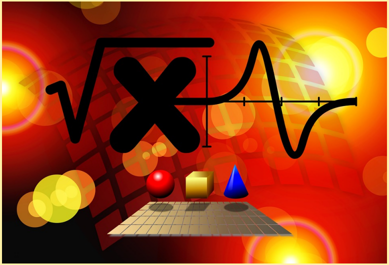
Crossing borders within mathematics

Pedram Hekmati



Can every natural number $n \in \mathbb{N}$

be written as a square of an integer?

$$n = a^2 , \qquad a \in \mathbb{Z}$$

Answer: No. **Counterexample:** $n = 2$.

Can every natural number $n \in \mathbb{N}$ be written as a sum of <u>two</u> squares of integers?

$$n = a^2 + b^2 \; , \qquad a, b \in \mathbb{Z}$$

Answer: No. **Counterexample:** $n = 3$.

Can every natural number $n \in \mathbb{N}$ be written as a sum of <u>three</u> squares of integers?

$$n = a^2 + b^2 + c^2 \,, \qquad a, b, c \in \mathbb{Z}$$

Answer: No. **Counterexample:** $n = 7$.

Can every natural number $n \in \mathbb{N}$ be written as a sum of four squares of integers?

$$n = a^2 + b^2 + c^2 + d^2 \,, \qquad a, b, c, d \in \mathbb{Z}$$

Answer: Yes! **(Lagrange's four-square theorem, 1770)**

In how many ways can n be written as a sum of four squares?

Let $c(n)$ denote this number.

$$
\begin{aligned}
\textit{Example}: \quad 1 &= 1^2+0^2+0^2+0^2 \\
&= 0^2+1^2+0^2+0^2 \\
&= 0^2+0^2+1^2+0^2 \\
&= 0^2+0^2+0^2+1^2 \\
&= (-1)^2+0^2+0^2+0^2 \\
&= 0^2+(-1)^2+0^2+0^2 \\
&= 0^2+0^2+(-1)^2+0^2 \\
&= 0^2+0^2+0^2+(-1)^2
\end{aligned}
$$

$$\Rightarrow \quad c(1) = 8.$$

NUMBER THEORY $\Longrightarrow$ COMBINATORICS

We seek a formula for

$$c(n) = |\{(a, b, c, d) \in \mathbb{Z}^4 \mid n = a^2 + b^2 + c^2 + d^2\}|$$

Jacobi's four-square theorem, 1834.

Look at

$$\eta(q) = \sum_{a \in \mathbb{Z}} q^{a^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \dots$$

Then

$$\eta^4(q) = \sum_{a,b,c,d \in \mathbb{Z}} q^{a^2+b^2+c^2+d^2} = \sum_{n=0}^{\infty} c(n) q^n.$$

COMBINATORICS $\Longrightarrow$ FOURIER ANALYSIS

Set $q = e^{\pi i z} \quad \Rightarrow \quad \eta(z) = \sum_{a \in \mathbb{Z}} e^{\pi i a^2 z}.$

$\eta(z)$ **is a holomorphic function if** $Im(z) > 0.$

$$\boxed{\eta^4(z+2) = \eta^4(z), \qquad \eta^4\left(-\frac{1}{z}\right) = -z^2\eta^4(z).}$$

How many holomorphic functions with these properties?

FOURIER ANALYSIS $\Longrightarrow$ COMPLEX GEOMETRY

$\eta^4(z) = \sum_{n=0}^{\infty} c(n) e^{\pi i n z}$ **is the only one up to scalar multiplication!**

Strategy

1. **Construct a function $f(z)$ with the above properties. Then**

$$f(z) = c \cdot \eta^4(z).$$

2. **Determine the constant c and compare Fourier coefficients.**

Use the Eisenstein series to construct $f(z)$.

$$G_2(z) = \sum_{m,n\in\mathbb{Z}} \frac{1}{(mz+n)^2}$$

$$G_2(z+1) = G_2(z), \qquad G_2\left(-\frac{1}{z}\right) = z^2 G_2(z) + 2\pi i z.$$

Define

$$f(z) = 2G_2(2z) - \frac{1}{2} G_2\left(\frac{z}{2}\right).$$

The function $f(z)$ has the Fourier series

$$f(z) = 3\zeta(2) + 4\pi^2 \sum_{n=1}^{\infty} \sigma(n)(e^{\pi inz} - 4e^{4\pi inz})$$

with $\sigma(n) = \sum_{d|n} d.$

Since $c(0) = 1$, **the scaling constant is** $c = 3\zeta(2) = \dfrac{\pi^2}{2}$:

$$f(z) = c \cdot \eta^4(z) = \frac{\pi^2}{2} \sum_{n=0}^{\infty} c(n) e^{\pi inz}.$$

Jacobi's formula (1834)

$$c(n) = 8 \sum_{4 \nmid d | n} d$$

Note that $c(n) \geq 8$ **for all** $n \in \mathbb{N}$.

$\Rightarrow$ **Lagrange's four-square theorem (1770) is a corollary.**

NUMBER THEORY $\Rightarrow$ COMBINATORICS $\Rightarrow$ FOURIER ANALYSIS

$\Rightarrow$ COMPLEX GEOMETRY $\Rightarrow$ NUMBER THEORY