}, (\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0] \end{aligned}$$

where we have identified the G-equivariant maps $X \times G \to Y$ with maps $X \to Y$. Now, since $(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$ is simply connected, which follows from path connectedness and the fact that each loop is unbased nullhomotopic by $[S^1, (\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0] = \widetilde{K}^0(S^1) = 0$, this is equal to the based homotopy classes of maps $S^{n+i} \to (\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$, which is equal to $\pi_{n+i}((\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0)$. The same argument shows that $\widetilde{K}^i(S^n) = \pi_{n+i}(\mathrm{Fred}_0)$. Now, by Proposition 2.3.10 we see that the K-theory of S^n twisted by the trivial twist δ is equal to the topological K-theory of S^n, and therefore $\pi_i((\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0) = \pi_i((\mathrm{Fred})_0)$. Thus

$$\pi_i(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}) = \pi_i((\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0) = \pi_i((\mathrm{Fred})_0) = \pi_i(\mathrm{Fred}),$$

and as the homotopy groups of the standard Fredholm operators are well-known, this completes the proof. □

We will now develop a tensor product on $(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$, the Fredholm operators on $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ of index zero or equivalently the connected component of the identity of $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$. For $S, T \in (\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$, we define

$$S \widehat{\otimes} T = S \otimes I + I \otimes T, \tag{4.1.4}$$

which is a Fredholm operator of index zero on $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \otimes \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$. Here, we use our isomorphism $\psi : \mathcal{O}_\infty \otimes \mathcal{K} \to (\mathcal{O}_\infty \otimes \mathcal{K}) \otimes (\mathcal{O}_\infty \otimes \mathcal{K})$ from earlier to induce an isomorphism $\psi_{\mathcal{H}} : \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \to \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \otimes \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ using the same notation as in (2.5.1). Thus we may view $S \widehat{\otimes} T$ as an element of $(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$, so this yields a well-behaved tensor product operation on $(\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}})_0$.

Although this tensor product has a simple form, the general product on $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ is more difficult to write explicitly. We desire the index to be multiplicative with respect to tensor products, but if the definition (4.1.4) were used in general then $S \widehat{\otimes} T$ would have index $\dim(\ker(S))\dim(\ker(T)) - \dim(\ker(S^*))\dim(\ker(T^*))$, which is not equal to $\mathrm{ind}(S)\,\mathrm{ind}(T)$ in general. Thus the tensor product must be adapted to work for the general case, which requires the use of a $\mathbb{Z}_2$-graded Hilbert module. We follow the work of Jänich [Jän65] in making the following definition.

Note that given an operator $T \in \mathcal{B}(\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \otimes \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}})$, we can conjugate by $\psi_{\mathcal{H}}$ to obtain an operator $\psi_{\mathcal{H}}^{-1} T \psi_{\mathcal{H}} \in \mathcal{B}(\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}})$. We use this in the following.

Definition 4.1.2. Let S and T be Fredholm operators on the $\mathbb{Z}_2$-graded Hilbert module $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$. The product of S and T is

$$S\widehat{\otimes}T = \begin{pmatrix} \psi_{\mathcal{H}}^{-1} S\otimes I\psi_{\mathcal{H}} & -\psi_{\mathcal{H}}^{-1} I\otimes T^*\psi_{\mathcal{H}} \\ \psi_{\mathcal{H}}^{-1} I\otimes T\psi_{\mathcal{H}} & \psi_{\mathcal{H}}^{-1} S^*\otimes I\psi_{\mathcal{H}} \end{pmatrix}. \tag{4.1.5}$$

Letting $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} = \mathcal{H}_0 \oplus \mathcal{H}_1$ denote the $\mathbb{Z}_2$-grading, although for instance $\psi_{\mathcal{H}}^{-1} S\otimes I\psi_{\mathcal{H}}$ is a map $\mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$, it is identified with a map $\mathcal{H}_0 \to \mathcal{H}_0$ via an inclusion $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \otimes \mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}} \subset \mathcal{H}_0$. The details of this will not concern us.

First of all, we require this tensor product to indeed produce a Fredholm operator. In order to move towards a product on higher twisted K-theory, we also desire this product to have properties such as associativity and commutativity up to homotopy. We will not provide the details of these, but the proofs in the case of the equivalent product on ordinary Fredholm operators can be found in Lemma 2 of [Jän65].

Lemma 4.1.3. *The product of Fredholm operators on $\mathcal{H}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ satisfies the following properties.*

(i) The product of Fredholm operators is a Fredholm operator;

(ii) The index of the product of Fredholm operators is the product of the indices;

(iii) For $A, B, C \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$, the operators $A\widehat{\otimes}(B\widehat{\otimes}C)$ and $(A\widehat{\otimes}B)\widehat{\otimes}C$ are homotopic through Fredholm operators;

(iv) For $A, B, C \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$, the operators $A\widehat{\otimes}(B\circ C)$ and $(A\widehat{\otimes}B)\circ(A\widehat{\otimes}C)$ are homotopic through Fredholm operators;

(v) There exists an $E \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ such that $E\widehat{\otimes}A$ and $A\widehat{\otimes}E$ are homotopic to A for all $A \in \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$.

This allows us to define the product (4.1.3) in the case that $m = n = 0$. Letting $F \in [\mathcal{E}_{\delta_X}, \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})}$, we pull F back to a map $p_X^*F : p_X^*\mathcal{E}_{\delta_X} \to \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$. We do the same for $G \in [\mathcal{E}_{\delta_Y}, \mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty\otimes\mathcal{K})}$, and then define the product of F and G to be the map

$$F\widehat{\otimes}G : (q_1, q_2, \gamma) \mapsto (p_X^*F(p_X^*q_1)\widehat{\otimes}p_Y^*G(p_Y^*q_2)) \cdot \gamma, \tag{4.1.6}$$

for $[(q_1, q_2, \gamma)] \in (p_X^*\mathcal{E}_{\delta_X} \times_{X\times Y} p_Y^*\mathcal{E}_{\delta_Y}) \times_{\mathrm{Ad}_{\psi^{-1}}} \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$, and where $p_X^* : p_X^*\mathcal{E}_{\delta_X} \to \mathcal{E}_{\delta_X}$ is the map induced by pullback of bundles. We observe that this product is clearly $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-equivariant as desired. Now, we must ensure that this product is well-defined on the tensor product bundle.

Lemma 4.1.4. *The product $F\widehat{\otimes}G$ defined in (4.1.6) is well-defined.*

Proof. Let $(q_1, q_2) \in p_X^*\mathcal{E}_{\delta_X} \times_{X\times Y} p_Y^*\mathcal{E}_{\delta_Y}$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. Then

$$\begin{aligned}(F\widehat{\otimes}G)(q_1\cdot\alpha, q_2\cdot\beta, \gamma) &= (p_X^*F(p_X^*q_1\cdot\alpha)\widehat{\otimes}p_Y^*G(p_Y^*q_2\cdot\beta))\cdot\gamma \\ &= ((p_X^*F(p_X^*q_1)\cdot\alpha)\widehat{\otimes}(p_Y^*G(p_Y^*q_2)\cdot\beta))\cdot\gamma\end{aligned}$$

using the $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-equivariance of F and G. Now, using the definition of the conjugation action of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ on $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ we see that

$$\begin{aligned}&(F\widehat{\otimes}G)(q_1\cdot\alpha, q_2\cdot\beta, \gamma) \\ =\ & ((\alpha^{-1}{}_{\mathcal{H}}p_X^*F(p_X^*q_1)\alpha_{\mathcal{H}})\widehat{\otimes}(\beta^{-1}{}_{\mathcal{H}}p_Y^*G(p_Y^*q_2)\beta_{\mathcal{H}}))\cdot\gamma \\ =\ & \begin{pmatrix}\psi_{\mathcal{H}}^{-1}\alpha^{-1}{}_{\mathcal{H}}p_X^*F(p_X^*q_1)\alpha_{\mathcal{H}}\otimes I\psi_{\mathcal{H}} & -\psi_{\mathcal{H}}^{-1}I\otimes(\beta^{-1}{}_{\mathcal{H}}p_Y^*G(p_Y^*q_2)\beta_{\mathcal{H}})^*\psi_{\mathcal{H}} \\ \psi_{\mathcal{H}}^{-1}I\otimes\beta^{-1}{}_{\mathcal{H}}p_Y^*G(p_Y^*q_2)\beta_{\mathcal{H}}\psi_{\mathcal{H}} & \psi_{\mathcal{H}}^{-1}(\alpha^{-1}{}_{\mathcal{H}}p_X^*F(p_X^*q_1)\alpha_{\mathcal{H}})^*\otimes I\psi_{\mathcal{H}}\end{pmatrix}\cdot\gamma \\ =\ & \left(\psi_{\mathcal{H}}^{-1}\circ(\alpha^{-1}{}_{\mathcal{H}}\otimes\beta^{-1}{}_{\mathcal{H}})\begin{pmatrix}p_X^*F(p_X^*q_1)\otimes I & -I\otimes(p_Y^*G(p_Y^*q_2))^* \\ I\otimes p_Y^*G(p_Y^*q_2) & (p_X^*F(p_X^*q_1))^*\otimes I\end{pmatrix}(\alpha_{\mathcal{H}}\otimes\beta_{\mathcal{H}})\circ\psi_{\mathcal{H}}\right)\cdot\gamma \\ =\ & \left(\psi_{\mathcal{H}}^{-1}\circ(\alpha^{-1}{}_{\mathcal{H}}\otimes\beta^{-1}{}_{\mathcal{H}})\circ\psi_{\mathcal{H}}(p_X^*F(p_X^*q_1)\widehat{\otimes}p_Y^*G(p_Y^*q_2))\psi^{-1}{}_{\mathcal{H}}\circ(\alpha_{\mathcal{H}}\widehat{\otimes}\beta_{\mathcal{H}})\circ\psi_{\mathcal{H}}\right)\cdot\gamma \\ =\ & (p_X^*F(p_X^*q_1)\widehat{\otimes}p_Y^*G(p_Y^*q_2))\cdot\mathrm{Ad}_{\psi^{-1}}(\alpha,\beta)\gamma \\ =\ & (F\widehat{\otimes}G)(q_1, q_2, \mathrm{Ad}_{\psi^{-1}}(\alpha,\beta)\gamma).\end{aligned}$$

So the product is well-defined as required. □

To extend this product to the groups of higher degree, we sacrifice the level of explicitness that we have used so far. In order to define a product between higher twisted K-theory groups of degree m and n, we need a product

$$\boxtimes : \Omega^m\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}} \times \Omega^n\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}} \to \Omega^{m+n}\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}} \tag{4.1.7}$$

which restricts to the tensor product $\widehat{\otimes}$ when $m = n = 0$. This product, however, is not simply defined to be the pointwise version of $\widehat{\otimes}$ as this would not yield an element of the $(m+n)$th iterated loop space of $\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ as is required. While we do not have an explicit form for the map $\boxtimes$, its existence is established by that of the analogous map on the ordinary Fredholm operators, which comes from the ring structure on topological K-theory.

Lemma 4.1.5. *There is a product $\boxtimes$ as in (4.1.7) which is associative, commutative and unital up to homotopy, and which is equal to the product (4.1.5) when $m = n = 0$.*

Then using this map, we are able to define our general external product (4.1.3) to be

$$F\boxtimes G : (q_1, q_2, \gamma) \mapsto (p_X^*F(p_X^*q_1)\boxtimes p_Y^*G(p_Y^*q_2))\cdot\gamma. \tag{4.1.8}$$

This product is then $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-equivariant, and the properties of the product stated in Lemma 4.1.5 carry over to the external product on higher twisted K-theory.

Proposition 4.1.6. *The external product on higher twisted K-theory (4.1.2) is associative, commutative and unital.*

The product also reduces to the ordinary external product in topological K-theory as discussed in, for instance, Section 2.4 of [Ati67].

Proposition 4.1.7. *When trivial twists are taken over X and Y, the external product (4.1.2) reduces to the ordinary external product in topological K-theory.*

Proof. When δ_X and δ_Y are trivial, the bundles $\mathcal{E}_{\delta_X}$ and $\mathcal{E}_{\delta_Y}$ are both trivial principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles. This means that

$$K^n(X, \delta_X) = [X \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}), \Omega^n \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} = [X, \Omega^n \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}], \quad (4.1.9)$$

where $F : X \to \Omega^n \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ is obtained from $\widetilde{F} : X \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \to \Omega^n \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ via $F(x) = \widetilde{F}(x, I)$. Now, $p_X^* \mathcal{E}_{\delta_X}$ and $p_Y^* \mathcal{E}_{\delta_Y}$ are both trivial principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over $X \times Y$, and the fibred product of these is the trivial $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}) \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $X \times Y$. Taking the associated bundle via $\mathrm{Ad}_{\psi^{-1}}$ then yields the trivial $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $X \times Y$. Thus, making the identifications (4.1.9), the product (4.1.8) is simply

$$(x, y, \gamma) \mapsto (F(x) \boxtimes G(y)) \cdot \gamma.$$

Furthermore, identifying $[X \times Y \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}), \Omega^{m+n} \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$ with the group $[X \times Y, \Omega^{m+n} \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]$ removes the $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ component from the map, and hence it reduces to $(x, y) \mapsto F(x) \boxtimes G(y)$. Finally, identifying $[X, \Omega^n \mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]$ with $[X, \Omega^n \mathrm{Fred}]$ this is the same as the external product on topological K-theory by the formulation of the product as required. □

We now restrict to a less general setting to move towards a module structure on higher twisted K-theory. Letting $X = Y$ the product (4.1.8) becomes a map

$$K^m(X, \delta_1) \times K^n(X, \delta_2) \to K^{m+n}(X \times X, p_1^* \delta_1 + p_2^* \delta_2),$$

and pulling this back along the diagonal map $\Delta : X \to X \times X$ yields the more familiar ring-like product

$$K^m(X, \delta_1) \times K^n(X, \delta_2) \to K^{m+n}(X, \delta_1 + \delta_2), \quad (4.1.10)$$

where $\Delta^*(p_1^* \delta_1 + p_2^* \delta_2) = (p_1 \circ \Delta)^* \delta_1 + (p_2 \circ \Delta)^* \delta_2 = \delta_1 + \delta_2$ using the functoriality of higher twisted K-theory. Note that this is not a graded ring structure on $K^*(X, \delta)$, rather it equips higher twisted K-theory with a ring-like structure where multiplication of classes corresponds to addition of twists. While topological K-theory does possess a graded ring structure, our results here are similar to the classical twisted case in which there is no graded ring structure as such but a product map of the same form as (4.1.10).

We are able to simplify the expression (4.1.8) for the product (4.1.10) by noting that $\Delta^* p_1^* F = F$ and $\Delta^* p_2^* G = G$ as was shown above for the twists δ_1 and δ_2. So the formula (4.1.8) in this case simply reduces to

$$F \boxtimes G : (q_1, q_2, \gamma) \mapsto (F(q_1) \boxtimes G(q_2)) \cdot \gamma. \tag{4.1.11}$$

Using Proposition 4.1.6 we see that the graded ring properties from the setting of topological K-theory carry over to higher twisted K-theory, with the caveat mentioned earlier that this does not equip $K^*(X, \delta)$ with a ring structure for a fixed δ, rather the twist must be allowed to change as per the product formula (4.1.10).

In the classical setting, there is a subset of twists known as multiplicative twists for which this product truly does provide the twisted K-theory group $K^*(X, \delta)$ with a ring structure for a fixed δ. In fact, it is using this class of twists that the results linking the equivariant twisted K-theory of certain Lie groups with representation theory are obtained in [FHT11a, FHT13, FHT11b], and so this provides the opportunity for future work in describing a class of multiplicative twists for higher twisted K-theory and finding links with the theorem of Freed, Hopkins and Teleman.

We are able to further restrict the product (4.1.10) to equip higher twisted K-theory with the structure of a graded module over topological K-theory. In particular, letting one of the twists be trivial we obtain

$$K^m(X) \times K^n(X, \delta) \to K^{m+n}(X, \delta).$$

This map will be a further simplification of the product (4.1.11), where we identify $K^m(X)$ with $[X \times \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}), \Omega^m \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]^{\operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})} = [X, \Omega^m \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]$ and observe that the tensor product of $\mathcal{E}_\delta$ and the trivial bundle $X \times \operatorname{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ will be isomorphic to $\mathcal{E}_\delta$ because the classes of the bundles are additive with respect to tensor product. This allows us to express the module structure as

$$F \boxtimes G : q \mapsto F(\pi(q)) \boxtimes G(q) \tag{4.1.12}$$

for $q \in \mathcal{E}_\delta$, where $F \in [X, \Omega^m \operatorname{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}]$ and $\pi : \mathcal{E}_\delta \to X$ is the bundle projection. The module axioms then follow from Proposition 4.1.6.

Proposition 4.1.8. *Higher twisted K-theory forms a graded module over topological K-theory.*

We will also explore this graded module structure from a different perspective. We briefly discuss an explicit form for a product on higher twisted K-theory in the case that $m = n = 0$ via operator algebraic K-theory. Recall from Definition 1.2.8 that the K-theory of a unital C^*-algebra A in degree 0 is the Grothendieck group of the monoid of projections in the infinite matrix algebra $M_\infty(A)$. Recall further that this extends to a definition of K-theory for general C^*-algebras A as the kernel of the map induced by projection from the unitisation A^+ to $\mathbb{C}$. Therefore an element in $K_0(A)$ can be represented by a pair (p,q) such that p and q are projections in $M_n(A^+)$ and $p-q \in M_n(A)$ for some n, and this element will be denoted $[p]-[q] \in K_0(A)$.

We use this formulation to define a product on higher twisted K-theory. For X a locally compact Hausdorff space and δ_X a twist represented by a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle $\mathcal{E}_{\delta_X}$, we see that an element of $K^0(X,\delta) = K_0(C_0(X,\mathcal{E}_{\delta_X}))$ is represented by a pair (p,q) with $p,q \in M_n(C_0(X,\mathcal{E}_{\delta_X})^+)$ such that $p-q \in M_n(C_0(X,\mathcal{E}_{\delta_X}))$ for some n. Letting Y, δ_Y and $\mathcal{E}_{\delta_Y}$ be defined in the same way, we express a product of the same form as (4.1.2) in the case $m=n=0$ as follows. While we believe this to be the same as the product developed using Fredholm operators, we have not worked out the details.

$$([p_1]-[p_2],[q_1]-[q_2]) \mapsto [\pi_X^* p_1 \otimes \pi_Y^* q_1 + \pi_X^* p_2 \otimes \pi_Y^* q_2] - [\pi_X^* p_1 \otimes \pi_Y^* q_2 + \pi_X^* p_2 \otimes \pi_Y^* q_1].$$

The properties of the p_i and q_j will carry over to ensure that the element on the right-hand side is in the correct K-theory group. In particular, since $p_i \in M_n(C_0(X,\mathcal{E}_{\delta_X})^+)$ and $q_j \in M_{n'}(C_0(Y,\mathcal{E}_{\delta_Y})^+)$, we have $\pi_X^* p_i \otimes \pi_Y^* q_j \in M_{nn'}(C_0(X\times Y, \pi_X^*\mathcal{E}_{\delta_X} \otimes \pi_Y^*\mathcal{E}_{\delta_Y}))$, and since $p_1 - p_2 \in M_n(C_0(X,\mathcal{E}_{\delta_X}))$ and $q_1 - q_2 \in M_{n'}(C_0(Y,\mathcal{E}_{\delta_Y}))$ we have

$$\begin{aligned}
&(\pi_X^* p_1 \otimes \pi_Y^* q_1 + \pi_X^* p_2 \otimes \pi_Y^* q_2) - (\pi_X^* p_1 \otimes \pi_Y^* q_2 + \pi_X^* p_2 \otimes \pi_Y^* q_1)\\
=\ &\pi_X^* p_1 \otimes (\pi_Y^*(q_1 - q_2)) - \pi_X^* p_2 \otimes (\pi_Y^*(q_1 - q_2))\\
=\ &\pi_X^*(p_1 - p_2) \otimes \pi_Y^*(q_1 - q_2) \in M_{n''}(C_0(X\times Y, \pi_X^*\mathcal{E}_{\delta_X} \otimes \pi_Y^*\mathcal{E}_{\delta_Y}))
\end{aligned}$$

for $n'' = \max(n,n')$ where equivalent projection matrices of the appropriate size may have been used. Note that as in the Fredholm case, we are able to obtain a very explicit form for the product map in degree 0. When moving to higher degrees, however, the descriptions of elements in the operator algebraic K-theory groups are not so simple, and so we will not extend this product to higher degrees.

We are able to restrict this product to view it as a map (4.1.10) as was done in the Fredholm case. Once again, the combination of the diagonal map and the projection maps will cancel out, providing a simpler expression for the product map as follows.

$$([p_1]-[p_2],[q_1]-[q_2]) \mapsto [p_1 \otimes q_1 + p_2 \otimes q_2] - [p_1 \otimes q_2 + p_2 \otimes q_1]. \tag{4.1.13}$$

Once again we will not extend this form of the product to higher degrees, but this provides a simple expression for the product in degree zero where the associativity and commutativity of the product are made clearer.

We will use this more explicit product to explore the graded module structure of higher twisted K-theory over topological K-theory. This formula allows us to give a more enlightening proof of Proposition 4.1.8, even if only in the case $m=n=0$.

Proof of Proposition 4.1.8 for $m = n = 0$. Let $[p_1] - [p_2], [q_1] - [q_2]$ represent classes in $K^0(X)$ and $[r_1] - [r_2], [s_1] - [s_2]$ represent classes in $K^0(X, \delta)$.

(i) Left distributivity:

$$\begin{aligned}([p_1] - [p_2], ([r_1] - [r_2]) + ([s_1] - [s_2])) &= ([p_1] - [p_2], [r_1 + s_1] - [r_2 + s_2]) \\ &\mapsto [p_1 \otimes r_1 + p_1 \otimes s_1 + p_2 \otimes r_2 + p_2 \otimes s_2] \\ &\quad - [p_1 \otimes r_2 + p_1 \otimes s_2 + p_2 \otimes r_1 + p_2 \otimes s_1] \\ &= [p_1 \otimes r_1 + p_2 \otimes r_2] - [p_1 \otimes r_2 + p_2 \otimes r_1] \\ &\quad + [p_1 \otimes s_1 + p_2 \otimes s_2] - [p_1 \otimes s_2 + p_2 \otimes s_1].\end{aligned}$$

(ii) Right distributivity:

$$\begin{aligned}(([p_1] - [p_2]) + ([q_1] - [q_2]), [r_1] - [r_2]) &= ([p_1 + q_1] - [p_2 + q_2], [r_1] - [r_2]) \\ &\mapsto [p_1 \otimes r_1 + q_1 \otimes r_1 + p_2 \otimes r_2 + q_2 \otimes r_2] \\ &\quad - [p_1 \otimes r_2 + q_1 \otimes r_2 + p_2 \otimes r_1 + q_2 \otimes r_1] \\ &= [p_1 \otimes r_1 + p_2 \otimes r_2] - [p_1 \otimes r_2 + p_2 \otimes r_1] \\ &\quad + [q_1 \otimes r_1 + q_2 \otimes r_2] - [q_1 \otimes r_2 + q_2 \otimes r_1].\end{aligned}$$

(iii) Compatibility with scalar multiplication:

$$\begin{aligned}&(([p_1] - [p_2]) \cdot ([q_1] - [q_2]), [r_1] - [r_2]) \\ =\ &([p_1 \otimes q_1 + p_2 \otimes q_2] - [p_1 \otimes q_2 + p_2 \otimes q_1], [r_1] - [r_2]) \\ \mapsto\ &[(p_1 \otimes q_1 + p_2 \otimes q_2) \otimes r_1 + (p_1 \otimes q_2 + p_2 \otimes q_1) \otimes r_2] \\ &- [p_1 \otimes q_1 + p_2 \otimes q_2) \otimes r_2 + (p_1 \otimes q_2 + p_2 \otimes q_1) \otimes r_1] \\ =\ &[p_1 \otimes (q_1 \otimes r_1 + q_2 \otimes r_2) + p_2 \otimes (q_1 \otimes r_2 + q_2 \otimes r_1)] \\ &- [p_1 \otimes (q_1 \otimes r_2 + q_2 \otimes r_1) + p_2 \otimes (q_1 \otimes r_1 + q_2 \otimes r_2)].\end{aligned}$$

(iv) The pair $(1, 0)$ in $K^0(X)$ acts as the identity element:

$$([1] - [0], [r_1] - [r_2]) \mapsto [1 \otimes r_1 + 0 \otimes r_2] - [1 \otimes r_2 + 0 \otimes r_1] = [r_1] - [r_2].$$

□

While these products and the existence of a graded module structure on higher twisted K-theory are not directly relevant to performing computations, by exploring them we are able to gain further insight into the structure of higher twisted K-theory.

4.2 Spectral sequences

Spectral sequences are useful computational tools available when dealing with cohomology theories and their twisted counterparts, and so it is expected that spectral sequences exist in the setting of higher twisted K-theory. Indeed, the arguments to obtain the existence of both the Atiyah–Hirzebruch and the Segal spectral sequences are standard in the literature, but we will work through the details of this for the Atiyah–Hirzebruch spectral sequence for the sake of completeness. Note that the particularly interesting results in this section are regarding the differentials in these sequences, and these results are unique to higher twisted K-theory. In spite of this, due to the difficulty in determining the differentials in these spectral sequences, are results are still weak and can be built upon allowing for further computations to be performed.

4.2.1 Atiyah–Hirzebruch spectral sequence

There is an Atiyah–Hirzebruch spectral sequence in topological K-theory, as well as an analogous twisted Atiyah–Hirzebruch spectral sequence in twisted K-theory constructed both by Rosenberg [Ros89] and by Atiyah and Segal [AS06]. Both sequences have the same E_2-term, but the difference lies in the differentials. We will show that there is yet another analogous spectral sequence in the higher twisted setting which shares the same E_2-term, but again the difference is in the differentials. While the differentials are very difficult to determine, we are able to obtain limited information by relating the differentials to the twisting cohomology class.

We will provide a very brief introduction to spectral sequences, but the reader who has not worked with them before may wish to consult a standard reference in homology to become more familiar with their use, for instance [CE99] or [BT82]. Spectral sequences are computational tools which are used to compute extraordinary cohomology groups from ordinary cohomology. For instance, the Atiyah–Hirzebruch spectral sequence in topological K-theory uses the ordinary cohomology of a space to compute its topological K-theory groups, and the variation for twisted K-theory uses ordinary cohomology to compute twisted K-theory.

To give a brief introduction to how they work, a spectral sequence for cohomology is a doubly infinite sequence $\{E_n^{p,q}\}_{p,q\in\mathbb{Z}}$ for each integer $n \geq 1$, where the differentials in the sequence tell you how to move from the E_n-term to the E_{n+1}-term. Often there will be an explicit formula for a certain term of the spectral sequence, for instance in the Atiyah–Hirzebruch spectral sequence there is an explicit expression for $E_2^{p,q}$ for all $p, q \in \mathbb{Z}$, and the differentials provide the remaining information for how to obtain the following terms in the sequence. The question that remains is how this sequence is used to compute extraordinary cohomology groups. We will only be considering spectral sequences that are fairly well-behaved, which firstly means that our sequences will be locally eventually constant, i.e. there will be a value N such that $E_n^{p,q} = E_N^{p,q}$ for every $p, q \in \mathbb{Z}$ for all $n \geq N$, and thus we may define the limit of the sequence $E_\infty^{p,q}$ to be this value. In this situation, we say that the spectral sequence collapses at the E_N-term. We will also only be using sequences that converge strongly, which is a technical definition but which means that we may compute the extraordinary cohomology groups from the E_∞-term as follows.

We say that the spectral sequence $\{E_n^{p,q}\}$ converges strongly to the graded group $\{H^i\}_{i\in\mathbb{Z}}$ if there is a filtration $(H^j)^i$ of H^i for all $i \in \mathbb{Z}$, i.e. a nested sequence of groups $(H^{j+1})^i \subset (H^j)^i \subset \cdots \subset H^i$, which is uniquely determined by short exact sequences of the form

$$0 \to (H^{j+1})^i \to (H^j)^i \to E_\infty^{j,i-j} \to 0. \tag{4.2.1}$$

For our locally eventually constant spectral sequences, there will be some $J \in \mathbb{Z}$ such that $(H^j)^i = H^i$ for all $i \in \mathbb{Z}$ for all $j \leq J$ and so determining this filtration via the short exact sequence will fully determine the graded group H^i. This shows where the extension problems in using spectral sequences for computations become apparent, as in general these short exact sequences are very difficult to use to determine H^i.

Note that analogous statements are true for spectral sequences in homology, where the sequence is of the form $\{E_{p,q}^n\}$ and the differentials go in the opposite direction.

In order to construct such a twisted Atiyah–Hirzebruch spectral sequence which is applicable to higher twisted K-theory, we follow the approach described in [CE99]. This requires a filtration of the twisted K-theory group, which is simple in the case of ordinary topological K-theory as the relative K-theory groups of the skeletal filtration of X may be used, but requires slight modification for our purposes.

Let X be a finite CW complex with p-skeleton X^p. We aim to filter $K^n(X,\delta) = K_n(A_\delta)$ by defining

$$K_p^n(X,\delta) = \ker[K_n(A_\delta) \xrightarrow{r_*} K_n(A_\delta|_{X^{p-1}})]$$

where r_* is the map induced on K-theory by restriction of sections $r : A_\delta \to A_\delta|_{X^{p-1}}$. We claim that this truly is a filtration of $K^n(X,\delta)$. In order to prove this, we need the following standard theorem.

Theorem 4.2.1 (Chapter XV Section 7 [CE99])**.** *Assume that for each pair of integers (p,q) such that $\infty \leq p \leq q \leq \infty$ a module $H(p,q)$ is given over a fixed ring. Suppose that for two pairs (p,q) and (p',q') such that $p \leq p'$ and $q \leq q'$ there is a homomorphism $H(p',q') \to H(p,q)$ defined, and furthermore given a triple (p,q,r) such that $-\infty \leq p \leq q \leq r \leq \infty$ there is a connecting homomorphism $\delta : H(p,q) \to H(q,r)$ defined. Suppose further that the following axioms are satisfied:*

(SP.1) The map $H(p,q) \to H(p,q)$ is the identity;

(SP.2) If $(p, q) \leq (p', q') \leq (p'', q'')$ *then the diagram*

$$\begin{array}{ccc} H(p'', q'') & \longrightarrow & H(p, q) \\ & \searrow \quad \nearrow & \\ & H(p', q') & \end{array}$$

commutes;

(SP.3) If $(p, q, r) \leq (p', q', r')$ *then the diagram*

$$\begin{array}{ccc} H(p', q') & \xrightarrow{\delta'} & H(q', r') \\ \downarrow & & \downarrow \\ H(p, q) & \xrightarrow{\delta} & H(q, r) \end{array}$$

commutes;

(SP.4) For each triple (p, q, r) *with* $-\infty \leq p \leq q \leq r \leq \infty$ *the sequence*

$$\cdots \to H(q, r) \to H(p, r) \to H(p, q) \xrightarrow{\delta} H(q, r) \to \cdots$$

is exact;

(SP.5) For a fixed q *the direct system of modules*

$$H(q, q) \to H(q-1, q) \to \cdots \to H(p, q) \to \cdots$$

has $H(-\infty, q)$ *as direct limit.*

Then $F^pH = \ker[H(-\infty, \infty) \to H(-\infty, p)]$ *is a filtration of* H. □

In order to apply this theorem, we define $H(p, q) = K_*(A_\delta|_{X^{q-1} \setminus X^{p-1}})$ where we specify $-\infty \leq p \leq q \leq \infty$, and note that $X^r = \emptyset$ for $r < 0$ and $X^r = X$ for r greater than or equal to the dimension of X. This means that

$$K_p^n(X, \delta) = \ker[H(-\infty, \infty) \to H(-\infty, p)],$$

so proving that these $H(p, q)$ satisfy the axioms posed in the theorem would prove that we do have a filtration of the higher twisted K-theory group.

Since these spaces $H(p, q)$ are defined to be operator algebraic K-theory groups of subalgebras of A_δ, they are thus submodules of $K^*(X, \delta)$. We must now develop the maps assumed in the theorem.

Given two pairs (p,q) and (p',q') such that $(p,q) \leq (p',q')$, meaning $p \leq p'$ and $q \leq q'$, we define a map $H(p',q') \to H(p,q)$ as follows. Since operator algebraic K-theory forms a covariant functor, we may define a map $A_\delta|_{X^{q'-1}\setminus X^{p'-1}} \to A_\delta|_{X^{q-1}\setminus X^{p-1}}$ and then take the corresponding map in K-theory. Given an element of $A_\delta|_{X^{q'-1}\setminus X^{p'-1}}$, i.e. a continuous section $s : X^{q'-1} \setminus X^{p'-1} \to \mathcal{E}_\delta|_{X^{q'-1}\setminus X^{p'-1}}$ vanishing at infinity, we define a section of the corresponding bundle over $X^{q-1} \setminus X^{p-1}$ to be

$$s'(x) = \begin{cases} s(x) & \text{if } x \in (X^{q'-1} \setminus X^{p'-1}) \cap (X^{q-1} \setminus X^{p-1}); \\ 0 & \text{if } x \notin (X^{q'-1} \setminus X^{p'-1}) \cap (X^{q-1} \setminus X^{p-1}); \end{cases}$$

where 0 denotes the zero element in the fibre over x. We claim that s' is, in fact, a continuous map since the assumption that s vanishes at infinity implies that the norm of s can be made arbitrarily small in a neighbourhood of the excised set $X^{p'-1}$. Thus s can be extended continuously by defining it to be zero on $X^{p'-1} \setminus X^{p-1}$. Note also that s' vanishes at infinity because s vanishes at infinity: for any $\epsilon > 0$ there is a compact subspace $Q \subset X^{q'-1} \setminus X^{p'-1}$ such that $\|s(x)\| < \epsilon$ for all $x \in (X^{q'-1} \setminus X^{p'-1}) \setminus Q$; take $Q' = Q \cap X^{q-1}$ to be the compact set outside of which $\|s'(x)\| < \epsilon$. Hence this yields the desired map $H(p',q') \to H(p,q)$.

Next, given a triple (p,q,r) with $-\infty \leq p \leq q \leq r \leq \infty$, we require a connecting map $\delta : H(p,q) \to H(q,r)$. This map can be obtained by considering the six-term exact sequence in operator algebraic K-theory, by defining a short exact sequence

$$0 \to A_\delta|_{X^{r-1}\setminus X^{q-1}} \to A_\delta|_{X^{r-1}\setminus X^{p-1}} \to A_\delta|_{X^{q-1}\setminus X^{p-1}} \to 0.$$

The nontrivial maps in this sequence are obtained in the same way as described above, associated to the pairs $(p,r) \leq (q,r)$ and $(p,q) \leq (p,r)$ respectively. This sequence is then exact by the definition of the maps described above.

We must now show that this sequence of modules and maps satisfies the axioms posed in Theorem 4.2.1.

Lemma 4.2.2. *The modules and maps defined above satisfy the axioms (SP.1) to (SP.5).*

Proof.

(SP.1-2) These are clear from the definition of the map described above.

(SP.3) This follows from the naturality of the connecting map in the six-term exact sequence in K-theory.

(SP.4) This follows from the definition of δ being the connecting map in the six-term exact sequence in K-theory.

(SP.5) It is straightforward to prove that $H(-\infty, q) = K_*(A_\delta|_{X^{q-1}})$ is the direct limit using the univeral property of direct limits and the continuity of K-theory for C^*-algebras through direct limits in Propositions 6.2.9 and 7.1.7 of [WO93].

□

Then Theorem 4.2.1 implies that $K^n_p(X, \delta)$ is a filtration of $K^n(X, \delta)$ and by defining

$$\begin{aligned} Z^p_r &= \operatorname{im}[H(p, p+r) \to H(p, p+1)]; \\ B^p_r &= \operatorname{im}[H(p-r+1, p) \to H(p, p+1)]; \\ E^p_r &= Z^p_r / B^p_r; \end{aligned}$$

we obtain a spectral sequence strongly converging to $K^*(X, \delta)$ by Chapter XV Proposition 4.1 of [CE99] combined with comments from Chapter XV Section 7. We give a formal statement of the existence of the spectral sequence in Theorem 4.2.3, but first we obtain a simple expression for the E_2-term. We see that $E^{p,q}_1 = K_{p+q}(A_\delta|_{X^p \setminus X^{p-1}})$, but $X^p \setminus X^{p-1}$ consists of the p-cells from which X is constructed, and since the interior of any p-cell is homeomorphic to $\mathbb{R}^p$, this implies that A_δ is trivial over these p-cells. Therefore we see that $E^{p,q}_1 = K^{p+q}(X^p \setminus X^{p-1}) = K^{p+q}(X^p, X^{p-1})$, and so

$$E^{p,q}_1 = \sum_i K^{p+q}(\sigma^p_i, \dot{\sigma}^p_i)$$

where the σ^i_p are the p-cells of X with boundary $\dot{\sigma}^p_i$. Then we have $\sigma^p_i / \dot{\sigma}^p_i = S^p$, and thus

$$K^{p+q}(\sigma^p_i, \dot{\sigma}^p_i) \cong \widetilde{K}^{p+q}(S^p) \cong \widetilde{K}^q(S^0) \cong K^q(x_0)$$

where x_0 is the space consisting of a single point. Therefore

$$E^{p,q}_1 = \sum_i K^q(x_0) = C^p(X, K^q(x_0))$$

by definition. Finally, since $E^{p,q}_2$ is by definition the cohomology of the $E^{p,q}_1$ with the usual coboundary maps, we see that $E^{p,q}_2 = H^p(X, K^q(x_0))$.

We have proved the following.

Theorem 4.2.3. *Let X be a CW complex with δ a twist over X. There exists an Atiyah–Hirzebruch spectral sequence converging strongly to $K^*(X, \delta)$ with $E^{p,q}_2 = H^p(X, K^q(x_0))$.*

□

This result as it is, however, is not very useful in calculating higher twisted K-theory groups. What we need is a description of the differentials in the spectral sequence in order to perform explicit computations. In the untwisted case originally treated in [AH59], it was found that the first nontrivial differential was given by the Steenrod operation $Sq^3 : H^p(X,\mathbb{Z}) \to H^{p+3}(X,\mathbb{Z})$. This was then extended to the standard twisted setting in [Ros89] and [AS04], where it was discovered that the first nontrivial differential was given by the Steenrod operation twisted by the class $\delta \in H^3(X,\mathbb{Z})$, i.e. the differential is expressed by $Sq^3 - (-) \cup [\delta] : H^p(X,\mathbb{Z}) \to H^{p+3}(X,\mathbb{Z})$. We obtain an analogous result in this setting, where we are now forced to restrict to the case that the twist δ can be represented by a cohomology class. We also lose some information in passing to the higher twisted setting, as the higher differentials of even the Atiyah–Hirzebruch spectral sequence in topological K-theory are not well-understood.

Theorem 4.2.4. *In the setting of the Atiyah–Hirzebruch spectral sequence, if a twist δ can be represented by a class $\delta \in H^{2n+1}(X,\mathbb{Z})$ then the d_{2n+1} differential is the differential d'_{2n+1} in the spectral sequence for ordinary topological K-theory twisted by δ, i.e. the map $d_{2n+1} : H^p(X,\mathbb{Z}) \to H^{p+2n+1}(X,\mathbb{Z})$ is given by $d_{2n+1}(x) = d'_{2n+1}(x) - x \cup \delta$.*

Proof. We follow the argument given in [AS04]. By definition, the d_{2n+1} differential must be a universal cohomology operation raising degree by $2n+1$, defined for spaces with a given class $\delta \in H^{2n+1}(X,\mathbb{Z})$. Standard arguments in homotopy theory show that these operations are classified by

$$H^{p+2n+1}(K(\mathbb{Z},p) \times K(\mathbb{Z},2n+1),\mathbb{Z}),$$

where the $K(\mathbb{Z},p)$ factor represents cohomology operations raising degree by $2n+1$ and the $K(\mathbb{Z},2n+1)$ factor comes from X being equipped with a class $\delta \in H^{2n+1}(X,\mathbb{Z})$. This cohomology group is isomorphic to

$$H^{p+2n+1}(K(\mathbb{Z},p),\mathbb{Z}) \oplus H^{p+2n+1}(K(\mathbb{Z},2n+1),\mathbb{Z}) \oplus \mathbb{Z}$$

where the third summand is generated by the product of the generators of $H^p(K(\mathbb{Z},p),\mathbb{Z})$ and $H^{2n+1}(K(\mathbb{Z},2n+1),\mathbb{Z})$. The only factor which will actually result in an operation $H^p(X,\mathbb{Z}) \to H^{p+2n+1}(X,\mathbb{Z})$ is the first, and so we conclude that the differential is given by $d_{2n+1}(x) = d'_{2n+1}(x) + kx \cup \delta$ where $k \in \mathbb{Z}$, since the operation must agree with the standard spectral sequence in the case that $\delta = 0$. We determine the integer k by explicitly computing the spectral sequence for $X = S^{2n+1}$ as follows, applying the same methods as in [AS04].

The filtration for this case is particularly simple, with $X^0 = X^1 = \cdots = X^{2n}$ consisting of a single point and $X^{2n+1} = S^{2n+1}$. Then the spectral sequence reduces to the long exact sequence for the pair (X, X^0), and the d_{2n+1} differential is the boundary map $K^0(X^0, \delta|_{X^0}) \to K^1(X, X_0; \delta)$. Equivalently, using the excision property of higher twisted K-theory applied to the compact pair (S^{2n+1}, D^{2n+1}_+) and excising the interior of D^{2n+1}_+ we see that $K^1(X, X_0; \delta) \cong K^1(D^{2n+1}_-, S^{2n}; \delta|_{D^{2n+1}_-})$ and so d_{2n+1} can be viewed as the boundary map $K^0(D_+^{2n+1}, \delta|_{D_+^{2n+1}}) \to K^1(D_-^{2n+1}, S^{2n}; \delta|_{D_-^{2n+1}})$. This map is the passage from top-left to bottom-right in the commutative diagram

$$\begin{array}{ccc} K^0(D_+^{2n+1}, \delta|_{D_+^{2n+1}}) & \longrightarrow & K^1(S^{2n+1}, D_+^{2n+1}; \delta) \\ \downarrow & & \downarrow \\ K^0(S^{2n}, \delta|_{S^{2n}}) & \longrightarrow & K^1(D_-^{2n+1}, S^{2n}; \delta|_{D_-^{2n+1}}). \end{array}$$

By studying the six-term exact sequence in higher twisted K-theory associated to the pair (D_-^{2n+1}, S^{2n}), it is clear that the lower horizontal map takes the generator $(1, 0)$ of $K^0(S^{2n})$, corresponding to the trivial line bundle over S^{2n}, to 0 and the generator $(0, 1)$, corresponding to the n-fold reduced external product of $(H - 1)$ with H the tautological line bundle over S^2, to the generator of $K^1(D_-^{2n+1}, S^{2n}; \delta|_{D_-^{2n+1}})$. All that remains is to determine the left-hand vertical map. This is done in the proof of Proposition 5.1.1, in particular this is the top horizontal map in (5.1.3) because in order to identify $K^0(S^{2n}, \delta|_{S^{2n}})$ with $\mathbb{Z} \oplus \mathbb{Z}$ we are using the trivialisation of δ over D_-^{2n+1} as opposed to D_+^{2n+1}. The map is shown to be $n \mapsto (n, -Nn)$ where the twist $\delta \in H^{2n+1}(S^{2n+1}, \delta)$ is given by $N \in \mathbb{Z}$ times a generator. Hence the composition sends $1 \in \mathbb{Z}$ to $-N \in \mathbb{Z}$. Since we see that $d_{2n+1}(1) = -N$ then we may conclude that $k = -1$ as required. □

Whilst this is not quite as explicit as the differential in the classical twisted case, it is still useful as it is known that all differentials in the Atiyah–Hirzebruch spectral sequence for ordinary topological K-theory are torsion operators [Arl92]. Since this result is only applicable when the twist can be represented by cohomology, then it will frequently be the case that the space has torsion-free cohomology and so these torsion differentials will have no effect.

Atiyah and Segal are also able to show in [AS04] that the higher differentials of the spectral sequence are given rationally by higher Massey products by generalising the Chern character to the twisted setting. This work can likely be generalised to the higher twisted setting, but since it only gives the differentials rationally it is not highly applicable to computations.

We give an idea of how the convergence of this spectral sequence and the short exact sequence (4.2.1) can be used to compute higher twisted K-theory. The E_∞-term of the Atiyah–Hirzebruch spectral sequence for higher twisted K-theory consists of two alternating rows, one of which is $E_\infty^{p,0}$ for $0 \leq p \leq N$ for some upper bound N and the other of which is a row of zeroes. Suppose that $N = 2n$ is even for the purpose of illustration. Then applying the 2-periodicity in the rows of the spectral sequence and the short exact sequence (4.2.1), we see that the filtration of $K^0(X,\delta)$ is determined by

$$\begin{aligned}
&K^0_{2n}(X,\delta) \cong E^{2n,0}_\infty;\\
&0 \to K^0_{2n}(X,\delta) \to K^0_{2n-2}(X,\delta) \to E^{2n-2,0}_\infty \to 0;\\
&0 \to K^0_{2n-2}(X,\delta) \to K^0_{2n-4}(X,\delta) \to E^{2n-2,0}_\infty \to 0;\\
&\qquad\qquad\vdots\\
&0 \to K^0_{2}(X,\delta) \to K^0_{0}(X,\delta) \to E^{0,0}_\infty \to 0;\\
&K^0(X,\delta) \cong K^0_0(X,\delta);
\end{aligned}$$

and similarly $K^1(X,\delta)$ is determined by

$$\begin{aligned}
&K^1_{2n-1}(X,\delta) \cong E^{2n-1,0}_\infty;\\
&0 \to K^1_{2n-1}(X,\delta) \to K^1_{2n-3}(X,\delta) \to E^{2n-3,0}_\infty \to 0;\\
&\qquad\qquad\vdots\\
&0 \to K^1_{3}(X,\delta) \to K^1_{1}(X,\delta) \to E^{1,0}_\infty \to 0;\\
&K^1(X,\delta) \cong K^1_1(X,\delta).
\end{aligned}$$

This makes it very clear how to compute the higher twisted K-theory groups from the E_∞-term of the spectral sequence, and also illustrates the number of extension problems that must be solved to do so. We will make use of this method in our computations in Chapter 5.

4.2.2 Segal spectral sequence

A more powerful version of the Atiyah–Hirzebruch spectral sequence is the Segal spectral sequence, which we will use for computing higher twisted K-theory in more complicated settings. One may work through the details of the construction as we have done for the Atiyah–Hirzebruch spectral sequence via a skeletal filtration which induces a filtration of the higher twisted K-theory group, but we will not go through the details again. At this point, we also bring higher twisted K-homology back into the picture, because it is in the Segal spectral sequence for higher twisted K-homology that the strongest information about the differentials can be easily obtained.

Theorem 4.2.5. *Let $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ be a fibre bundle of CW complexes, and suppose that a twist δ over E can be represented by a class $\delta \in H^{2n+1}(E,\mathbb{Z})$. Then there is a homological Segal spectral sequence*

$$H_p(B, K_q(F,\iota^*\delta)) \Rightarrow K_*(E,\delta)$$

and a corresponding cohomological Segal spectral sequence

$$H^p(B, K^q(F,\iota^*\delta)) \Rightarrow K^*(E,\delta).$$

These spectral sequences are strongly convergent if the ordinary (co)homology of B is bounded.

Proof. The proof follows from standard methods as did the proof of Theorem 4.2.3, for instance Rosenberg's proof of Theorem 3 in [Ros89] can be adapted which employs Segal's original proof in Proposition 5.2 of [Seg68]. □

Remark 4.2.1. The ordinary (co)homology of B will be bounded if B is weakly equivalent to a finite dimensional CW complex and this will cover all of the cases that we consider, so we obtain strong convergence from this spectral sequence.

Note that we refer to this as a Segal spectral sequence because the method of proof employs Segal's original techniques from [Seg68].

As mentioned above, there is more that can be said about the differentials in the homology spectral sequence and we present these details here explicitly. Note that the following theorem uses a Hurewicz map in higher twisted K-homology which we have not developed. We will not need to use this map explicitly at any time, and so we do not present the details of its construction. The map can be constructed in the same way as in classical twisted K-homology, which follows from the formulation of the Hurewicz map in ordinary K-homology for instance in Theorem II.14.1 of [Ada74].

Theorem 4.2.6. *In the setting of the homology Segal spectral sequence of Theorem 4.2.5, suppose that*

- $\iota^* : H^{2n+1}(E, \mathbb{Z}) \to H^{2n+1}(F, \mathbb{Z})$ *is an isomorphism, so that the twisting class δ on E can be identified with the restricted twisting class $\iota^*\delta$ on F,*
- *the differentials $d^2, \cdots, d^{r-1}$ leave $E^2_{r,0} = H_r(B, K_0(F, \iota^*\delta))$ unchanged, or equivalently $E^2_{r,0} = E^3_{r,0} = \cdots = E^r_{r,0}$, and*
- *there is a class $x \in E^2_{r,0}$ which comes from a class $\alpha \in \pi_r(B)$ under the Hurewicz map $\pi_r(B) \to H_r(B, K_0(F, \iota^*\delta))$.*

Then $d^r(x) \in E^r_{0,r-1}$ is the image of α under the composition of the boundary map $\partial : \pi_r(B) \to \pi_{r-1}(F)$ in the long exact sequence of the fibration and the Hurewicz map $\pi_{r-1}(F) \to K_{r-1}(F, \iota^*\delta)$.

Proof. In order to prove this, we note that since the class x was not changed by the differentials $d^2, \cdots, d^{r-1}$ and since the twisting class comes from the fibre, we can take, without loss of generality, B to be S^r and then $E = (\mathbb{R}^r \times F) \cup F$, where $\mathbb{R}^r \times F$ is π^{-1} of the open r-cell in B. In this special case, as noted by Rosenberg in the proof of Theorem 6 [Ros89] the spectral sequence comes from the long exact sequence

$$\cdots \to K_r(F, \iota^*\delta) \xrightarrow{\iota_*} K_r(E, \delta) \to K_r(E, F, \delta) \cong K_0(F, \iota^*\delta) \xrightarrow{\partial} K_{r-1}(F, \iota^*\delta) \to \cdots$$

where we identify $K_0(F, \iota^*\delta)$ with $H_r(B, K_0(F, \iota^*\delta))$. Hence the differential d^r is simply the boundary map in this sequence, and the result follows from the naturality of the Hurewicz homomorphism which implies the commutativity of the diagram

$$\begin{array}{ccc} \pi_r(B) & \xrightarrow{\quad\partial\quad} & \pi_{r-1}(F) \\ {\scriptstyle \text{Hurewicz}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Hurewicz}} \\ H_r(B, K_0(F, \iota^*\delta)) & \xrightarrow{\ \partial\ } & K_{r-1}(F, \iota^*\delta). \end{array}$$

□

Although we are only interested in computing higher twisted K-theory groups, there are some settings in which it is equivalent to compute the higher twisted K-homology, hence why this theorem will prove useful.

Proposition 4.2.7. *Assume that the algebra of sections vanishing at infinity of any algebra bundle with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ over a locally compact space X is contained in the bootstrap category of C^*-algebras defined in Definition 22.3.4 of [Bla86]. If the higher twisted K-theory of X is a direct sum of finite torsion groups, then the higher twisted K-theory and higher twisted K-homology of X are isomorphic with a degree shift.*

Proof. The higher twisted K-theory and K-homology groups can be related by the universal coefficient theorem in KK-theory as in Theorem 23.1.1 of [Bla86], which states that

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK^*(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

is a short exact sequence whose first map has degree 0 and second map has degree 1 with respect to the grading, if A and B are seperable and A is in the bootstrap category of C^*-algebras defined in Definition 22.3.4 of [Bla86]. In order to obtain higher twisted K-homology as the KK-group in this sequence, we let A be the space of sections of the algebra bundle representing the twist and let $B = \mathcal{O}_\infty$. Then assuming that A is in the bootstrap category, the short exact sequence becomes

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K^{n+1}(X, \delta), K_0(\mathcal{O}_\infty)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K^n(X, \delta), K_1(\mathcal{O}_\infty) \to K_n(X, \delta) \\ \to \mathrm{Hom}_{\mathbb{Z}}(K^n(X, \delta), K_0(\mathcal{O}_\infty) \oplus \mathrm{Hom}_{\mathbb{Z}}(K^{n+1}(X, \delta), K_1(\mathcal{O}_\infty) \to 0 \end{aligned}$$

for each n. Using the K-theory of $\mathcal{O}_\infty$, this reduces to

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K^{n+1}(X, \delta), \mathbb{Z}) \to K_n(X, \delta) \to \mathrm{Hom}_{\mathbb{Z}}(K^n(X, \delta), \mathbb{Z}) \to 0.$$

Now, if $K^n(X, \delta) \cong \bigoplus_k \mathbb{Z}_{m_{k,n}}$ for some finite sequence $\{m_{k,n}\}$ and $n = 0, 1$ as per our assumption, then the Hom group will be trivial and the Ext group will become

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}\left(\bigoplus_k \mathbb{Z}_{m_{k,n}}, \mathbb{Z}\right) &= \bigoplus_k \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{m_{k,n}}, \mathbb{Z}) \\ &= \bigoplus_k \mathbb{Z}_{m_{k,n}} \\ &= K^n(X, \delta),\end{aligned}$$

since the Ext functor is additive in the first variable and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_m, G) \cong G/mG$ by properties in Section 3.1 of [Hat00]. Therefore the short exact sequence provides an isomorphism $K^{n+1}(X, \delta) \cong K_n(X, \delta)$ as required. □

Remark 4.2.2. We highlight the assumption made in the statement of this Proposition; that the algebra of sections vanishing at infinity of any algebra bundle with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$ over a locally compact space X is contained in the bootstrap category of C^*-algebras defined in Definition 22.3.4 of [Bla86] and thus does satisfy the universal coefficient theorem. However, this assumption is valid when $\mathcal{O}_\infty \otimes \mathcal{K}$ is replaced by $\mathcal{K}$, and it is true for both $C_0(X)$ for any locally compact Hausdorff space as well as for $\mathcal{O}_\infty \otimes \mathcal{K}$. In fact, it is conjectured that every seperable nuclear C^*-algebra satisfies the universal coefficient theorem in KK-theory. We note that this Proposition is only used to prove one result; Theorem 5.1.8 in the following chapter.

As we will see in Chapter 5, these results will allow for the higher twisted K-theory of Lie groups to be computed in some cases, which is a difficult task even in the classical twisted setting.

Chapter 5

Computations in Higher Twisted K-theory

The final chapter of this thesis is dedicated to computation. This will allow for the techniques developed in previous chapters to come together, such as the Mayer–Vietoris sequence, the spectral sequence, the Hilbert module picture and the clutching construction. As higher twisted K-theory forms a generalisation of both topological and classical twisted K-theory, we will see that computations for these variants of K-theory will fall out as a result of our computations. To determine which spaces we can work with, we recall that we are able to work with twists most effectively when they can be identified with cohomology, and thus we limit ourselves to working with these spaces. This is the case when the space has torsion-free cohomology by Pennig and Dadarlat's work, and so in the first section we restrict our attention such spaces including spheres, products of spheres and certain Lie groups. We subsequently extend this view in the second section to spaces whose cohomology may have torsion but we show that it does not affect the representation of twists as cohomology classes, such as real projective space, Lens spaces and certain $SU(2)$-bundles.

As explained in Section 2.6, computations in higher twisted K-theory may be of physical interest in the realms of string theory and M-theory. While we will not give explicit physical descriptions of our computations here, further research into the relationship between higher twisted K-theory and physics may help to provide insight into both fields and allow these results to lead to a greater understanding of M-theory.

5.1 Torsion-free spaces

In the torsion-free setting we are able to apply Theorem 2.4.1 relating twists of K-theory to cohomology, and so we begin by working with this class of spaces.

5.1.1 Spheres

Spheres are some of the simplest topological spaces, particularly for our purposes as only the zeroth and the top degree cohomology groups are non-trivial and we have an explicit description of the bundles of interest over the spheres via the clutching construction. Because of this, we will begin by computing the higher twisted K-theory of the odd-dimensional spheres, and this should reduce to known results in the case that trivial twists are used or in the case of classical twists over S^3. We also use a Mayer–Vietoris sequence and the Hilbert module picture to provide a geometric expression for the generator of the K-theory group. Note that we restrict our attention to odd-dimensional spheres because there are no non-trivial twists over even-dimensional spheres by Theorem 2.4.1.

As explained in Section 3.1, the correspondence between principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over S^{2n+1} representing twists and $H^{2n+1}(S^{2n+1}, \mathbb{Z})$ is given by the clutching construction. Since both $\pi_{2n}(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ and $H^{2n+1}(S^{2n+1}, \mathbb{Z})$ are isomorphic to the integers, taking a class $[\delta] = N[\delta_0] \in H^{2n+1}(S^{2n+1}, \mathbb{Z})$ with $[\delta_0]$ a generator and $N \in \mathbb{Z}$ corresponds to the principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle $\mathcal{E}_\delta$ over S^{2n+1} constructed via the gluing map $[f] = N[f_0] \in \pi_{2n}(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K}))$ with $[f_0]$ the corresponding generator under the isomorphism.

Note that throughout the remainder of the chapter, we refer to "the generator" of various cohomology groups isomorphic to $\mathbb{Z}$. The choice of generator is unimportant here, because if a twist is N times one generator then it will be $-N$ times the other. The integer N only appears in our results in the form $\mathbb{Z}_N$, and since the groups $\mathbb{Z}_N$ and $\mathbb{Z}_{-N}$ are isomorphic, the result will be true regardless of the choice of generator.

Proposition 5.1.1. *Let $\delta \in H^{2n+1}(S^{2n+1}, \mathbb{Z})$ be a twist of K-theory for S^{2n+1} which is N times the generator. The higher twisted K-theory of S^{2n+1} is then*

$$K^0(S^{2n+1}, \delta) = 0 \quad \textit{and} \quad K^1(S^{2n+1}, \delta) = \mathbb{Z}_N$$

if $N \neq 0$, or

$$K^0(S^{2n+1}) = \mathbb{Z} \quad \textit{and} \quad K^1(S^{2n+1}) = \mathbb{Z}$$

when $N = 0$.

Proof. Given a twist δ as above, we construct a short exact sequence of C^*-algebras including the algebra of sections of the associated algebra bundle, allowing the higher twisted K-theory groups to be computed via a six-term exact sequence. Lemma 3.1.4 shows that the algebra bundle $\mathcal{A}_\delta$ associated to the principal bundle $\mathcal{E}_\delta$ is also formed using the clutching construction. Then as shown in Lemma 3.1.2, the space of continuous sections of this algebra is of the form

$$A_\delta = \{(h_+, h_-) \in C(D_+^{2n+1} \amalg D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}) : h_+(x) = f(x)(h_-(x)) \,\forall\, x \in S^{2n}\}.$$

Then we may define the short exact sequence

$$0 \to A_\delta \xrightarrow{\iota} C(D_+^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}) \oplus C(D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\pi} C(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0,$$

where ι denotes inclusion and $\pi(h_+, h_-)(x) = h_+(x) - f(x)(h_-(x))$ to make the sequence exact. We denote the algebra in the middle of this sequence by $C(D_+^{2n+1} \amalg D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K})$ for brevity, and then applying the six-term exact sequence gives

$$\begin{array}{ccccc} K_0(A_\delta) & \xrightarrow{\iota_*} & K_0(C(D_+^{2n+1} \amalg D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K})) & \xrightarrow{\pi_*} & K_0(C(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K})) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K})) & \xleftarrow{\pi_*} & K_1(C(D_+^{2n+1} \amalg D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K})) & \xleftarrow{\iota_*} & K_1(A_\delta). \end{array}$$

We are able to simplify several terms in this sequence using trivialisations of the algebra bundle. Firstly, since the hemispheres D_+^{2n+1} and D_-^{2n+1} are contractible, the algebra bundle $\mathcal{A}_\delta$ will be trivialisable over these hemispheres. To be more specific, using a trivialsiation t_+ over the upper hemisphere we are able to identify $K_n(C(D_+^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}))$ with $\mathbb{Z}$ for $n = 0$ and 0 for $n = 1$, and similarly trivialising via t_- over the lower hemisphere identifies $K_0(C(D_-^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}))$ with $\mathbb{Z}$ for $n = 0$ and 0 for $n = 1$.

We may also simplify the terms involving the equatorial sphere S^{2n}, since the restriction of $\mathcal{A}_\delta$ to S^{2n} will be necessarily trivialisable due to S^{2n} having trivial odd-degree cohomology. At this point we must make a choice of trivialisation, since we have both t_+ and t_- which can trivialise $\mathcal{A}_\delta$ over S^{2n}. We choose t_+, and in doing so we identify $K_0(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K})$ with $\mathbb{Z} \oplus \mathbb{Z}$ for $n = 0$ and 0 for $n = 1$.

This reduces the six-term exact sequence to

$$0 \to K^0(S^{2n+1}, \delta) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \oplus \mathbb{Z} \to K^1(S^{2n+1}, \delta) \to 0.$$

The only map to determine here is π_*, and to do so we must study the differing trivialisations of $\mathcal{A}_\delta$ over S^{2n} since π_* is a priori a map between higher twisted K-theory groups. Using the Mayer–Vietoris sequence in Proposition 2.3.9, the map π_* is given by the difference $j_+^* - j_-^*$, where $j_\pm : S^{2n} \to D_\pm^{2n+1}$ is inclusion and $j_\pm^*$ denotes the induced map on higher twisted K-theory described in Subsection 2.2.2. Since we have trivialised the bundle over S^{2n} using t_+, we will need to take the differing trivialisations into account when determining the map j_-^*. The trivialisations of $\mathcal{A}_\delta$ over S^{2n} fit into the commuting diagram

$$\begin{array}{ccc} K^0(S^{2n}, \delta|_{S^{2n}}) & \underset{(t_+)_*}{\xrightarrow{\cong}} & K^0(S^{2n}) \\ \cong \downarrow (t_-)_* & \nearrow & \\ K^0(S^{2n}) & & \end{array} \tag{5.1.1}$$

where the map $(t_+)_* \circ (t_-)_*^{-1}$ must be determined to change coordinates from D_-^{2n+1} to D_+^{2n+1}. In order to do this, we write the trivialisations explicitly.

Firstly, observe that the restriction of $\mathcal{A}_\delta$ to D_+^{2n+1} is the quotient of

$$(D_+^{2n+1} \times (\mathcal{O}_\infty \otimes \mathcal{K})) \amalg (S^{2n} \times (\mathcal{O}_\infty \otimes \mathcal{K}))$$

under the usual equivalence relation on the equatorial sphere. Then t_+ will be the map sending the class of (x, o) to the class of $(x, f(x)^{-1}(o))$. Note that this is well-defined, because an element on $S^{2n} \times (\mathcal{O}_\infty \otimes \mathcal{K})$ can be represented either as (x_+, o) or $(x_-, f(x)o)$ with x_+ and x_- representing the point $x \in S^{2n}$ in the respective hemispheres, and these representatives will be mapped to $(x_+, f(x)^{-1}o)$ and (x_-, o) respectively, both of which represent the same equivalence class in $\mathcal{A}_\delta|_{D_+^{2n+1}}$.

It can similarly be shown that t_- sends the class of (x, o) to the class of $(x, f(x)(o))$ and this is well-defined. So taking the equivalence relation on S^{2n} into account, these trivialisations differ by the transition function

$$t_+ \circ (t_-)^{-1} : S^{2n} \times (\mathcal{O}_\infty \otimes \mathcal{K}) \to S^{2n} \times (\mathcal{O}_\infty \otimes \mathcal{K})$$

given by $(x, v) \mapsto (x, f(x)^{-1}(v))$.

These trivialisations also induce maps $(t_\pm)_* : C(D_\pm^{2n+1}, \mathcal{A}_\delta|_{D_\pm^{2n+1}}) \to C(S^{2n}, \mathcal{A}_\delta|_{S^{2n}})$ on the section algebras in the obvious way, and composition gives

$$(t_+ \circ (t_-)^{-1})_* : C(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K}) \to C(S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K})$$

sending $g : S^{2n} \to \mathcal{O}_\infty \otimes \mathcal{K}$ to the map $S^{2n} \ni x \mapsto f(x)^{-1}(g(x))$. These maps then in turn induce maps between operator algebraic K-theory groups as in the commutative diagram (5.1.1).

We will now determine the maps ${j_\pm}^*$ induced on higher twisted K-theory. Firstly for j_+ we have the commutative diagram

$$\begin{array}{ccc} K^*(D_+^{2n+1}, \delta) & \xrightarrow{{j_+}^*} & K^*(S^{2n}, \delta) \\ \Big\downarrow{\scriptstyle (t_+)_*} & & \Big\downarrow{\scriptstyle (t_+)_*} \\ K^*(D_+^{2n+1}) & \longrightarrow & K^*(S^{2n}) \end{array} \tag{5.1.2}$$

where the lower map $K^*(D_+^{2n+1}) \to K^*(S^{2n})$ is the map induced by j_+ on ordinary K-theory. Thus we see that j_+^* is the same as the map induced by j_+ on ordinary K-theory, which is $j_+^*(m) = (m, 0)$. This is as expected, because both bundles are trivialised via t_+. On D_-^{2n+1}, however, we must change coordinates via the transition function $t_+ \circ (t_-)^{-1}$ so that we are trivialising the bundle over S^{2n} via t_+ rather than t_-. This gives the diagram

$$\begin{array}{ccc} K^*(D_-^{2n+1}, \delta) & \xrightarrow{{j_-}^*} & K^*(S^{2n}, \delta) \\ \Big\downarrow{\scriptstyle (t_-)_*} & & \Big\downarrow{\scriptstyle (t_+)_*} \\ K^*(D_-^{2n+1}) & \longrightarrow & K^*(S^{2n}) \end{array} \tag{5.1.3}$$

and thus j_-^* can be viewed as the map induced by j_- on ordinary K-theory followed by $(t_+ \circ (t_-)^{-1})_*$. Since $(t_+ \circ (t_-)^{-1})_*$ is multiplication by f^{-1}, we seek the map induced by the composition

$$C(D_-^{2n+1}, \mathcal{A}_\delta|_{D_-^{2n+1}}) \xrightarrow{\text{res}} C(S^{2n}, \mathcal{A}_\delta|_{S^{2n}}) \xrightarrow{\times f^{-1}} C(S^{2n}, \mathcal{A}_\delta|_{S^{2n}}).$$

In the classical case when $N = 0$ this is the map $n \mapsto (n, 0)$, but if $N \neq 0$ then the second component of this map is non-trivial, resulting in $n \mapsto (n, -Nn)$ with the factor of $-N$ corresponding to the multiplication by f^{-1}.

Thus $\pi_*(m, n) = (m, 0) - (n, -Nn) = (m - n, Nn)$, which has trivial kernel and whose cokernel is $(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \oplus N\mathbb{Z}) \cong \mathbb{Z}_N$ when $N \neq 0$. So we are able to conclude via the exact sequence that $K^0(S^{2n+1}, \delta) = 0$ while $K^1(S^{2n+1}, \delta) \cong \mathbb{Z}_N$. Note that if $N = 0$ we instead have $\pi_*(m, n) = (m - n, 0)$ with kernel and cokernel $\mathbb{Z}$ corresponding to the standard topological K-theory of S^{2n+1}. □

While this computation shows that the higher twisted K-theory of the odd-dimensional spheres agrees with the classical notion of twisted K-theory for S^3, it is desirable to have an explicit geometric representative for the generator of this group. In order to obtain this, we shift our viewpoint to the equivalent definition of higher twisted K-theory in terms of generalised Fredholm operators presented in Theorem 2.5.1. Firstly, we need a lemma allowing us to view the higher twisted K-theory of the odd-dimensional spheres in a slightly different way.

Lemma 5.1.2. *The higher twisted K-theory group $K^1(S^{2n+1}, \delta)$ can be expressed as*

$$\pi_0(\{(h_+, h_-) \in C(D_+^{2n+1} \amalg D_-^{2n+1}, \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}) : h_+(x) = f(x) \cdot h_-(x) \,\forall\, x \in S^{2n}\}).$$

Proof. Recall that $K^1(S^{2n+1}, \delta) = \pi_0(C(\mathcal{E}_\delta, \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})})$. Since $\mathcal{E}_\delta$ is constructed via the clutching construction, a map in $C(\mathcal{E}_\delta, \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})^{\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})}$ can be viewed as a pair of maps $h_\pm : D_\pm^{2n+1} \to \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ satisfying $h_+(x) = f(x) \cdot h_-(x)$ for all $x \in S^{2n}$. Conversely, given any such pair of maps $h_\pm$, these maps may be glued together to form the equivariant map $h : \mathcal{E}_\delta \to \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}$ via

$$h([x, v])(t)(o) = \begin{cases} h_+(x)(t)(v \cdot o) & \text{if } x \in D_+^{2n+1}; \\ h_-(x)(t)(v \cdot o) & \text{if } x \in D_-^{2n+1}; \end{cases}$$

where $x \in S^{2n+1}$, $v \in \mathcal{O}_\infty \otimes \mathcal{K}$, $t \in S^1$ and $o \in \mathcal{H}_{\mathcal{O}_\infty \otimes \mathcal{K}}$. Firstly, note that h is well-defined. If $[(x, v)]$ is chosen with $x \in S^{2n}$, then the two possible representatives of this point are $(x, v) \in D_+^{2n+1} \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ and $(x, f(x)v) \in D_-^{2n+1} \times \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$. But $h_-(x)(t)(f(x)v \cdot o) = (f(x) \cdot h_-(x))(t)(v \cdot o)$ by the definition of the action, and thus this is equal to $h_+(x)(t)(v \cdot o)$ by the compatibility of the maps on the equatorial sphere. Furthermore, the map h is $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-equivariant again by the definition of the action as required. □

Using this viewpoint, we are able to construct a representative for the generator of the higher twisted K-theory group. In our description of the generator, we use an isomorphism between the hemisphere D^{2n+1}_+ with its boundary identified to a point and the sphere S^{2n+1} to view a map on S^{2n+1} as a map on D^{2n+1}_+ which is constant on the boundary. We illustrate this for clarity in Figure 5.1.1 in the case of S^2 which can be visualised.

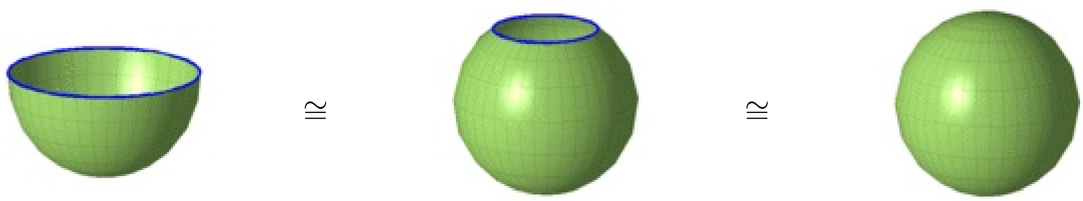


Figure 5.1.1: Illustration of the isomorphism between the hemisphere D^2_+ with its boundary identified to a point and the 3-sphere S^2. Adapted from [Bha11].

Proposition 5.1.3. *The generator of $K^1(S^{2n+1}, \delta)$ can be represented by the pair of maps $h_\pm$ where h_+ is obtained by taking the generator $k \in \pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$ and viewing this as a map $D^{2n+1}_+ \to \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty\otimes\mathcal{K}}$ which is constant on the equatorial sphere via the isomorphism displayed in Figure 5.1.1, and h_- is defined to be a loop which remains constant at the identity operator.*

Proof. In order to obtain a generator, we use a different short exact sequence of C^*-algebras to obtain a six-term exact sequence in K-theory. Here we take the sequence

$$0 \to C_0(\mathbb{R}^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\iota} A_\delta \xrightarrow{\pi} C(x_0, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0$$

for $x_0 \in S^{2n+1}$ defined so that $S^{2n+1} \setminus \{x_0\} \cong \mathbb{R}^{2n+1}$, with the obvious maps for $x_0 \notin S^{2n}$. Note that we may take sections of the trivial bundle over $\mathbb{R}^{2n+1}$ since there are no non-trivial principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundles over $\mathbb{R}^{2n+1}$, and similarly for $\{x_0\}$. This gives rise to the six-term exact sequence

$$\begin{array}{ccccc} 0 = K^0(\mathbb{R}^{2n+1}) & \xrightarrow{\iota_*} & K^0(S^{2n+1}, \delta) & \xrightarrow{\pi_*} & K^0(\{x_0\}) = \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \partial \\ 0 = K^1(\{x_0\}) & \xleftarrow[\pi_*]{} & K^1(S^{2n+1}, \delta) & \xleftarrow[\iota_*]{} & K^1(\mathbb{R}^{2n+1}) = \mathbb{Z}, \end{array}$$

where twisted K-theory groups equipped with the trivial twisting have been identified with their untwisted counterparts, and this reduces to

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota_*} K^1(S^{2n+1}, \delta) \to 0.$$

By Proposition 5.1.1 we know that $K^1(S^{2n+1}, \delta) = \mathbb{Z}_N$ and so ι_* is a surjective map from $\mathbb{Z}$ to $\mathbb{Z}_N$. This means that it must be given by reduction modulo N and hence the generator of $K^1(S^{2n+1}, \delta)$ is the image of the generator of $K^1(\mathbb{R}^{2n+1}) \cong \mathbb{Z}$ under ι_*. The map ι_* can be interpreted by making the following identifications:

$$\begin{aligned} K^1(\mathbb{R}^{2n+1}) &= \widetilde{K}^1(S^{2n+1}) \\ &\cong K^1(S^{2n+1}) \\ &\cong [S^{2n+1}, \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}] \\ &= \pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}), \end{aligned}$$

where we use the fact that the ordinary topological K-theory of S^{2n+1} is the same as the higher twisted K-theory of S^{2n+1} with trivial twist.

In order to realise the reduction modulo N map from $\pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$, we let $[k : S^{2n+1} \to \Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}}] \in \pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$ be the generator and by identifying S^{2n+1} with $D_+^{2n+1}/\sim$ as illustrated in Figure 5.1.1, we view k as a map h_+ on D_+^{2n+1} which is constant at the identity on the equatorial sphere. Then defining a map h_- on D_-^{2n+1} to be a loop which is constant at the identity gives a pair $[h_\pm] \in K^1(S^{2n+1}, \delta)$ via Lemma 5.1.2. Applying this process with M times the generator of $\pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$ yields an element of $K^1(S^{2n+1}, \delta)$ which is M mod N times the generator. Thus the generator of $K^1(S^{2n+1}, \delta)$ is obtained by applying this process to the generator k itself as required. □

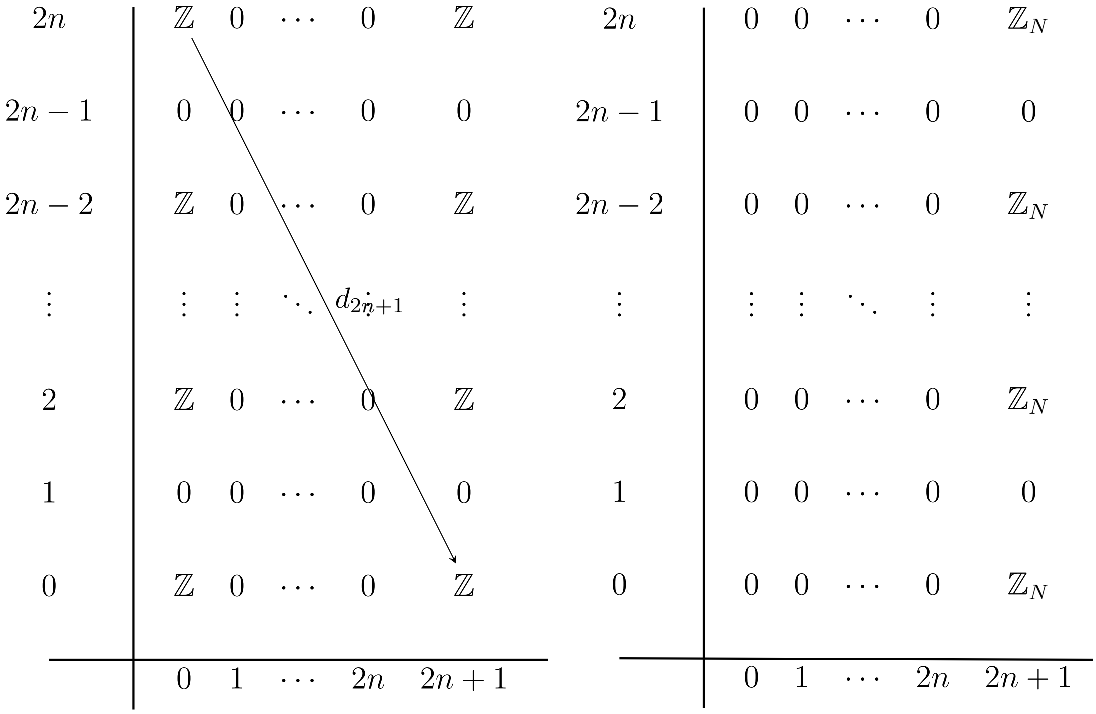
Note that the choice of generator of $\pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$ is once again unimportant here, and will yield two different generators 1 and $N-1$ of $\mathbb{Z}_N$.

This formulation of the generator agrees with that of Mickelsson in the classical twisted setting [Mic02], and the existence of such an explicit generator in terms of the generator of $\pi_{2n+1}(\Omega\,\mathrm{Fred}_{\mathcal{O}_\infty \otimes \mathcal{K}})$ may have a physical interpretation which could be used to further investigate relevant areas of physics.

It should be noted that obtaining explicit expressions for the generators of higher twisted K-theory groups is difficult in general, as in this case we relied on applying the Mayer–Vietoris sequence as well as a useful identification of a topological K-theory group with a homotopy group. This will not be possible in other cases, and so further work can be done in finding more general methods to express the generators of higher twisted K-theory groups.

To complete our computations for odd-dimensional spheres, we provide an alternative, more straightforward proof of Proposition 5.1.1 using the twisted Atiyah–Hirzebruch spectral sequence.

Proof of Proposition 5.1.1. We use the spectral sequence in Theorem 4.2.3. Since the spheres have integral cohomology $H^p(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ if and only if $p = 0$ or $2n+1$ and is trivial otherwise, we see that $E_2^{p,q} \cong E_r^{p,q}$ for $2 \leq r \leq 2n$. The only non-zero differential



(a) E_2-term (b) E_∞-term

Figure 5.1.2: Atiyah–Hirzebruch spectral sequence for S^{2n+1}.

is then $d_{2n+1} : H^0(S^{2n+1}, \mathbb{Z}) \to H^{2n+1}(S^{2n+1}, \mathbb{Z})$, as displayed on the left side of Figure 5.1.2. By Theorem 4.2.4, this differential is given by $d_{2n+1}(x) = d'_{2n+1}(x) - x \cup \delta$ where d'_{2n+1} is some torsion operator, i.e. the image of d'_{2n+1} is torsion. Thus the differential is simply cup product with δ, meaning that the $E_{2n+1} \cong \cdots \cong E_\infty$ term is as shown on the right of Figure 5.1.2. Then by the convergence of the spectral sequence, we may once again conclude that $K^0(S^{2n+1}, \delta) = 0$ while $K^1(S^{2n+1}, \delta) \cong \mathbb{Z}_N$. □

While this computation is much more manageable, it cannot be used to obtain any information about the generators of the higher twisted K-theory groups. Due to the simplicity of this method, however, it will prove useful in computing the higher twisted K-theory of more complicated spaces.

5.1.2 Products of spheres

We may generalise the results of the previous section by considering products of spheres. In theory, it is possible to consider any product of spheres consisting of at least one odd-dimensional sphere, since the product will have torsion-free cohomology and non-trivial cohomology in at least one odd degree. In practice, however, without developing more general techniques we are limited to a smaller class of products.

One such product that we can compute via the same methods as used to prove Proposition 5.1.1 is $S^{2m} \times S^{2n+1}$ for $m, n \geq 1$. This space has two non-trivial odd-degree cohomology groups: one in degree $2n+1$ and one in degree $2m+2n+1$, both of which are isomorphic to $\mathbb{Z}$. The clutching construction is only applicable to those twists coming from $(2n+1)$-classes, since a $(2m+2n+1)$-class would correspond to a gluing map from S^{2m+2n} to the automorphism group of $\mathcal{O}_\infty \otimes \mathcal{K}$ and this will give a bundle over $S^{2m+2n+1}$ as opposed to $S^{2m} \times S^{2n+1}$. In spite of this, we can use a modified version of our clutching construction to deal with the twists of degree $2n+1$ and then use a Mayer–Vietoris sequence to compute the higher twisted K-theory groups.

Proposition 5.1.4. *Let $\delta \in H^{2n+1}(S^{2m} \times S^{2n+1}, \mathbb{Z})$ be a twist of K-theory for $S^{2m} \times S^{2n+1}$ which is N times the generator. The higher twisted K-theory of $S^{2m} \times S^{2n+1}$ is then*

$$K^0(S^{2m} \times S^{2n+1}, \delta) = 0 \quad \textit{and} \quad K^1(S^{2m} \times S^{2n+1}, \delta) = \mathbb{Z}_N \oplus \mathbb{Z}_N$$

if $N \neq 0$, or

$$K^0(S^{2m} \times S^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z} \quad \textit{and} \quad K^1(S^{2m} \times S^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}$$

when $N = 0$.

Proof. In order to construct a bundle representing to a class in $H^{2n+1}(S^{2m} \times S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$, we modify our previous approach by taking trivial bundles over $S^{2m} \times D^{2n+1}_{\pm}$ and modifying the gluing map $[f] \in \pi_{2n}(\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})) \cong H^{2n+1}(S^{2n+1})$ to $\widetilde{f} : S^{2m} \times S^{2n} \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ which is constant over the S^{2m} factor, i.e. $\widetilde{f}(x, y) = f(y)$. Then we obtain a principal $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$-bundle over $S^{2m} \times S^{2n+1}$ which pulls back to a trivial bundle over S^{2m} (since there are no non-trivial bundles over this space) and to the clutching bundle as constructed previously over S^{2n+1}. Lemma 3.1.2 implies that the algebra of sections of the associated algebra bundle with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ will be of the form

$$\begin{aligned} A_\delta &= C(S^{2m} \times S^{2n+1}, \mathcal{E}_{\widetilde{f}}) \\ &= \{(h_+, h_-) \in C((S^{2m} \times D^{2n+1}_+) \amalg (S^{2m} \times D^{2n+1}_-))\mathcal{O}_\infty \otimes \mathcal{K}) : h_+(x, y) = f(y)(h_-(x, y))\}, \end{aligned}$$

and so we form the short exact sequence of C^*-algebras

$$0 \to A_\delta \xrightarrow{\iota} C((S^{2m} \times D^{2n+1}_+) \amalg (S^{2m} \times D^{2n+1}_-), \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\pi} C(S^{2m} \times S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0$$

with the obvious maps. Then the corresponding six-term exact sequence in K-theory is

$$\begin{array}{ccccc} K_0(A_\delta) & \xrightarrow{\iota_*} & K_0(C((S^{2m} \times D^{2n+1}_+) \amalg (S^{2m} \times D^{2n+1}_-))) & \xrightarrow{\pi_*} & K_0(C(S^{2m} \times S^{2n})) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(S^{2m} \times S^{2n})) & \xleftarrow{\pi_*} & K_1(C((S^{2m} \times D^{2n+1}_+) \amalg (S^{2m} \times D^{2n+1}_-))) & \xleftarrow{\iota_*} & K_1(A_\delta), \end{array}$$

where we have once again used trivialisations of the algebra bundle to identify higher twisted K-theory groups with topological K-theory groups. This sequence reduces to

$$0 \to K^0(S^{2m} \times S^{2n+1}, \delta) \to \mathbb{Z}^4 \xrightarrow{\pi_*} \mathbb{Z}^4 \to K^1(S^{2m} \times S^{2n+1}, \delta) \to 0,$$

where the map π_* can be analysed in the same way as in the proof of Proposition 5.1.1. Studying trivialisations of the bundle over $S^{2m} \times D^{2n+1}_{\pm}$ yields similar expressions for the transition functions as found in the proof of Proposition 5.1.1, and studying the same type of commutative diagrams as (5.1.2) and (5.1.3) gives insight into the induced maps on higher twisted K-theory. We find that the map on higher twisted K-theory induced by inclusion $j_+ : S^{2m} \times D^{2n+1}_+ \to S^{2m} \times S^{2n+1}$ is the same as the map induced in topological K-theory, which is

$$\mathbb{Z}^2 \cong K^0(S^{2m} \times D^{2n+1}_+) \to K^0(S^{2m} \times S^{2n}) \cong \mathbb{Z}^4$$

defined by $(m, n) \mapsto (m, 0, n, 0)$. The map induced by j_- is again the ordinary map in K-theory followed by the map induced by multiplication by f^{-1}, which in this case will be $(o, p) \mapsto (o, -oN, p, -pN)$ as previously, where the two factors of N appear since there are two even-dimensional spheres involved here. Therefore the map π_* of interest can be expressed as $\pi_*(m, n, o, p) = (m - o, oN, n - p, pN)$ which has trivial kernel and whose cokernel is $\mathbb{Z}_N \oplus \mathbb{Z}_N$ when $N \neq 0$ as required. When $N = 0$, the map π_* has both kernel and cokernel isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ as required. □

Here we have two different generators of order N for the K^1-group, and so it would be of interest to explicitly write down these generators. The Mayer–Vietoris technique used for S^{2n+1}, however, does not generalise to this case and so this would require the development of further machinery.

We may also partially verify our computation using the Atiyah–Hirzebruch spectral sequence.

Example 5.1.1. We will present the computation for the case $S^2 \times S^5$ for simplicity, but the other cases follow via the same argument. The E_2-term of this spectral sequence is displayed on the left of Figure 5.1.3, where the only non-zero differential is d_5 which acts as cup product with the twisting class, i.e. multiplication by N. The resulting E_∞-term is then displayed on the right of Figure 5.1.3.

From the spectral sequence is is easy to conclude that $K^0(S^{2m} \times S^{2n+1}, \delta) = 0$, but the K^1 group cannot be determined without solving the extension problem

$$0 \to \mathbb{Z}_N \to K^1(S^{2m} \times S^{2n+1}, \delta) \to \mathbb{Z}_N \to 0.$$

We see that the true answer is one of N non-equivalent solutions to this extension problem, and so once again it is due to the simplicity of this example that the complete solution can be found via the Mayer–Vietoris sequence.

4	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	0	1	2	3	4	5	6	7

(a) E_2-term

4	0	0	0	0	0	$\mathbb{Z}_N$	0	$\mathbb{Z}_N$
3	0	0	0	0	0	0	0	0
2	0	0	0	0	0	$\mathbb{Z}_N$	0	$\mathbb{Z}_N$
1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	$\mathbb{Z}_N$	0	$\mathbb{Z}_N$
	0	1	2	3	4	5	6	7

(b) E_∞-term

Figure 5.1.3: Atiyah–Hirzebruch spectral sequence for $S^2 \times S^5$.

There is yet another way that we may verify this computation, which is by applying a Künneth theorem in C^*-algebraic K-theory.

Example 5.1.2. We use the Künneth theorem given in Theorem 23.1.3 of [Bla86], which states that

$$0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

is a short exact sequence if A belongs in the bootstrap category of C^*-algebras defined in Definition 22.3.4 of [Bla86]. In our case, we let A denote the continuous complex valued functions on S^{2m} – which belongs in the bootstrap category – so that $K_*(A) = K^*(S^{2m})$, and we take B to be the algebra of sections $\mathcal{A}_\delta$ as in the proof of Proposition 5.1.4 so that $K_*(B) = K^*(S^{2n+1}, \delta)$. Since $K_*(A)$ is torsion-free, this means that the Tor term will be trivial and thus we obtain an isomorphism $K_*(A) \otimes K_*(B) \cong K_*(A \otimes B)$. Furthermore, since the algebra bundle $\mathcal{E}_{\tilde{f}}$ over $S^{2m} \times S^{2n+1}$ is trivial over the factor of S^{2m}, we see that the sections of the bundle can be split into

$$C(S^{2m} \times S^{2n+1}, \mathcal{E}_{\tilde{f}}) = C(S^{2m}, \mathcal{O}_\infty \otimes \mathcal{K}) \otimes C(S^{2n+1}, \mathcal{E}_{\tilde{f}}|_{S^{2n+1}}) \cong C(S^{2m}) \otimes C(S^{2n+1}, \mathcal{E}_f).$$

Therefore $A \otimes B$ is isomorphic to the space of sections of $\mathcal{E}_{\tilde{f}}$, and so we conclude that $K_*(A \otimes B) = K^*(S^{2m} \times S^{2n+1}, \delta)$. Hence the isomorphism given by the Künneth theorem verifies that

$$K^0(S^{2m} \times S^{2n+1}, \delta) \cong ((\mathbb{Z} \oplus \mathbb{Z}) \otimes 0) \oplus (0 \otimes \mathbb{Z}_N) = 0$$

and

$$K^1(S^{2m} \times S^{2n+1}, \delta) \cong ((\mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z}_N) \oplus (0 \otimes 0) = \mathbb{Z}_N \oplus \mathbb{Z}_N$$

as required.

Although the higher twisted K-theory groups corresponding to elements of the degree $2m+2n+1$ integral cohomology of $S^{2m} \times S^{2n+1}$ cannot be determined using the same clutching construction and six-term exact sequence, this computation can be performed via the spectral sequence.

Proposition 5.1.5. *Let $\delta \in H^{2m+2n+1}(S^{2m} \times S^{2n+1}, \mathbb{Z})$ be a higher twist of K-theory for $S^{2m} \times S^{2n+1}$ which is N times the generator. The higher twisted K-theory of $S^{2m} \times S^{2n+1}$ is then*

$$K^0(S^{2m} \times S^{2n+1}, \delta) = \mathbb{Z} \quad \text{and} \quad K^1(S^{2m} \times S^{2n+1}, \delta) = \mathbb{Z} \oplus \mathbb{Z}_N$$

if $N \neq 0$, or

$$K^0(S^{2m} \times S^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad K^1(S^{2m} \times S^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}$$

when $N = 0$.

Proof. The E_2-term will be the same as that given in Figure 5.1.3, but the difference is that the differential only acts from H^0 to $H^{2m+2n+1}$. The E_∞-term will then retain the copies of $\mathbb{Z}$ in the $2m$ and $2n+1$ columns, while the $2m+2n+1$ column will again contain $\mathbb{Z}_N$ when $N \neq 0$. The result then follows for $N \neq 0$ without extension problems. When $N = 0$, there will be no non-trivial differentials in which case the E_∞-term will be the same as the E_2-term and the result follows. □

Remark 5.1.1. We cannot verify this computation using the Künneth theorem as we could for Proposition 5.1.4, because we cannot express the space of sections of the bundle representing $\delta \in H^{2m+2n+1}(S^{2m} \times S^{2n+1}, \mathbb{Z})$ as a tensor product. Suppose we were to decompose δ into the cup product of $\delta_{2m} \in H^{2m}(S^{2m}, \mathbb{Z})$ and $\delta_{2n+1} \in H^{2n+1}(S^{2n+1}, \mathbb{Z})$ via a Künneth theorem in cohomology, and then we let A be the space of sections of the bundle over S^{2m} represented by δ_{2m} and B be the space of sections of the bundle over S^{2n+1} represented by δ_{2n+1}. Then A and B would be exactly as in Example 5.1.2, since there are no non-trivial algebra bundles over S^{2m} with fibres isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$, and so the tensor product algebra $A \otimes B$ would be the same as it was in Example 5.1.2. Therefore we cannot use the same approach to verify this computation.

Of course there are many other possible products of spheres that can be investigated, and the spectral sequence can be used in a straightforward way to draw conclusions about the higher twisted K-theory groups. In spite of this, most cases involve some non-trivial extension problems to be solved and so it is difficult to obtain results for products of spheres in full generality using current techniques.

5.1.3 Lie groups

A great deal of work has been done by many mathematicians and physicists in computing the twisted K-theory of Lie groups in the classical setting, including Hopkins, Braun [Bra04], Douglas [Dou06] and Rosenberg [Ros17]. In the case of $SU(n)$, the twisted K-groups were explicitly computed and as a consequence it was shown that the higher differentials in the twisted Atiyah–Hirzebruch spectral sequence are non-zero in general, suggesting that this technique will not yield general results for the higher twisted K-groups of $SU(n)$. Nevertheless, it is possible to compute these groups via the Atiyah–Hirzebruch spectral sequence in a special case.

We compute the higher twisted K-theory of $SU(n)$ up to extension problems for δ a $2n-1$ twist. Although this does not fully describe the higher twisted K-theory groups, it gives important information regarding torsion and the maximum order of elements in the groups. To illustrate the general technique, we will explicitly compute the higher twisted K-theory of $SU(3)$ for a 5-twist, and later use this in the general computation. Note that since we have proved in Proposition 2.3.10 that higher twisted K-theory with the trivial twist agrees with topological K-theory and we have seen several examples of this already in this hapter, we will henceforth only consider non-trivial twists.

Lemma 5.1.6. *Let $\delta \in H^5(SU(3),\mathbb{Z})$ be a twist of K-theory for $SU(3)$ which is $N \neq 0$ times the generator. The 5-twisted K-theory of $SU(3)$ is then*

$$K^0(SU(3),\delta) = \mathbb{Z}_N \quad \text{and} \quad K^1(SU(3),\delta) = \mathbb{Z}_N.$$

Proof. The E_2-page of the twisted Atiyah–Hirzebruch spectral sequence in this case is as follows.

2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$
	0	1	2	3	4	5	6	7	8

The d_3 differential is given by the Steenrod operation Sq^3 which necessarily annihilates $H^0(SU(3),\mathbb{Z})$ by Theorem 4L.12 of [Hat00], but also annihilates $H^5(SU(3),\mathbb{Z})$ since the image of Sq^3 is a 2-torsion element by definition. Hence the only non-trivial differential in this spectral sequence is $d_5(x) = d_5'(x) - x \cup \delta$. The torsion operator d_5' will have no effect on the cohomology, and cup product with $-\delta$ will be multiplication by $-N$ on both $H^0(SU(3),\mathbb{Z})$ and $H^3(SU(3),\mathbb{Z})$. Hence the E_∞-term is as shown below.

2	0	0	0	0	0	$\mathbb{Z}_N$	0	0	$\mathbb{Z}_N$
1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	$\mathbb{Z}_N$	0	0	$\mathbb{Z}_N$
	0	1	2	3	4	5	6	7	8

Thus we may conclude that $K^0(SU(3),\delta) \cong \mathbb{Z}_N$ and $K^1(SU(3),\delta) \cong \mathbb{Z}_N$ as required. □

Note that this computation works specifically for 5-twists δ, as unlike when taking a 3-twist there are no non-trivial higher differentials to consider. Furthermore, there is no extension problem to solve and so for the case of $SU(3)$ this is a complete computation.

This computation directly generalises to the case of $2n-1$ twists on $SU(n)$, although here we only obtain the result up to extension problems and so we can only comment on torsion in the group.

Lemma 5.1.7. *Let $\delta \in H^{2n-1}(SU(n),\mathbb{Z})$ be a twist of K-theory for $SU(n)$ which is $N \neq 0$ times the generator. The $(2n-1)$-twisted K-theory of $SU(n)$ is then a finite abelian group with all elements having order a divisor of a power of N.*

Proof. We use the same Atiyah–Hirzebruch spectral sequence approach as in the proof of Lemma 5.1.6. The differentials d_j for $j < 2n-1$ are trivial, as they are given by torsion operations. The differential d_{2n-1} is cup product with $-\delta$, which is multiplication by $-N$ for each of the 2^{n-2} maps $\mathbb{Z}(\bigwedge c_i) \to \mathbb{Z}(\bigwedge c_i)\wedge c_{2n-1}$ where the $c_{2i-1} \in H^{2i-1}(SU(n),\mathbb{Z})$ for $i = 2,\cdots,n$ denote the primitive generators, and the higher differentials are zero. At this stage, there are 2^{n-3} extension problems to solve for $n > 3$, but no extension problems for $n = 3$ which is how the previous result was obtained. In spite of this, since every group in the E_∞-term of the spectral sequence is $\mathbb{Z}_N$, we can conclude that the higher twisted K-theory groups will be torsion with all elements having order a divisor of a power of N, even if the extension problems cannot be solved to determine the explicit torsion. □

To be more explicit about the extension problems involved, we consider the case of a 7-twist on $SU(4)$ as an example.

Example 5.1.3. Following the proof of Lemma 5.1.7, we have to solve a single extension problem both for the odd degree and even degree groups of the form

$$0 \to \mathbb{Z}_N \to K^i(SU(4),\delta) \to \mathbb{Z}_N \to 0.$$

Although this extension problem has $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_N,\mathbb{Z}_N) \cong \mathbb{Z}_N$ inequivalent solutions, we can conclude that $K^i(SU(4),\delta)$ is a torsion group whose elements have order a divisor of N^2.

While an explicit computation of the higher twisted K-theory of $SU(n)$ is quite difficult in general, we can generalise some results of Rosenberg to the higher twisted setting in order to obtain more non-trivial structural information about these groups. It is at this point that we must turn to the more powerful Segal spectral sequence given in Theorem 4.2.5 so that Theorem 4.2.6 may be employed.

Theorem 5.1.8. *For any non-zero $\delta \in H^5(SU(n+1), \mathbb{Z})$ given by N times the generator with N relatively prime to $n!$ ($n > 1$), the graded group $K^*(SU(n+1), \delta)$ is isomorphic to $\mathbb{Z}_N$ tensored with an exterior algebra on $n-1$ odd generators.*

Proof. We proceed by induction on n. First, note that the case $n = 2$ has already been proved in Lemma 5.1.6, as we have shown that $K^0(SU(3), \delta) \cong \mathbb{Z}_N \cong K^1(SU(3), \delta)$ so that $K^*(SU(3), \delta)$ is of the form $\mathbb{Z}_N$ tensored with $\mathbb{Z}c$ for some odd generator c. Then by Proposition 4.2.7, the same is true for the higher twisted K-homology groups. So assume $n > 2$ and that the result holds for smaller values of n. Take the Segal spectral sequence in higher twisted K-homology associated to the classical fibration

$$SU(n) \xhookrightarrow{\iota} SU(n+1) \to S^{2n+1},$$

which gives

$$E^2_{p,q} = H_p(S^{2n+1}, K_q(SU(n), \iota^*\delta)) \Rightarrow K_*(SU(n+1), \delta).$$

Note that since the map ι^* induced on ordinary cohomology by inclusion is an isomorphism in degree 5, we may identify $\iota^*\delta \in H^5(SU(n), \mathbb{Z})$ with $\delta \in H^5(SU(n+1), \mathbb{Z})$. Since N is relatively prime to $(n-1)!$, by the inductive assumption we have

$$K_q(SU(n), \delta) \cong \mathbb{Z}_N \otimes \wedge(x_1, \cdots, x_{n-2})$$

for some odd generators x_i. We aim to show that this spectral sequence collapses. The only potentially non-zero differential is d^{2n+1}, which is related to the homotopical non-triviality of the fibration as explained in Theorem 4.2.6. To determine the explicit differential, we need to understand the Hurewicz maps mentioned in the theorem and the long exact sequence in homotopy for the fibration $SU(n) \xrightarrow{\iota} SU(n+1) \to S^{2n+1}$. This long exact sequence contains

$$\pi_{2n+1}(SU(n+1)) \to \pi_{2n+1}(S^{2n+1}) \xrightarrow{\partial} \pi_{2n}(SU(n)) \to \pi_{2n}(SU(n+1)),$$

and so we see that the boundary map $\partial : \mathbb{Z} \to \mathbb{Z}_{n!}$ has kernel of index $n!$. Now, the Hurewicz map of interest is

$$\pi_{2n}(SU(n)) \to K_{2n}(SU(n), \delta) \cong K_0(SU(n), \delta).$$

Although this map is difficult to describe explicitly, since this is a map $\mathbb{Z}_{n!} \to \mathbb{Z}_N$ then if $\gcd(N, n!) = 1$ this map must be trivial and hence the differential is trivial. Thus if

$\gcd(N, n!) = 1$ then the spectral sequence collapses and $K_*(SU(n+1), \delta)$ is isomorphic to $\mathbb{Z}_N$ tensored with an exterior algebra on $n-1$ odd generators since the E^∞-term of the spectral sequence will consist of $\mathbb{Z}_N \otimes \wedge(x_1, \cdots, x_{n-2})$ in the zeroth and $(2n+1)$th columns which will become $K_0(SU(n+1), \delta)$ and $K_1(SU(n+1), \delta)$ respectively.

In order to conclude that the same is true for higher twisted K-theory, we see that the E_2-term of the Segal spectral sequence in higher twisted K-theory consists only of finite torsion groups, and even though we do not have information about the differentials in this sequence we may conclude that the E_∞-term will also consist only of finite torsion groups and thus the limit of the sequence is a direct sum of torsion groups. Therefore we are in the setting of Proposition 4.2.7, and we may use this to obtain the result for higher twisted K-theory from the computation for higher twisted K-homology. $\square$

Remark 5.1.2. We reiterate our concerns raised in Remark 4.2.2, that the proof of Proposition 4.2.7 relies on an assumption which is based on a conjecture in C^*-algebra theory. Therefore this theorem should be viewed in light of this assumption.

We also have a structural theorem which is applicable in a more general setting, but which provides slightly less information about the higher twisted K-theory groups.

Theorem 5.1.9. *If $\delta_k \in H^k(SU(n), \mathbb{Z})$ is given by N times any primitive generator of $H^*(SU(n), \mathbb{Z})$ (all of which have odd degree) then $K^*(SU(n), \delta_k)$ is a finite abelian group and all elements have order a divisor of a power of N.*

Proof. Again we proceed by induction on n, and observe that this has already been proved for the base case in Lemma 5.1.7. Hence we need only show that under this assumption it is true for $\delta_{2n-1} \in H^{2n-1}(SU(n+1), \mathbb{Z})$. To do this, we once again use the classical fibration over S^{2n+1} and apply the Segal spectral sequence, this time simply in higher twisted K-theory, and obtain

$$E_2^{p,q} = H^p(S^{2n+1}, K^q(SU(n), \delta_{2n-1})) \Rightarrow K^*(SU(n+1), \delta_{2n-1}).$$

But $K^q(SU(n), \delta_{2n-1})$ is torsion with all elements of order a divisor of a power of N by the inductive assumption, and so the same is true for E_2 and thus E_∞. Finally, even if there are non-trivial extension problems to solve in order to obtain $K^*(SU(n+1), \delta_{2n-1})$, the result is still true as argued in the proof of Lemma 5.1.7. $\square$

In special cases, this result yields particularly useful information about these groups. We can observe that if $N = 1$ then $K^*(X, \delta)$ vanishes identically, and if $N = p^r$ is a prime power then $K^*(X, \delta)$ is a p-primary torsion group.

Whilst these results are not completely general, they do provide some insight into the complicated behaviour of the higher twisted K-theory of $SU(n)$. Furthermore, if the Hurewicz map could be better understood then this may allow for better descriptions of the higher twisted K-groups of $SU(n)$ in general. Rosenberg is able to draw more general conclusions in the classical twisted case by using a universal coefficient theorem of Khorami [Kho11]. This universal coefficient theorem relies on techniques that we have not developed, but given a greater understanding of the cohomology groups of $\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{K})$ this could allow for the theorem, and thus the results in this section, to be generalised.

To conclude this section, we take a brief look at another of the classical groups whose higher twisted K-theory we can gain insight into using analogous methods to those above – the symplectic groups $Sp(n)$. We are using the compact symplectic group, defined to be the intersection of the unitary group $U(2n)$ and the $2n \times 2n$ symplectic matrices

$$\left\{ M \in M_{2n\times 2n}(\mathbb{C}) : M^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Since the cohomology of $Sp(n)$ is given by $H^*(Sp(n), \mathbb{Z}) \cong \wedge[x_3, x_7, \cdots, x_{4n-1}]$ where the $x_i \in H^i(Sp(n), \mathbb{Z})$, the computation of $K^*(Sp(n), \delta)$ for $\delta \in H^{4n-1}(Sp(n), \mathbb{Z})$ proceeds in the same manner as that for the $(2n-1)$-twisted K-theory of $SU(n)$.

Example 5.1.4. For instance, taking a 7-twist on $Sp(2)$ we obtain the following E_2-page of the twisted Atiyah–Hirzebruch spectral sequence.

2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$
	0	1	2	3	4	5	6	7	8	9	10

As argued earlier, the only non-zero differential will be d_7 which will be cup product with the negation of the twisting class, or equivalently multiplication by $-N$ as a map $\mathbb{Z} \to \mathbb{Z}$. Thus the only non-zero terms of the E_∞-page will be $E_\infty^{2i,7}$ and $E_\infty^{2i,10}$ which will be $\mathbb{Z}_N$. Hence the higher twisted K-groups are given by $K^0(Sp(2), \delta) \cong \mathbb{Z}_N \cong K^1(Sp(2), \delta)$.

From this, we see that the computation for the $(4n-1)$-twisted K-theory of $Sp(n)$ follows the same methods as the $(2n-1)$-twisted K-theory of $SU(n)$, and the same result is obtained (possibly with a different solution to the extension problem). So once again we obtain that $K^i(Sp(n), \delta)$ is a torsion group whose elements have order a divisor of a power of N.

As was the case for $SU(n)$, we may use this to obtain information about the general higher twisted K-groups of $Sp(n)$. We follow the exact same argument using the classical fibration $Sp(n) \to Sp(n+1) \to S^{4n+3}$ to replace $SU(n) \to SU(n+1) \to S^{2n+1}$, and this proves the following.

Proposition 5.1.10. *If $\delta_k \in H^k(Sp(n), \mathbb{Z})$ is given by N times any primitive generator of $H^*(Sp(n), \mathbb{Z})$ then $K^*(Sp(n), \delta_k)$ is a finite abelian group and all elements have order a divisor of a power of N.* □

This approach can also be applied to other compact, simply connected, simple Lie groups such as G_2, and similar approaches exist for some non-simply connected groups such as the projective special unitary groups as used in [MR16].

5.2 Torsion spaces

As discussed at the end of Section 2.4, it is possible to identify the twists of K-theory with cohomology classes even when the cohomology is not torsion-free in some cases. We perform computations with a range of spaces for which this is the case.

5.2.1 Real projective space

We begin with real projective space, and in particular odd-dimensional real projective space $\mathbb{R}P^{2n+1}$. This has integral cohomology groups

$$H^p(\mathbb{R}P^{2n+1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 2n+1; \\ \mathbb{Z}_2 & \text{if } 0 < p < 2n+1 \text{ is even;} \\ 0 & \text{else;} \end{cases} \tag{5.2.1}$$

and by the universal coefficient theorem it can be shown that the $\mathbb{Z}_2$-cohomology is

$$H^p(\mathbb{R}P^{2n+1}, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq p \leq 2n+1; \\ 0 & \text{else.} \end{cases}$$

Recall from Section 2.4 that the full set of twists of K-theory over a space X is given by the first group in a generalised cohomology theory $E^1_{\mathcal{O}_\infty}(X)$, which is computed via a spectral sequence. In order to determine whether we may identify the twists of K-theory over $\mathbb{R}P^{2n+1}$ with all odd-degree cohomology classes, we use this spectral sequence to compute $E^1_{\mathcal{O}_\infty}(\mathbb{R}P^{2n+1})$. The E_2-term of this spectral sequence is as follows.

	0	1	2	3	4	$\cdots$	$2n$	$2n+1$
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
-1	0	0	0	0	0	$\cdots$	0	0
-2	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}$
-3	0	0	0	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$-2n+1$	0	0	0	0	0	$\cdots$	0	0
$-2n$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}$

The only possibly non-trivial higher differentials in this spectral sequence will be maps $H^{odd}(\mathbb{R}P^{2n+1}, \mathbb{Z}_2) \to H^{even}(\mathbb{R}P^{2n+1}, \mathbb{Z})$ or $H^{even}(\mathbb{R}P^{2n+1}, \mathbb{Z}) \to H^{2n+1}(\mathbb{R}P^{2n+1}, \mathbb{Z})$, the latter necessarily being zero since the target is torsion-free. Furthermore, since the twists are determined specifically by the first group in this generalised cohomology theory, the only groups of interest in this spectral sequence are those circled and thus the only differentials that may have an effect are those from $H^1(\mathbb{R}P^{2n+1}, \mathbb{Z}_2)$. It is known, however, that the classical twists of K-theory are those coming from $H^1(X, \mathbb{Z}_2)$ and $H^3(X, \mathbb{Z})$, and as such these groups always form a subgroup of $E^1_{\mathcal{O}_\infty}(X)$. This means that the differentials leaving $H^1(\mathbb{R}P^{2n+1}, \mathbb{Z}_2)$ are all trivial, and thus $E^1_{\mathcal{O}_\infty}(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. As the H^1 twists are studied in the classical case, the higher twists of interest are those coming from the $H^{2n+1}(\mathbb{R}P^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ factor. The same argument may be applied to even-dimensional real projective space $\mathbb{R}P^{2n}$, but this has no non-trivial odd-dimensional cohomology groups and so the only twists of K-theory are those coming from $H^1(\mathbb{R}P^{2n}, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

We will now compute the higher twisted K-theory of $\mathbb{R}P^{2n+1}$ for a $(2n+1)$-twist δ. We provide two different proofs of this proposition using the Mayer–Vietoris sequence, the latter of which can be generalised to Lens spaces.

Proposition 5.2.1. *Let $\delta \in H^{2n+1}(\mathbb{R}P^{2n+1}, \mathbb{Z})$ be a twist of K-theory for $\mathbb{R}P^{2n+1}$ which is $N \neq 0$ times the generator. The higher twisted K-theory of $\mathbb{R}P^{2n+1}$ is then*

$$K^0(\mathbb{R}P^{2n+1}, \delta) = \mathbb{Z}_{2^n} \quad \text{and} \quad K^1(\mathbb{R}P^{2n+1}, \delta) = \mathbb{Z}_N.$$

Proof 1. Letting $\mathcal{A}_\delta$ be the algebra bundle with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ over $\mathbb{R}P^{2n+1}$ corresponding to the twist δ, we consider the short exact sequence of C^*-algebras

$$0 \to C_0(\mathbb{R}^{2n+1}, \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\iota} C(\mathbb{R}P^{2n+1}, \mathcal{A}_\delta) \xrightarrow{\pi} C(\mathbb{R}P^{2n}, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0,$$

where the maps ι and π come from viewing $\mathbb{R}^{2n+1}$ as the quotient of $\mathbb{R}P^{2n+1}$ by $\mathbb{R}P^{2n}$. The corresponding six-term exact sequence is

$$\begin{array}{ccccc} K^0(\mathbb{R}^{2n+1}) & \xrightarrow{\iota_*} & K^0(\mathbb{R}P^{2n+1}, \delta) & \xrightarrow{\pi_*} & K^0(\mathbb{R}P^{2n}) \\ \partial\uparrow & & & & \downarrow\partial \\ K^1(\mathbb{R}P^{2n}) & \xleftarrow[\pi_*]{} & K^1(\mathbb{R}P^{2n+1}, \delta) & \xleftarrow[\iota_*]{} & K^1(\mathbb{R}^{2n+1}), \end{array}$$

where higher twisted K-theory groups have been identified with topological K-theory groups via trivialisations, and this simplifies to

$$0 \to K^0(\mathbb{R}P^{2n+1}, \delta) \xrightarrow{\pi_*} \mathbb{Z}_{2^n} \oplus \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota_*} K^1(\mathbb{R}P^{2n+1}, \delta) \to 0.$$

In this case, the connecting map is given by $\partial(m, n) = nN$ as in 8.3 of [BCM⁺02] which has kernel $\mathbb{Z}_{2^n}$ and cokernel $\mathbb{Z}_N$ when $N \neq 0$ as required. □

Although we have not discussed equivariant higher twisted K-theory, the definitions and properties of the classical case generalise immediately and we use this in the following proof.

Proof 2. Viewing $\mathbb{R}P^{2n+1}$ as the quotient $S^{2n+1}/\mathbb{Z}_2$, we take the short exact sequence

$$0 \to C_0(\mathbb{R} \times S^{2n}, \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\iota} C(S^{2n+1}, \mathcal{A}_\delta) \xrightarrow{\pi} C(\{x_0, x_1\}, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0$$

where $S^{2n+1} \backslash \{x_0, x_1\} \cong \mathbb{R} \times S^{2n}$. The associated six-term exact sequence in $\mathbb{Z}_2$-equivariant K-theory is

$$\begin{array}{ccccc} K^0_{\mathbb{Z}_2}(\mathbb{R} \times S^{2n}) & \xrightarrow{\iota_*} & K^0_{\mathbb{Z}_2}(S^{2n+1}, \delta) & \xrightarrow{\pi_*} & K^0_{\mathbb{Z}_2}(\{x_0, x_1\}) \\ \partial\uparrow & & & & \downarrow\partial \\ K^1_{\mathbb{Z}_2}(\{x_0, x_1\}) & \xleftarrow[\pi_*]{} & K^1_{\mathbb{Z}_2}(S^{2n+1}, \delta) & \xleftarrow[\iota_*]{} & K^1_{\mathbb{Z}_2}(\mathbb{R} \times S^{2n}), \end{array}$$

where trivialisations have been used to identify higher twisted K-theory groups with topological K-theory groups. Here, since $\mathbb{Z}_2$ acts freely on $\{x_0, x_1\}$ we observe that $K^*_{\mathbb{Z}_2}(\{x_0, x_1\}) = K^*(\{x_0\})$. Furthermore, since $\mathbb{Z}_2$ acts freely on S^{2n+1} we also have $K^*_{\mathbb{Z}_2}(S^{2n+1}, \delta) \cong K^*(\mathbb{R}P^{2n+1}, \delta)$. Now, in order to determine the equivariant K-theory of $\mathbb{R} \times S^{2n}$, the untwisted version of this six-term exact sequence may be applied analogously to [BCM⁺02] to determine that $K^0_{\mathbb{Z}_2}(\mathbb{R} \times S^{2n}) = \mathbb{Z}_{2^n}$ while $K^1_{\mathbb{Z}_2}(\mathbb{R} \times S^{2n}) = \mathbb{Z}$. Note also that the class δ on S^{2n+1} is identified with a class on $\mathbb{R}P^{2n+1}$ under the isomorphism induced by the projection map. Thus this sequence reduces to

$$0 \to \mathbb{Z}_{2^n} \xrightarrow{\iota_*} K^0(\mathbb{R}P^{2n+1}, \delta) \xrightarrow{\pi_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota_*} K^1(\mathbb{R}P^{2n+1}, \delta) \to 0,$$

where the connecting map $\partial : \mathbb{Z} \to \mathbb{Z}$ is once again multiplication by N as in [BCM+02]. This allows us to again conclude that $K^0(\mathbb{R}P^{2n+1}, \delta) \cong \mathbb{Z}_{2^n}$ and $K^1(\mathbb{R}P^{2n+1}, \delta) \cong \mathbb{Z}_N$ as required. □

Note that this computation agrees with the classical case of the 3-twisted K-theory of $\mathbb{R}P^3$.

Although the Atiyah–Hirzebruch spectral sequence proved useful in simple computations for spheres and products of spheres, it is not so helpful in this case due to the torsion in the cohomology of $\mathbb{R}P^{2n+1}$ as we will illustrate below.

Example 5.2.1. We attempt to apply the twisted Atiyah–Hirzebruch spectral sequence to compute the higher twisted K-theory of $\mathbb{R}P^{2n+1}$, and the E_2-term is as follows.

2	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}$
1	0	0	0	0	0	$\cdots$	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}$
	0	1	2	3	4	$\cdots$	$2n$	$2n+1$

Here, the only differentials between non-trivial groups are maps $\mathbb{Z}_2 \to \mathbb{Z}$ which are necessarily zero, and $d_{2n+1} : H^0(\mathbb{R}P^{2n+1}, \mathbb{Z}) \to H^{2n+1}(\mathbb{R}P^{2n+1}, \mathbb{Z})$ which is simply given by cup product with $-\delta$ since these groups are torsion-free. Thus the E_∞-term of the spectral sequence is as follows.

2	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}_N$
1	0	0	0	0	0	$\cdots$	0	0
0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$	$\mathbb{Z}_2$	$\mathbb{Z}_N$
	0	1	2	3	4	$\cdots$	$2n$	$2n+1$

From this, we may conclude that $K^1(\mathbb{R}P^{2n+1}, \delta) \cong \mathbb{Z}_N$, but in order to determine the degree zero group $K^0(\mathbb{R}P^{2n+1}, \delta)$ we must solve $n-1$ extension problems. Assuming that all extension problems are trivial, we conclude that $K^0(\mathbb{R}P^{2n+1}, \delta) \cong \mathbb{Z}_2^n$. This agrees with the result for $\mathbb{R}P^3$ that $K^0(\mathbb{R}P^3) \cong \mathbb{Z}_2$, but as we have proved in Proposition 5.2.1 this is not the correct expression for the degree zero group. Obtaining the true solution by solving the extension problems here would be much more difficult than the method using the Mayer–Vietoris sequence.

5.2.2 Lens spaces

The equivariant computation in Proof 2 of Proposition 5.2.1 generalises nicely to Lens spaces. While Lens spaces of the form $S^3/\mathbb{Z}_p$ for p prime are particularly common, there exists a notion of higher Lens space $L(n,p) = S^{2n+1}/\mathbb{Z}_p$ where $\mathbb{Z}_p$ identified with the p^{th} roots of unity in $\mathbb{C}$ acts on $S^{2n+1} \subset \mathbb{C}^{2n}$ by scaling. In this notation, the Lens space $S^3/\mathbb{Z}_p$ is $L(1,p)$ and real projective space $\mathbb{R}P^{2n+1}$ can be viewed as $L(n,2)$. Since these higher Lens spaces are quotients of S^{2n+1}, they have non-trivial higher odd-degree cohomology groups and thus it is natural to consider their higher twisted K-theory.

As we did for $\mathbb{R}P^{2n+1}$, we must determine the group $E^1_{\mathcal{O}_\infty}(L(n,p))$ to see whether the twists of K-theory for $L(n,p)$ may be identified with all odd-degree cohomology classes. To do so, we observe that the cohomology groups of $L(n,p)$ are the same as those of $\mathbb{R}P^{2n+1}$ in (5.2.1), but with $\mathbb{Z}_2$ replaced with $\mathbb{Z}_p$. Without needing to determine $H^*(L(n,p),\mathbb{Z}_2)$ we are then able to follow the exact same argument as for $\mathbb{R}P^{2n+1}$ to conclude that $E^1_{\mathcal{O}_\infty}(L(n,p)) \cong H^1(L(n,p),\mathbb{Z}_2) \oplus H^{2n+1}(L(n,p),\mathbb{Z})$. Thus we can view the twists of K-theory over $L(n,p)$ as integral $(2n+1)$-classes.

Proposition 5.2.2. *Let $\delta \in H^{2n+1}(L(n,p),\mathbb{Z})$ be a twist of K-theory for $L(n,p)$ which is $N \neq 0$ times the generator. The higher twisted K-theory of $L(n,p)$ is then*

$$K^0(L(n,p),\delta) = \mathbb{Z}_{p^n} \quad \textit{and} \quad K^1(L(n,p),\delta) = \mathbb{Z}_N.$$

Proof. Letting $A \subset L(n,p)$ consist of a $\mathbb{Z}_p$-orbit and $\mathcal{A}_\delta$ denote the algebra bundle with fibre $\mathcal{O}_\infty \otimes \mathcal{K}$ over $L(n,p)$ corresponding to the twist δ, we take the short exact sequence

$$0 \to C_0(S^{2n+1} \setminus A, \mathcal{O}_\infty \otimes \mathcal{K}) \xrightarrow{\iota} C(S^{2n+1}, \mathcal{E}_\delta) \xrightarrow{\pi} C(A, \mathcal{O}_\infty \otimes \mathcal{K}) \to 0.$$

The associated six-term exact sequence in $\mathbb{Z}_p$-equivariant higher twisted K-theory is

$$\begin{array}{ccccc}
K^0_{\mathbb{Z}_p}(S^{2n+1} \setminus A) & \xrightarrow{\iota_*} & K^0_{\mathbb{Z}_p}(S^{2n+1},\delta) & \xrightarrow{\pi_*} & K^0_{\mathbb{Z}_p}(A) \\
\partial \uparrow & & & & \downarrow \partial \\
K^1_{\mathbb{Z}_p}(A) & \xleftarrow[\pi_*]{} & K^1_{\mathbb{Z}_p}(S^{2n+1},\delta) & \xleftarrow[\iota_*]{} & K^1_{\mathbb{Z}_p}(S^{2n+1} \setminus A),
\end{array}$$

where higher twisted K-theory groups have been identified with topological K-theory groups via trivialisations. Now, since $\mathbb{Z}_p$ acts freely on the compact set A we have $K^*_{\mathbb{Z}_p}(A) = K^*(\{x_0\})$, and similarly $K^*_{\mathbb{Z}_p}(S^{2n+1}, \delta) = K^*(L(n,p), \delta)$, where once again $\delta \in H^{2n+1}(S^{2n+1}, \mathbb{Z})$ is identified with $\delta \in H^{2n+1}(L(n,p), \mathbb{Z})$ via the isomorphism induced by the projection map. It remains to determine $K^*_{\mathbb{Z}_p}(S^{2n+1} \setminus A)$. To do so, we require some basic properties of equivariant K-theory which carry over from the standard case. Firstly, by identifying $K^*(S^{2n+1}, A)$ with $K^*(S^{2n+1} \setminus A)$, we have the short exact sequences

$$0 \to K^m_{\mathbb{Z}_p}(S^{2n+1} \setminus A) \to K^m_{\mathbb{Z}_p}(S^{2n+1}) \to K^m_{\mathbb{Z}_p}(A) \to 0$$

for $m = 0, 1$. Since the $\mathbb{Z}_p$-equivariant K-theory of A is known, this implies that $K^n_{\mathbb{Z}_p}(S^{2n+1} \setminus A) \cong \widetilde{K}^n_{\mathbb{Z}_p}(S^{2n+1})$, which is simply isomorphic to $\widetilde{K}^n(L(n,p))$. By a computation analogous to that for $\mathbb{R}P^{2n+1}$ using Corollary 2.7.6 of [Ati67], we also observe that $K^0(L(n,p)) \cong \mathbb{Z}_{p^n} \oplus \mathbb{Z}$ while $K^1(L(n,p)) \cong \mathbb{Z}$. Thus the sequence reduces to

$$0 \to \mathbb{Z}_{p^n} \xrightarrow{\iota_*} K^0(L(n,p), \delta) \xrightarrow{\pi_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota_*} K^1(L(n,p), \delta) \to 0,$$

where the connecting map $\partial : \mathbb{Z} \to \mathbb{Z}$ is multiplication by N as in 8.3 of [BCM$^+$02]. Hence $K^0(L(n,p), \delta) \cong \mathbb{Z}_{p^n}$ and $K^1(L(n,p), \delta) \cong \mathbb{Z}_N$ as required. □

This provides a generalisation of the result for $\mathbb{R}P^{2n+1}$ in Proposition 5.2.1 as well as for the 3-dimensional Lens spaces in Section 8.4 of [BCM$^+$02].

5.2.3 $SU(2)$-bundles

It is of great interest in both string theory and mathematical gauge theory to investigate $SU(2)$-bundles over manifolds M. The link with string theory appears because these bundles arise in spherical T-duality; a generalisation of T-duality investigated in a series of papers by Bouwknegt, Evslin and Mathai [BEM15a, BEM15b, BEM18] and briefly discussed in Section 2.6. As shown by the authors, this form of duality provides a map between certain conserved charges in type IIB supergravity and string compactifications. They also find links between spherical T-duality and higher twisted K-theory. Specifically in the case that M is a compact oriented 4-manifold, the authors compute the 7-twisted K-theory of a principal $SU(2)$-bundle over M in terms of its second Chern class up to an extension problem. The ordinary 3-twisted K-theory can be computed using standard techniques, which leaves the 5-twisted K-theory to be computed. We will further assume that M has torsion-free cohomology, which is true if the additional assumption that M is simply connected is made but this is not necessary.

Note that we are not requiring the $SU(2)$-bundle P over M to be a principal bundle. The computation will be valid for both principal $SU(2)$-bundles as used in [BEM15a] as well as oriented non-principal $SU(2)$-bundles used in [BEM15b], the former of which correspond to unit sphere bundles of quaternionic line bundles while the latter of which correspond to unit sphere bundles of rank 4 oriented real Riemannian vector bundles. The reason that we need not distinguish between these types of bundles is that there is a Gysin sequence in each case allowing the cohomology of P to be computed. Given an $SU(2)$-bundle of either of these forms $\pi : P \to M$, there is a Gysin sequence of the form

$$\cdots \to H^k(M,\mathbb{Z}) \xrightarrow{\pi^*} H^k(P,\mathbb{Z}) \xrightarrow{\pi_*} H^{k-3}(M,\mathbb{Z}) \xrightarrow{\cup e(P)} H^{k+1}(M,\mathbb{Z}) \to \cdots$$

where $e(P)$ denotes the Euler class of E which may be identified with the second Chern class of the associated vector bundle in the principal bundle case. In this sequence, the pushforward π_* is defined by using Poincaré duality to change from cohomology to homology, using the pushforward in homology and once again employing Poincaré duality to switch back, which explains the degree shift. We can use this to compute the integral cohomology of P in terms of that of M. In what follows, we will assume that all exact sequences split in order to compute the higher twisted K-theory up to extension problems. Although this will not be true in all cases, it still yields meaningful results as in [BEM15a, BEM15b].

Firstly, we assume that $e(P) = 0$ in which case the obstruction to $\pi : P \to M$ having a section vanishes and therefore the Gysin sequence splits at π_*. This yields

$$H^k(P,\mathbb{Z}) \cong H^k(M,\mathbb{Z}) \oplus H^{k-3}(M,\mathbb{Z}).$$

Thus we obtain

$$H^k(P,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 7; \\ H^k(M,\mathbb{Z}) & \text{if } k = 1, 2; \\ H^3(M,\mathbb{Z}) \oplus \mathbb{Z} & \text{if } k = 3; \\ H^1(M,\mathbb{Z}) \oplus \mathbb{Z} & \text{if } k = 4; \\ H^{k-3}(M,\mathbb{Z}) & \text{if } k = 5, 6. \end{cases}$$

If $e(P) = j \in \mathbb{Z}$ with $j \neq 0$ on the other hand, then the cup product with $e(P)$ from $H^0(M,\mathbb{Z})$ to $H^4(M,\mathbb{Z})$ will be multiplication by j. We obtain the same integral cohomology of P as above in all degrees except 3 and 4, and to compute these we use the Gysin sequence as follows:

$$0 \to H^3(M,\mathbb{Z}) \xrightarrow{\pi^*} H^3(P,\mathbb{Z}) \xrightarrow{\pi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup e(P)} H^4(M,\mathbb{Z}) \xrightarrow{\pi^*} H^4(P,\mathbb{Z}) \xrightarrow{\pi_*} H^1(M,\mathbb{Z}) \to 0.$$

Since the cup product here has trivial kernel, we conclude that $H^3(P,\mathbb{Z}) \cong H^3(M,\mathbb{Z})$. Similarly, the cokernel is $\mathbb{Z}_j$ and hence $H^4(P,\mathbb{Z}) \cong H^1(M,\mathbb{Z}) \oplus \mathbb{Z}_j$ assuming that the sequence is split at $\pi_* : H^4(P,\mathbb{Z}) \to H^1(M,\mathbb{Z})$.

In the case that $e(P) = 0$, we see that P has torsion-free cohomology and thus the 5-twists given by $H^5(P,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$ can be considered. If $e(P) = j \neq 0$, however, then there is torsion in the cohomology of P and so it must be determined whether all of the elements of $H^5(P,\mathbb{Z})$ correspond to twists. The E_2-term of the Atiyah–Hirzebruch spectral sequence used by Pennig and Dadarlat to compute $E^1_{\mathcal{O}_\infty}(P)$ is as follows.

	0	1	2	3	4	5	6	7
0	$\mathbb{Z}_2$	$H^1(P,\mathbb{Z}_2)$	$H^2(P,\mathbb{Z}_2)$	$H^3(P,\mathbb{Z}_2)$	$H^4(P,\mathbb{Z}_2)$	$H^5(P,\mathbb{Z}_2)$	$H^6(P,\mathbb{Z}_2)$	$\mathbb{Z}_2$
-1	0	0	0	0	0	0	0	0
-2	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}_j$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$
-3	0	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}_j$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$

Although there is torsion in $H^4(P,\mathbb{Z})$ which will affect the computation of $E^*_{\mathcal{O}_\infty}(P)$, this will not have an effect on $E^1_{\mathcal{O}_\infty}(P)$ much like was the case for $\mathbb{R}P^{2n+1}$. Since the differentials are torsion operators then $d_3 : H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$ will be zero, and as $H^1(P,\mathbb{Z}_2)$ makes up a subset of the twists then the differential $d_3 : H^1(P,\mathbb{Z}_2) \to H^4(P,\mathbb{Z})$ will also necessarily be zero. Thus we may conclude that the twists of K-theory are given by odd-degree cohomology in this case, and so it is sensible to use 5-twists and 7-twists for P.

In order to compute the twisted K-theory groups themselves, the twisted Atiyah–Hirzebruch spectral sequence may be used. The E_2-term for $e(P) = 0$ is shown below.

2	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})\oplus\mathbb{Z}$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})\oplus\mathbb{Z}$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$
	0	1	2	3	4	5	6	7

Similarly, the E_2-term of the analogous spectral sequence for $e(P) = j \neq 0$ is below.

2	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}_j$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	$H^1(M,\mathbb{Z})$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z})\oplus\mathbb{Z}_j$	$H^2(M,\mathbb{Z})$	$H^3(M,\mathbb{Z})$	$\mathbb{Z}$
	0	1	2	3	4	5	6	7

In both cases, we may argue that the differential Sq^3 is zero as follows. Firstly, Sq^3 of course annihilates $H^k(P, \mathbb{Z})$ for $0 \leq k \leq 2$ by Theorem 4L.12 of [Hat00] as well as $5 \leq k \leq 7$ since the image of the map is a trivial cohomology group, which leaves only $k = 3$ and $k = 4$ to be considered. But the image of Sq^3 is a $\mathbb{Z}_2$ torsion element by definition, and there is no torsion in $H^6(P, \mathbb{Z})$ or $H^7(P, \mathbb{Z})$ and hence this differential must be zero. Similarly, the torsion part of d_5 will annihilate all cohomology classes, leaving only the cup product with the twisting class $\delta \in H^5(P, \mathbb{Z})$ to be considered.

To determine how the cup product with the twisting class affects the cohomology, the isomorphisms between the cohomology of M and P need to be viewed more explicitly. Observe that $\pi_* : H^k(P, \mathbb{Z}) \to H^{k-3}(M, \mathbb{Z})$ is an isomorphism for $k = 5, 6, 7$ and similarly $\pi^* : H^k(P, \mathbb{Z}) \to H^k(M, \mathbb{Z})$ is an isomorphism for $k = 0, 1, 2$. Fixing a twist $\delta \in H^5(P, \mathbb{Z})$, this is equivalent to $(\pi_*)^{-1}(\eta)$ for some $\eta \in H^2(M, \mathbb{Z})$. Then taking $\alpha_k \in H^k(M, \mathbb{Z})$ for $k = 0, 1, 2$ such that $\pi^*(\alpha_k) \in H^k(P, \mathbb{Z})$, we see that $\pi_*(\delta \cup \pi^*(\alpha_k)) \in H^k(M, \mathbb{Z})$ will be the image of the cup product in the cohomology of M. But by a property of pushforwards, this is equal to $\pi_*(\delta) \cup \alpha_k = \eta \cup \alpha_k$. Hence the cup product with δ on the cohomology of P is equivalent to cup product with η on the cohomology of M. Since this cup product is injective on $H^0(P, \mathbb{Z})$, all higher differentials will be zero and thus the E_∞-term of the spectral sequences are as shown in Figure 5.2.1 for $e(P) = 0$ and 5.2.2 for $e(P) = j \neq 0$.

	0	1	2	3	4	5	6	7
2	0	$\ker \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\ker \cup\eta\vert_{H^2(M,\mathbb{Z})}$	$H^3(M,\mathbb{Z}) \oplus \mathbb{Z}$	$H^1(M,\mathbb{Z}) \oplus \mathbb{Z}$	$H^2(M,\mathbb{Z})/\eta\mathbb{Z}$	$\operatorname{coker} \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\operatorname{coker} \cup\eta\vert_{H^2(M,\mathbb{Z})}$
1	0	0	0	0	0	0	0	0
0	0	$\ker \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\ker \cup\eta\vert_{H^2(M,\mathbb{Z})}$	$H^3(M,\mathbb{Z}) \oplus \mathbb{Z}$	$H^1(M,\mathbb{Z}) \oplus \mathbb{Z}$	$H^2(M,\mathbb{Z})/\eta\mathbb{Z}$	$\operatorname{coker} \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\operatorname{coker} \cup\eta\vert_{H^2(M,\mathbb{Z})}$

Figure 5.2.1: E_∞-term of the AHSS for $e(P) = 0$.

	0	1	2	3	4	5	6	7
2	0	$\ker \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\ker \cup\eta\vert_{H^2(M,\mathbb{Z})}$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z}) \oplus \mathbb{Z}_j$	$H^2(M,\mathbb{Z})/\eta\mathbb{Z}$	$\operatorname{coker} \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\operatorname{coker} \cup\eta\vert_{H^2(M,\mathbb{Z})}$
1	0	0	0	0	0	0	0	0
0	0	$\ker \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\ker \cup\eta\vert_{H^2(M,\mathbb{Z})}$	$H^3(M,\mathbb{Z})$	$H^1(M,\mathbb{Z}) \oplus \mathbb{Z}_j$	$H^2(M,\mathbb{Z})/\eta\mathbb{Z}$	$\operatorname{coker} \cup\eta\vert_{H^1(M,\mathbb{Z})}$	$\operatorname{coker} \cup\eta\vert_{H^2(M,\mathbb{Z})}$

Figure 5.2.2: E_∞-term of the AHSS for $e(P) \neq 0$.

Finally we may conclude that, up to extension problems, the 5-twisted K-theory of P when $e(P) = 0$ is

$$K^0(P, \delta) = \ker \cup\eta|_{H^2(M,\mathbb{Z})} \oplus H^1(M) \oplus \mathbb{Z} \oplus \operatorname{coker} \cup\eta|_{H^1(M,\mathbb{Z})};$$
$$K^1(P, \delta) = \ker \cup\eta|_{H^1(M,\mathbb{Z})} \oplus H^3(M) \oplus \mathbb{Z} \oplus H^2(M, \mathbb{Z})/\eta\mathbb{Z} \oplus \operatorname{coker} \cup\eta|_{H^2(M,\mathbb{Z})};$$

and when $e(P) = j \neq 0$ is

$$K^0(P, \delta) = \ker \cup\eta|_{H^2(M,\mathbb{Z})} \oplus H^1(M) \oplus \mathbb{Z}_j \oplus \operatorname{coker} \cup\eta|_{H^1(M,\mathbb{Z})};$$
$$K^1(P, \delta) = \ker \cup\eta|_{H^1(M,\mathbb{Z})} \oplus H^3(M) \oplus H^2(M, \mathbb{Z})/\eta\mathbb{Z} \oplus \operatorname{coker} \cup\eta|_{H^2(M,\mathbb{Z})}.$$

To be more explicit about the extension problems involved, we illustrate how these higher twisted K-theory groups may differ if the extension problems are non-trivial.

Considering the case of $e(P) = j \neq 0$ above, determining the higher twisted K-theory group of P would require solving the following:

$$0 \to \mathbb{Z}_j \to A \to \operatorname{coker} \cup\eta|_{H^1(M,\mathbb{Z})} \to 0;$$
$$0 \to \ker \cup\eta|_{H^2(M,\mathbb{Z})} \to K^0(P, \delta) \to A \to 0.$$

So the direct sums shown in the equations above should really be viewed as these extension problems.

Of course these higher twisted K-theory groups are heavily dependent on the ring structure of the cohomology of M, but given a specific 4-manifold M with torsion-free cohomology, i.e. a simply connected 4-manifold satisfying the previous assumptions, this determines the 5-twisted K-theory of P up to extension problems.

Example 5.2.2. We apply the formulas given above to the manifold $M = S^2 \times S^2$. This space has trivial cohomology in degrees 1 and 3, and $H^2(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. To specify the ring structure on cohomology, the cup product of the generators of the two factors of $\mathbb{Z}$ in $H^2(S^2 \times S^2, \mathbb{Z})$ is the generator of $H^4(S^2 \times S^2, \mathbb{Z})$.

Consider a twist $\delta \in H^2(M, \mathbb{Z})$ given by $\delta = (L\alpha_0, N\beta_0)$ with α_0, β_0 the generators and $L, N \neq 0$. Then the cup product map $H^2(M, \mathbb{Z}) \to H^4(M, \mathbb{Z})$ is $(a, b) \mapsto Lb + Na$, which letting k denote the greatest common divisor of L and N has kernel $\{(Li/k, Ni/k) : i \in \mathbb{Z}\} \cong \mathbb{Z}$ and has cokernel $\mathbb{Z}_k$. From this, we see that when $e(P) = 0$ we have

$$K^0(P, \delta) = \mathbb{Z} \oplus \mathbb{Z};$$
$$K^1(P, \delta) = \mathbb{Z} \oplus \mathbb{Z}_L \oplus \mathbb{Z}_N \oplus \mathbb{Z};$$

and when $e(P) = j \neq 0$ the groups are

$$K^0(P, \delta) = \mathbb{Z} \oplus \mathbb{Z}_j;$$
$$K^1(P, \delta) = \mathbb{Z}_L \oplus \mathbb{Z}_N \oplus \mathbb{Z}.$$

Example 5.2.3. To give one further example, we let $M = \mathbb{C}P^2$ with $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ and such that the generator of $H^4(M, \mathbb{Z})$ is the square of the generator of $H^2(M, \mathbb{Z})$. Letting $\delta = N\delta_0$ with δ_0 the generator of $H^2(M, \mathbb{Z})$, we see that the twisted K-theory when $e(P) = 0$ is

$$K^0(P, \delta) = \mathbb{Z};$$
$$K^1(P, \delta) = \mathbb{Z} \oplus \mathbb{Z}_N \oplus \mathbb{Z}_N;$$

and when $e(P) = j \neq 0$ is

$$K^0(P, \delta) = \mathbb{Z}_j;$$
$$K^1(P, \delta) = \mathbb{Z}_N \oplus \mathbb{Z}_N.$$

Since there is no specified 5-class in the setting of spherical T-duality, the physical interpretation of these 5-twisted K-theory groups is not as clear as in the case of 7-twists. Nevertheless, the work of Bouwknegt, Evslin and Mathai provides a link between spherical T-duality and supergravity theories in Type IIB string theory, and so further research into this area may shed light on the physical meaning of the computations that we have performed. Tying these computations together with the physics may then provide further insight into certain aspects of string theory.

In this chapter we have brought together all of the critical results and techniques developed throughout the thesis, and in doing so we have achieved our aim of providing a comprehensive introduction to higher twisted K-theory. Many of the results presented in this chapter can be used as the starting point for further investigation, particularly in exploring techniques to compute the higher twisted K-theory of more general products of spheres, in obtaining more general results for the higher twisted K-theory of Lie groups and in exploring the link between higher twisted K-theory and string theory. More broadly, the results in this thesis provide a firm base upon which the rich area of higher twisted K-theory can continue to be explored.

Bibliography

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3 (Suppl. 1):3–38, 1964.

[Ada74] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, 1974. ISBN: 978-0226005249.

[AGG14] B. Antieau, G. Gómez, and D. Gepner. Actions of Eilenberg-MacLane spaces on K-theory spectra and uniqueness of twisted K-theory. *Trans. Am. Math. Soc.*, 366:3631–3648, 2014.

[AH59] M. F. Atiyah and F. Hirzebruch. Riemann–Roch Theorems for Differentiable Manifolds. *Bull. Am. Math. Soc.*, 65(4):276–281, 1959.

[AH04] Michael F. Atiyah and Michael J. Hopkins. A variant of K-theory, $K_{\pm}$. *Topology, Geometry and Quantum Field Theory*, 2004. Cambridge Univ. Press.

[Arl92] Dominique Arlettaz. The order of the differentials in the Atiyah–Hirzebruch spectral sequence. *K-theory*, 6(4):347–361, 1992.

[AS04] Michael Atiyah and Graeme Segal. Twisted K-theory. *Ukr. Mat. Visn.*, 1(3):1287–330, 2004.

[AS06] Michael Atiyah and Graeme Segal. Twisted K-theory and cohomology. *Nankai Tracts in Mathematics*, 11:5–43, 2006.

[Ati67] Michael Atiyah. *K-theory.* W. A. Benjamin Inc., 1967. ISBN: 978-0201407921.

[BCM+02] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, and D. Stevenson. Twisted K-theory and K-theory of bundle gerbes. *Comm. Math. Phys.*, 228:17–49, 2002.

[BEM04a] Peter Bouwknegt, Jarah Evslin, and Varghese Mathai. T-duality: Topology Change from H-flux. *Comm. Math. Phys.*, 249(2):383–415, 2004.

[BEM04b] Peter Bouwknegt, Jarah Evslin, and Varghese Mathai. Topology and H-flux of T-dual manifolds. *Phys. Rev. Lett.*, 92(18):181601, 3pp, 2004.

[BEM15a] P. Bouwknegt, J. Evslin, and V. Mathai. Spherical T-Duality. *Commun. Math. Phys.*, 337:909–954, 2015.

[BEM15b] P. Bouwknegt, J. Evslin, and V. Mathai. Spherical T-duality II: An infinity of spherical T-duals for non-principal-bundles. *J. Geom. Phys.*, 92:46–54, 2015.

[BEM18] P. Bouwknegt, J. Evslin, and V. Mathai. Spherical T-Duality and the spherical Fourier–Mukai transform. *J. Geom. Phys.*, 133:303–314, 2018.

[Bha11] Bhattacharya, Subhrajit. Disk to sphere using quotient space, 2011. [Online; accessed February 12, 2020].

[BKP03] David Benson, Alex Kumjian, and Chris Phillips. Symmetries of Kirchberg algebras. *Canadian Math. Bull.*, 46:509–528, 2003.

[Bla86] Bruce Blackadar. *K-Theory for Operator Algebras.* Springer, 1986. ISBN: 978-1-4613-9574-4.

[Bla06] Bruce Blackadar. *Operator Algebras.* Springer, 2006. ISBN: 978-3-540-28486-4.

[BM00] Peter Bouwknegt and Varghese Mathai. D-branes, B-fields and twisted K-theory. *J. High Energy Phys.*, 3:Paper 7, 11pp, March 2000.

[Bot59] Raoul Bott. The Stable Homotopy of the Classical Groups. *The Annals of Mathematics*, 70(2):313–337, 1959.

[Bra04] Volker Braun. Twisted K-theory of Lie groups. *J. High Energy Phys.*, 3:Paper 29, 15pp, March 2004.

[BT82] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer, 1982. ISBN: 0-387-90613-4.

[CE99] Henri Cartan and Samuel Eilenberg. *Homological Algebra.* Princeton University Press, 1999. ISBN: 0-691-04991-2.

[Coh98] Ralph L. Cohen. *The Topology of Fibre Bundles.* Dept. of Mathematics, Stanford Univesity, 1998.

[Cun77] Joachim Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[Cun81] Joachim Cuntz. K-theory for certain C^*-algebras. *Ann. Math.*, 113:181–197, 1981.

[Dav91] Kenneth R. Davidson. *C^*-algebras by Example.* American Mathematical Soc., 1991. ISBN: 0-8218-0599-1.

[DD63] Jacques Dixmier and Adrien Douady. Champs continus d'espaces Hilbertiens et de C^*-algébres [Continuous fields of Hilbert spaces and C^*-algebras]. *Bull. Soc. Math. France*, 91:227–284, 1963.

[DK70] Peter Donovan and Max Karoubi. Graded Brauer groups and K-theory with local coefficients. *Publications Mathématiques de l'IHÉS*, 38:5–25, 1970.

[Dou05] Christopher Lee Douglas. *Twisted Stable Homotopy Theory.* PhD thesis, Massachusetts Institute of Technology, 2005.

[Dou06] Christopher L. Douglas. On the twisted K-homology of simple Lie groups. *Topology*, 45(6):955–988, 2006.

[DP15a] Marius Dadarlat and Ulrich Pennig. A Dixmier–Douady theory for strongly self-absorbing C^*-algebras II: the Brauer group. *J. Noncommut. Geom.*, 9(4):1137–1154, 2015.

[DP15b] Marius Dadarlat and Ulrich Pennig. Unit spectra of K-theory from strongly self-absorbing C^*-algebras. *Algebr. Geom. Topol.*, 15(1):137–168, 2015.

[DP16] Marius Dadarlat and Ulrich Pennig. A Dixmier–Douady theory for strongly self-absorbing C^*-algebras. *J. Reine Angew. Math.*, 718:153–181, 2016.

[EFTW79] Masatoshi Enomoto, Masatoshi Fujii, Hiroaki Takehana, and Yasuo Watatani. Automorphisms on Cuntz Algebras, II. *Math. Japonica*, 24(4):463–468, 1979.

[EP19] David E. Evans and Ulrich Pennig. Equivariant higher twisted K-theory of $SU(n)$ for exponential functor twists. 2019. arXiv preprint math.KT/1906.08179.

[FHT11a] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[FHT11b] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted K-theory III. *Ann. Math.*, 174(2):947–1007, 2011.

[FHT13] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted K-theory II. *J. Am. Math. Soc.*, 26(3):595–644, 2013.

[Hat00] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2000. ISBN: 0-521-79540-0.

[Hat17] Allen Hatcher. Vector Bundles and K-theory. Unpublished book available at `pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`, November 2017.

[Huy05] Daniel Huybrechts. *Complex Geometry*. Springer, 2005. ISBN: 3-540-21290-6.

[Jän65] Klaus Jänich. Vektorraumbündel und der Raum der Fredholm-Operatoren [Vector bundles and the space of Fredholm operators]. *Math. Ann.*, 161:129–142, 1965.

[JS99] X. Jiang and H. Su. On a simple unital projectionless C^*-algebra. *Am. J. Math.*, 121:359–413, 1999.

[Kar68] Max Karoubi. Algèbres de Clifford et K-théorie [Clifford algebras and K-theory]. *Ann. Sci. Éc. Norm. Supér.*, 1(2):161–270, 1968.

[Kar05] Max Karoubi. *K-theory*. Springer, 2nd edition, 2005. ISBN: 978-3-540-79889-7.

[Kho11] Mehdi Khorami. A universal coefficient theorem for twisted K-theory. *Topology*, 4(3):535–542, 2011.

[KK97] Akitaka Kishimoto and Alexander Kumjian. Crossed products of Cuntz algebras by quasi-free automorphisms. *Operator Algebras and their Applications*, 13:173–192, 1997.

[Lee03] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2003. ISBN: 978-1-4419-9981-8.

[Mic02] Jouko Mickelsson. Gerbes, (twisted) K-theory, and the supersymmetric WZW model. *IRMA Lect. Math. Theor. Phys.*, 5:93–107, 2002.

[Mil56] John Milnor. On the construction of universal bundles II. *Ann. Math.*, 63:430–436, 1956.

[Min87] J. A. Mingo. K-theory and multipliers of stable C^*-algebras. *Trans. Am. Math. Soc.*, 299(1):397–411, 1987.

[MM17] Varghese Mathai and Richard B. Melrose. Geometry of pseudodifferential algebra bundles and Fourier integral operators. *Duke Math. J.*, 166(10):1859–1922, 2017.

[Moo04] Gregory Moore. K-theory from a physical perspective. *Topology, geometry and quantum field theory*, (308):194–234, 2004.

[MR16] Varghese Mathai and Jonathan Rosenberg. Group dualities, T-dualities and twisted K-theory. *J. Lond. Math. Soc.*, 97(1), 2016.

[MS03] Varghese Mathai and Danny Stevenson. Chern character in twisted K-theory: equivariant and holomorphic cases. *Comm. Math. Phys.*, 236(1):161–186, 2003.

[MS06a] Varghese Mathai and Danny Stevenson. On a generalized Connes–Hochschild–Kostant–Rosenberg theorem. *Adv. Math.*, 200(2):305–335, 2006.

[MS06b] Peter J. May and Johann Sigurdsson. *Parameterized Homotopy Theory.* American Mathematical Soc., 2006. ISBN: 978-0821839225.

[MST77] I. Madsen, V. Snaith, and J. Tornehave. Infinite loop maps in geometric topology. *Math. Proc. Cambridge Philos. Soc.*, 81(3):399–430, 1977.

[MW16] Jouko Mickelsson and Stefan Wagner. Third group cohomology and gerbes over Lie groups. *J. Geom. Phys.*, 108:49–70, 2016.

[Par08] Efton Park. *Complex Topological K-theory.* Cambridge Studies in Advaned Mathematics, 2008. ISBN: 978-0-521-85634-8.

[Pen15] Ulrich Pennig. A non-commutative model for higher twisted K-theory. *J. Topol.*, 9(1):27–50, 2015.

[Ros89] Jonathan Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. *J. Aus. Math. Soc.*, 47(3):368–381, 1989.

[Ros17] J. Rosenberg. A new approach to twisted K-theory of compact Lie groups. 2017. arXiv preprint math.KT/1708.05541.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Ètudes Sci. Publ. Math.*, 34:105–112, 1968.

[Swa62] Richard Swan. Vector bundles and projective modules. *Trans. Am. Math. Soc.*, 105(2):264–277, 1962.

[TW07] Andrew S. Toms and Wilhelm Winter. Strongly self-absorbing C^*-algebras. *Trans. Am. Math. Soc.*, 359(8):3999–4029, 2007.

[Ver88] Erik Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nucl. Phys. B*, 300(3):360–376, 1988.

[Wal04] Gerard Walschap. *Metric Structures in Differential Geometry.* Springer, 2004. ISBN: 978-1-4419-1913-7.

[Wei94] Charles Weibel. *An Introduction to Homological Algebra.* Cambridge University Press, 1994. ISBN: 0-521-43500-5.

[Wit00] Edward Witten. *D*-branes and K-theory. *J. High Energy Phys.*, 3:Paper 19, 41pp, March 2000.

[WO93] N. E. Wegge-Olsen. *K-theory and C^*-algebras*. Oxford University Press, 1993. ISBN: 978-0-19-859694-3.