

PURE MTH 3002 Topology & Analysis III (3246)

Assignment 1. Due Friday 15th March 2002

In all these questions when you are asked to give a proof you should be careful that you give a logical, carefully laid out argument. It should also be grammatically correct when read out in words. If you have problems or questions please come and see me or email me at michael.murray@adelaide.edu.au.

1. Show that the following relations are equivalence relations.

- (a) Let X and Y be matrices. Define $X \sim Y$ if there is an invertible matrix P such that $X = PYP^{-1}$.
- (b) Let S be the set of all pairs of integers (a, b) with $b \neq 0$. Define $(a, b) \sim (a', b')$ if $ab' = a'b$.
- (c) Let $f: X \rightarrow Y$ be a function between two sets. Define $x_1 \sim x_2$ if $f(x_1) = f(x_2)$.

2. For each of the following sets say whether they are bounded above or below or not bounded above or below. You are not required to prove what you say.

- (a) $\{1/n \mid n = 1, 2, 3, \dots\}$
- (b) $\{x \mid x^2 - 4 < 0\}$
- (c) $\{x \mid x^2 - 4 > 0\}$
- (d) $\{x^4 \mid x \in \mathbb{R}\}$

3. Find the sup or inf as indicated below. You are not required to prove anything.

- (a) $\inf(0, 51)$
- (b) $\sup\{x \mid x^2 - 4 < 0\}$
- (c) $\inf\{1/n \mid n = 1, 2, 3, \dots\}$
- (d) $\sup\{\sin(x) \mid x \in \mathbb{R}\}$

4. Let S and T be subsets of the real numbers which are bounded above. Assume that for every $s \in S$ there is a $t \in T$ with $s < t$. Show that $\sup(S) \leq \sup(T)$. Give an example where the hypothesis is satisfied but $\sup(S) = \sup(T)$.

5. Show that the sequence $x_n = 1/2^n$ is a Cauchy sequence. You may assume that for any K there is an m such that $2^m > K$.

6. Let $a_0.a_1a_2\dots$ be a decimal expansion, that is a_0 is an integer and a_1, a_2, \dots are numbers from the set $\{0, 1, \dots, 8, 9\}$. Define a sequence by $x_1 = a_0 + a_1/10$, $x_2 = x_1 + a_2/100$ so that $x_r = a_0 + \sum_{i=1}^r a_i/10^i$. Show that $\{x_i\}_{i=1}^\infty$ is a Cauchy sequence.

7. If $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are Cauchy sequences show that $\{x_i + y_i\}_{i=1}^\infty$ is a Cauchy sequence.

8. Find the limit of the sequences:

- (a) $x_n = \frac{n+2}{n^2+3n}$
- (b) $x_n = \frac{1}{n} \sin(n)$.

You should use the limit laws proved in class.

9. If $\{x_n\}_{n=1}^\infty$ is a sequence with limit x show that $\{5x_n\}_{n=1}^\infty$ has limit $5x$ *without* using the limit laws from class. I.e construct an argument using ϵ, N etc.

10. Let $C(\mathbb{Q})$ be the set of all Cauchy sequences with rational values. If $x = \{x_n\}_{n=1}^\infty$ and $y = \{y_n\}_{n=1}^\infty$ define a relation by $x \sim y$ if $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$. Show that \sim is an equivalence relation. We have seen in class that Cauchy sequences converge. Show that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$ if and only if $x \sim y$.

This last question indicates one way of defining the real numbers from the rational numbers. You can define the reals as the set of all equivalence classes of rational numbers. Some work is then required to define the positive real numbers, addition and multiplication and check they satisfy all the axioms including the least upper bound axiom.