Undergraduate Seminar
School of Mathematical Sciences

Tuesday, 4th September, 1.10pm
Seminar room 7.15 of Ingkarni Wardli

30-minute seminar, followed by sandwiches and juice

Dr Pedram Hekmati

Examples of counterexamples

Abstract: This aims to be an example of an exemplary talk on examples of celebrated counterexamples in mathematics. A famous example, for example, is Euler’s counterexample to Fermat’s conjecture in number theory.

Examples of counterexamples

Undergraduate Seminar

Pedram Hekmati
A counterexample is a specific instance of the falsity of a proposed conjecture.
Importance

- intuition for mathematical concepts

- probe boundaries of theorems and conjectures

- hints on how to improve a conjecture
Is every continuous function differentiable?

No! The Weierstrass function:
Original Conjecture: All prime numbers are odd

Counterexample: 2

New conjecture: All prime numbers greater than 2 are odd
Lack of counterexamples?
Fermat's last theorem

Conjecture (1637): No positive integers $x, y, z$ can satisfy

$$x^n + y^n = z^n$$

for any natural number $n$ greater than 2.

RESOLVED!

Complete proof by Andrew Wiles in 1995.
Poincaré conjecture

All closed simply connected 3-manifolds are homeomorphic to the 3-sphere.

(1854-1912)
Closed | Not closed
---|---
![Sphere](image1)
![Hole](image2)
![Cube](image3)

![Ellipse](image4)
![Rectangle](image5)
![Circle](image6)
Simply connected:

Not simply connected:
Homeomorphism:
RESOLVED!

Complete proof in 2003 by Grigory Perelman, as a consequence of Thurston's geometrisation conjecture.
Riemann hypothesis

Riemann zeta function:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

Conjecture (1859): All non-trivial zeros of the Riemann zeta function have real part 1/2.

UNRESOLVED!
Goldbach's Conjecture

Conjecture (1742): Every even integer greater than 2 can be expressed as the sum of two primes.
The Goldbach's conjecture proved

Agostino Prástaro

(Submitted on 13 Aug 2012 (v1), last revised 19 Aug 2012 (this version, v3))

We give a direct proof of the Goldbach's conjecture in number theory, formulated in the Euler's form. The proof is also constructive, since it gives a criterion to find two prime numbers $\geq 15$, such that their sum gives a fixed even number $\geq 25$ (A prime number is an integer that can be divided only for itself other than for 1. In this paper we consider 1 as a prime number). The proof is obtained by recasting the problem in the framework of the Commutative Algebra and Algebraic Topology.

Comments: 15 pages
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Which authors of this paper are endorsers?
Goldbach's weak conjecture: Every odd number greater than 7 can be expressed as the sum of three odd primes.
EVERY ODD NUMBER GREATER THAN 1 IS THE SUM OF AT MOST FIVE PRIMES

TERENCE TAO

ABSTRACT. We prove that every odd number \( N \) greater than 1 can be expressed as the sum of at most five primes, improving the result of Ramaré that every even natural number can be expressed as the sum of at most six primes. We follow the circle method of Hardy-Littlewood and Vinogradov, together with Vaughan’s identity; our additional techniques, which may be of interest for other Goldbach-type problems, include the use of smoothed exponential sums and optimisation of the Vaughan identity parameters to save or reduce some logarithmic losses, the use of multiple scales following some ideas of Bourgain, and the use of Montgomery’s uncertainty principle and the large sieve to improve the \( L^2 \) estimates on major arcs. Our argument relies on some previous numerical work, namely the verification of Richstein of the even Goldbach conjecture up to \( 4 \times 10^{14} \), and the verification of van de Lune and (independently) of Wedeniwski of the Riemann hypothesis up to height \( 3.29 \times 10^9 \).

1. Introduction

Two of most well-known conjectures in additive number theory are the even and odd Goldbach conjectures, which we formulate as follows:\[1\]

**Conjecture 1.1** (Even Goldbach conjecture). Every even natural number \( x \) can be expressed as the sum of at most two primes.

**Conjecture 1.2** (Odd Goldbach conjecture). Every odd number \( x \) larger than 1 can be expressed as the sum of at most three primes.

It was famously established by Vinogradov \[47\], using the Hardy-Littlewood circle method, that the odd Goldbach conjecture holds for all sufficiently large odd \( x \). Vinogradov’s argument can be made effective, and various explicit thresholds for “sufficiently large” have been given in the literature; in particular, Chen and Wang \[5\] established the odd Goldbach conjecture for all \( x \geq \exp(\exp(11.503)) \approx \exp(99012) \), and Liu & Wang \[24\] subsequently extended this result to the range \( x \geq \exp(3100) \). At the other extreme, by combining Richstein’s numerical verification \[48\] of the even Goldbach conjecture for...
How to come up with a conjecture?

- accumulated evidence

- generalisation to a larger set

- the 3 I's: inspiration - intuition - insight
Consequences of counterexamples

- need to *weaken* the conjecture

- need to *strengthen* the conjecture

- *vacuous* statement
Original Conjecture: All rectangles are squares.

Counterexample: 🟣

Weaker conjecture: All rectangles have four sides.
Original Conjecture: All rectangles are squares.

Strengthened conjecture: All shapes that are rectangles and have four sides of equal length are squares.
**Original Conjecture:** For any set, every infinite subset has precisely seven elements.

**Counterexample:** Natural numbers is an infinite subset of the integers.

**Problem:** Statement about the empty set.
Geometry
In 1906, Arthur Schönhflies showed that for every non-intersecting closed curve, the interior and exterior regions are homeomorphic to the interior and exterior regions defined by the circle.

True in higher dimensions?
Counterexample: Alexander Horned Sphere

In 1923, James Waddell Alexander II constructed a surface that is homeomorphic to the 2-sphere. But, the exterior region is not simply connected!
Algebra
In 1729, Christian Goldbach wrote a letter to Leonhard Euler:

"Note the observation by Fermat that all numbers of the form $2^{(2^r)+1}$, that is 3, 5, 17 etc., are primes, which he himself admits that he was not able to prove, and as far as I know, nobody else has proved it either."

(1707-1783)

Today, numbers of the form $2^{(2^r)+1}$ are called **Fermat numbers**, and the statement is called **Fermat's conjecture**.
In 1732, Euler published a 5-page paper with a counterexample and six additional conjectures of his own, the first of which is Fermat's little theorem:

For any prime number \( p \) and any integer \( x \), \( p \) divides the integer \( x^p - x \).

Euler gave the first published proof of this conjecture in 1736, but the result was already known to Fermat (without proof) and Leibniz (unpublished proof, 1683).
Counterexample: Euler's strategy was to determine the possible forms of the primes $p$ dividing the numbers $2^{2^r}+1$, e.g.

- if $p$ divides $2^4+1$, then $p=4k+1$
- if $p$ divides $2^{32}+1$, then $p=64k+1$

Now, $2^{32}+1 = 4,294,967,297$. By checking only 5 cases ($k=3,4,7,9,10$), he concluded that $4,294,967,297 = 641 \times 6,700,417$. 
Analysis
Problem with infinite sums: \[ \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty \]

We are interested in convergent sums: \[ \left| \sum_{n=1}^{\infty} a_n \right| < \infty \]

Now, clearly \[ \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n < \infty \]
Question

\[ b_n \leq a_n \text{ for all } n \text{ and } \left| \sum_{n=1}^{\infty} a_n \right| < \infty \implies \left| \sum_{n=1}^{\infty} b_n \right| < \infty? \]

Accumulate evidence

\[ a_n = \frac{1}{n^2}, \quad b_n = \frac{1}{n^3} \implies b_n \leq a_n \]

\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.645 \quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) \approx 1.206 \]
Counterexample

\[ a_n = 0, \quad b_n = -\frac{1}{n} \quad \Rightarrow \quad b_n \leq a_n \text{ for all } n \]

\[
\left| \sum_{n=1}^{\infty} a_n \right| = 0 < \infty \quad \text{but} \quad \left| \sum_{n=1}^{\infty} b_n \right| = \infty
\]
Strengthened Conjecture

\[ |b_n| \leq |a_n| \text{ for all } n \text{ and } \sum_{n=1}^{\infty} a_n < \infty \implies \sum_{n=1}^{\infty} b_n < \infty \]
Counterexample

\[ a_n = \frac{(-1)^{n+1}}{n}, \quad b_n = \frac{1}{n} \implies |b_n| \leq |a_n| \]

\[ \sum_{n=1}^{\infty} a_n = \ln(2) < \infty, \quad \sum_{n=1}^{\infty} b_n = \infty \]
Sums that converge, but not absolutely,

\[ \sum_{n=1}^{\infty} |a_n| = \infty \]

are called conditionally convergent.

In 1854, Riemann proved that a conditionally convergent sum can take any value by rearrangement of the terms in the sum!
Even if this wasn't an example of an exemplary talk, I sure hope it wasn't a counterexample.

Thank you!