Lecture 2: Relative symplectic structures and wild ramification

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Moduli Spaces, Representation Theory, and Quantization
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Outline

- joint with Dima Arinkin and Bertrand Toën
- Understand symplectic structures along the fibers of sheaves of derived stacks over a topological space.
- Apply to the geometry of the moduli of tame or wild local systems on a smooth variety over $\mathbb{C}$:
  - construct (shifted) Poisson structures;
  - describe their symplectic leaves.

Sheaves of derived stacks (i)

Problem: Define and construct symplectic structures on the stalks of a sheaf of derived stacks $\mathscr{F}$ over a space S.

Note:

- For this to make sense the sections of the sheaf $\mathscr{F}$ will have to satisfy representability conditions.
- The non-degeneracy condition on a stalkwise symplectic form will have to involve some notion of duality for complexes of sheaves of $\mathbb{C}$-vector spaces on S.

Covariant Verdier duality (i)

- X - locally compact Hausdorff space;
- $\mathsf{Sh}(X)$ - the ∞-category of complexes of sheaves of $\mathbb{C}$-vector spaces on X;
- $\mathsf{CoSh}(X)$ - the ∞-category of complexes of cosheaves of $\mathbb{C}$-vector spaces on X;

Theorem: [Lurie] The **Verdier duality functor**

$$\begin{aligned} \mathbf{V} : \ \mathsf{Sh}(X) &\longrightarrow \mathsf{CoSh}(X) \\ E &\longrightarrow (U \to \mathbb{H}^{\bullet}_{c}(U, E)) \end{aligned}$$

is an equivalence of ∞-categories

Covariant Verdier duality (ii)

Comments:

- The duality theorem holds for $\boxed{\text{sheaves and cosheaves}}$ with values in any stable ∞-category.
- $K_c^X := \mathbf{V}(\mathbb{C}_X)$ is the **Verdier dualizing cosheaf** on X.
- $K_X \cong \left(K_c^X\right)^\vee$ is the usual **Verdier dualizing sheaf**

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$(-)^\vee : \mathrm{CoSh}(X) \to \mathrm{Sh}(X)$ is the pointwise $\mathbb{C}$-linear dual

Covariant Verdier duality (iii)

Comments:

- For a continuous map $p : X \longrightarrow Y$ we have a push-foward/pull-back adjunctions on sheaves and cosheaves

$$p^* : \mathrm{Sh}(Y) \rightleftarrows \mathrm{Sh}(X) : p_*, \quad p_+ : \mathrm{CoSh}(X) \rightleftarrows \mathrm{CoSh}(Y) : p^+.$$

After conjugation by $\mathbf{V}$, p_+ becomes $p_! : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$ and p^+ becomes $p^! : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X)$.

- When p is proper we have:
  - $p_! \cong p_*$,
  - p defines a pullback on cohomology with compact supports which induces a map ocosheaves on Y:

$$\mathrm{cotr} : K_c^Y \to p_+ K_c^X.$$

Sheaves of derived stacks (ii)

- X - locally compact Hausdorff space;
- $\mathbf{dSt}(X)$ - the ∞-category of sheaves of derived stacks over X.
- For a continuous map $p : X \longrightarrow Y$ we again have the pull-back/push-forward adjunction

$$p^* : \mathbf{dSt}(Y) \rightleftarrows \mathbf{dSt}(X) : p_*.$$

If $\mathscr{F} \in \mathbf{dSt}(X)$, then the value of $\mathscr{F}$ on an open $U \subset X$ is itself a functor

$$\mathscr{F}_U : \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0} \to \mathrm{SSets}.$$

Relative closed forms (i)

If $\mathscr{F} \in \mathbf{dSt}(X)$, then

$$\begin{aligned} \boldsymbol{\mathcal{A}}_X^{p,cl}(\mathscr{F}) : \quad \mathscr{U}(X) &\longrightarrow \mathrm{Vect}_{\mathbb{C}} \\ U &\longrightarrow \mathcal{A}^{p,cl}(\mathscr{F}_U) \end{aligned}$$

is a copresheaf of complexes of $\mathbb{C}$-vector spaces on X. Write $\underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})$ for the associated cosheaf. It comes equipped with a canonical map

$$\underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F}) \to \boldsymbol{\mathcal{A}}_X^{p,cl}(\mathscr{F})$$

Relative closed forms (ii)

Definition: The complex of global **relative *n*-shifted closed *p*-forms** on $\mathscr{F}$ is the complex

$$\mathbb{A}_X^{p,cl}(\mathscr{F}, n) := \mathbb{R}\underline{Hom}_{\mathrm{CoSh}(X)}\left(K_c^X, \underline{\boldsymbol{\mathcal{A}}}^{p,cl}(\mathscr{F}[n])\right).$$

Variant: We can also consider forms with values in any complex of cosheaves $E \in \mathrm{CoSh}(X)$ by taking instead

$$\mathbb{A}_X^{p,cl}(\mathscr{F})^E := \mathbb{R}\underline{Hom}_{\mathrm{CoSh}(X)}\left(E, \underline{\boldsymbol{\mathcal{A}}}^{p,cl}(\mathscr{F})\right).$$

Push-forward (i)

Suppose $p : X \to Y$ is a proper map of locally compact Hausdorff spaces. We have a canonical map of complexes of cosheaves

$$p_+ \underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F}) \to \underline{\boldsymbol{\mathcal{A}}}_Y^{p,cl}(p_* \mathscr{F})$$

and if $\omega \in \mathbb{A}_X^{p,cl}(\mathscr{F}, n)$, represented by a map of cosheaves

$$\omega : K_c^X \to \underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})[n],$$

we get a well defined composition

$$\begin{tikzcd} K_c^Y \arrow[r, "\mathrm{cotr}"] \arrow[rrr, bend right, "\mathrm{tr} p_* \omega"'] & p_+ K_c^X \arrow[r, "p_+ \omega"] & p_+ \underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})[n] \arrow[r] & \underline{\boldsymbol{\mathcal{A}}}_Y^{p,cl}(p_* \mathscr{F})[n] \end{tikzcd}$$

Push-forward (ii)

Get a map of complexes of relative closed forms:

$$\begin{array}{ccc} \mathbb{A}^{p,cl}_{X}(\mathscr{F}, n) & \xrightarrow{\mathrm{tr} p_*} & \mathbb{A}^{p,cl}_{Y}(p_*\mathscr{F}, n), \\ \omega & \longrightarrow & \mathrm{tr} p_* \omega. \end{array}$$

Conclusion: A closed n-shifted 2-form ω on $\mathscr{F} \to X$ induces a closed n-shifted 2-form $\mathrm{tr} p_* \omega$ on $p_*\mathscr{F} \to Y$.

Problem: Understand non-degeneracy and show that $\mathrm{tr} p_*$ sends symplectic structures to symplectic structures.

Non-degeneracy (i)

For the non-degeneracy we need representability conditions on the stalks of $\mathscr{F} \in \mathbf{dSt}(X)$.

Definition: A sheaf of derived stacks $\mathscr{F} \in \mathbf{dSt}(X)$ is **locally formally representable** over X if for every open $U \subset X$ the derived stack $\mathscr{F}_U/\mathbb{C}$ is an unpointed formal moduli problem, i.e. for each $z \in \mathscr{F}_U(\mathbb{C})$ the restriction of $(\mathscr{F}_U, z)$ to augmented Artinian cdga$/\mathbb{C}$ is given by a dg Lie algebra $\mathscr{L}_{\mathscr{F}_U,z} \in \mathrm{dgLie}_{\mathbb{C}}$.

Non-degeneracy (ii)

Fix $U \subset X$ open and a closed point $z \in \mathscr{F}(\mathbb{C})$. The assignment

$$(V \subset U) \to \mathscr{L}_{\mathscr{F}_V, z_{|V}}$$

is a presheaf of dg Lie algebras on U. Passing to foberwise duals we get a copresheaves of complexes on U:

$$(V \subset U) \to \mathscr{L}^{\vee}_{\mathscr{F}_V, z_{|V}}$$

Let $\mathscr{L}_{\mathscr{F}, z} \in \mathrm{Sh}(U)$ and $\mathscr{L}^{\vee}_{\mathscr{F}, z} \in \mathrm{CoSh}(U)$ denote the associated sheaf and cosheaf respectively.

Non-degeneracy (iii)

By formal representability $\mathscr{F}$ has a relative cotanget complex which is naturally identified with $\mathscr{L}^{\vee}_{\mathscr{F},z}[-1]$ $\Longrightarrow$ get a canonical map

$$\underline{\boldsymbol{\mathcal{A}}}^{p,cl}_{U}(\mathscr{F})[n] \to \bigwedge\nolimits^{p}_{\mathbb{C}} \mathscr{L}^{\vee}_{\mathscr{F},z}[n-p]$$

of cosheaves on U. Any $\omega \in \mathbb{A}^{p,cl}_{X}(\mathscr{F}, n)$ thus gives a map

$$K^{U}_{c} \to \underline{\boldsymbol{\mathcal{A}}}^{2,cl}_{U}(\mathscr{F})[n] \to \bigwedge\nolimits^{2}_{\mathbb{C}} \mathscr{L}^{\vee}_{\mathscr{F},z}[n-2].$$

Dualizing and composing with the natural map $\mathscr{L}_{\mathscr{F},z} \to \mathscr{L}^{\vee\vee}_{\mathscr{F},z}$ gives a map of sheaves on U:

$$\omega^{\flat}_{z} : \bigwedge\nolimits^{2} \mathscr{L}_{\mathscr{F},z}[2] \to K_{U}[n].$$

Non-degeneracy (iv)

Definition: An n-shifted closed relative 2-form $\omega \in \mathbb{A}^{2,cl}_X(\mathscr{F}, n)$ is **non-degenerate** or **relative n-shifted symplectic** if for every open $U \subset X$ and every $z \in \mathscr{F}_U(\mathbb{C})$ the map $\omega_z^\flat$ induces a quasi-isomorphism

$$\mathscr{L}_{\mathscr{F},z}[1] \xrightarrow{\sim} R\underline{Hom}(\mathscr{L}_{\mathscr{F},z}[1], K_U[n])$$

in $\mathrm{Sh}(U)$.

Non-degeneracy (iv)

Definition: An n-shifted closed relative 2-form $\omega \in \mathbb{A}^{2,cl}_X(\mathscr{F}, n)$ is **non-degenerate** or **relative n-shifted symplectic** if for every open $U \subset X$ and every $z \in \mathscr{F}_U(\mathbb{C})$ the map $\omega^\flat_z$ induces a quasi-isomorphism

$$\underbrace{\mathscr{L}_{\mathscr{F},z}}_{\mathbb{T}_{\mathscr{F}_U,z}}[1] \xrightarrow{\sim} R\underline{Hom}(\mathscr{L}_{\mathscr{F},z}[1], K_U[n])$$

in $\mathrm{Sh}(U)$.

Push-forward (iii)

Theorem: [Arinkin-P-Toën] Let

$p : X \to Y$ be a continuous map of locally compact Hausdorff spaces,

$\mathscr{F} \in \mathbf{dSt}(X)$ be a locally formally representable sheaf of derived stacks over X, and

$\omega \in \mathbb{A}^{2,cl}_X(\mathscr{F}, n)$ be a relative closed n-shifted 2-form.

Then

- $p_*\mathscr{F} \in \mathbf{dSt}(Y)$ is locally formally representable;
- if ω is non-degenerate, then $\mathrm{tr} p_*\omega$ is also non-degenerate.

Hence $\left(\begin{array}{l} (\mathscr{F}, \omega) \text{ - } n\text{-shifted} \\ \text{symplectic} \end{array} \right) \Longrightarrow \left(\begin{array}{l} (p_*\mathscr{F}, \mathrm{tr} p_*\omega) \text{ -} \\ n\text{-shifted symplectic} \end{array} \right)$

Closed forms on diagrams of stacks (i)

Fix

- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_{\bullet}$ - a **diagram of shape I in** C.

Closed forms on diagrams of stacks (i)

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- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_\bullet$ - a **diagram of shape $\mathbf{I}$ in C**.

$$\boxed{\begin{array}{l}\text{a functor } F_\bullet : \mathbf{I} \to C, \\ F_\bullet = \{F_\alpha | \alpha \in \mathbf{I}\}\end{array}}$$

Closed forms on diagrams of stacks (i)

Fix

- $\mathbf{I}$ - a finitely presentable ∞-category;
- C - any category with finite limits.

Notation:

- $F_\bullet$ - a **diagram of shape I in** C.
- $\mathbf{I}^{\mathrm{tw}}$ - the category of **twisted arrows in I**.
- $(t, s) : \mathbf{I}^{\mathrm{tw}} \to \mathbf{I} \times \mathbf{I}^{op}$ - the natural functor.

ob($\mathbf{I}^{tw}$): maps $x \xrightarrow{\gamma} y \in \mathrm{mor}(I)$;

$\mathrm{Hom}_{\mathbf{I}^{tw}} \begin{pmatrix} x_1 & , & x_2 \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} \\ y_1 & & y_2 \end{pmatrix}$: commutative diagrams
$$\begin{tikzcd} x_1 \arrow[d, "\gamma_1"'] & x_2 \arrow[l, "u"'] \arrow[d, "\gamma_2"] \\ y_1 \arrow[r, "v"'] & y_2 \end{tikzcd}$$

Closed forms on diagrams of stacks (i)

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- $(t, s) : \mathbf{I}^{\mathrm{tw}} \to \mathbf{I} \times \mathbf{I}^{op}$ - the natural functor.
- $F_{\mathbf{I}} = \lim\limits_{\alpha \in \mathbf{I}} F_\alpha$.

Closed forms on diagrams of stacks (ii)

Fix $E_\bullet : \mathbf{I} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$, and $\mathscr{F}_\bullet : \mathbf{I} \longrightarrow \mathrm{dSt}_{\mathbb{C}}$.

Consider the functor

$$\begin{aligned} \mathcal{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} : \quad \mathbf{I}^{tw} & \longrightarrow \mathrm{Vect}_{\mathbb{C}} \\ \gamma & \longrightarrow \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)}, \end{aligned}$$

and define the **complex of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** as the complex

$$\boxed{\mathbb{A}^{p,cl}_{\mathbf{I}}(\mathscr{F}_\bullet)_{E_\bullet} = \lim_{\gamma \in \mathbf{I}^{tw}} \mathcal{A}^{p,cl}(\mathscr{F}_{s(\gamma)}) \otimes E_{t(\gamma)}.}$$

Closed forms on diagrams of stacks (iii)

The **space of closed p-forms on $\mathscr{F}_\bullet$ with values in $E_\bullet$** is defined as

$$\mathbf{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} = \left| \mathbf{DK}\left(\tau^{\leqslant 0}\mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}\right)\right|$$

and an **$E_\bullet$-valued closed p-form on $\mathscr{F}_\bullet$** is an element in $\pi_0\mathbf{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} = H^0(\mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet})$.

Note: The space of forms comes equipped with a natural **global sections morphism**

$$\Gamma : \mathbb{A}_{\mathsf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet} \to \mathcal{A}^{p,cl}(\mathscr{F}_{\mathsf{I}}) \otimes E_{\mathsf{I}}.$$

Cospecialization and gluing (i)

Suppose

- X - nice topological space (e.g. a CW complex);
- $\imath : Z \hookrightarrow X$ - closed subspace;
- $\jmath : U \hookrightarrow X$ - the complementary open
- $\mathscr{C}$ -an ∞-category with all small limits and colimits. subspace.

For any $F \in \mathrm{Sh}(X, \mathscr{C})$ write $F_{|Z} = \imath^* F$ and $F_{|U} = \jmath^* F$. Applying $\imath^*$ to the unit of the adjunction $\jmath^* \dashv \jmath_*$ yields a **cospecialization map** in $\mathrm{Sh}(Z, \mathscr{C})$:

$$\mathrm{cosp}_Z : F_{|Z} \to \imath^* \jmath_* \left(F_{|U} \right).$$

Cospecialization and gluing (ii)

The assignment $F \longrightarrow (F_{|U}, F_{|Z}, \mathrm{cosp}_Z)$ provides an equivalence of ∞-categories:

$$\mathrm{Sh}(X, \mathscr{C}) \xrightarrow{\cong} \mathrm{lax}^{\mathrm{op}} \lim \left[\mathrm{Sh}(U, \mathscr{C}) \xrightarrow{\imath^* \jmath_*} \mathrm{Sh}(Z, \mathscr{C}) \right],$$

i.e. $\mathrm{Sh}(X, \mathscr{C})$ can be viewed as the $\mathrm{lax}^{\mathrm{op}}$ limit of the functor $\imath^* \jmath_*$.

Key observation: Applying this gluing description to strata in a stratification leads to a combinatorial picture for closed forms and symplectic structures on constructible sheaves of stacks over a space.

Sheaves on a stratified space (i)

Suppose

- X is a nice stratified space (has a finite conical stratification).
- I is the finite poset labeling the strata of X.
- $X_\alpha \subset X$ is the stratum labeled by $\alpha \in I$. X_α will be viewed as a sratified subspace with a single stratum.
- $\mathrm{Sh}^{\mathrm{str}}(X)$ -sheaves of spaces which are constructible for the given stratification. By definition $\mathrm{Sh}^{\mathrm{str}}(X_\alpha)$ is the category of local systems of spaces over X_α.
- For every $\alpha \in I$, write $F_\alpha \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha)$ for the restriction of F to X_α.

Sheaves on a stratified space (ii)

Construction: Fix $\alpha \in I$, $F \in \mathsf{Sh}(X)$, and let

- $Z \subset X$ be a closed subset such that the stratum $X_\alpha \subset Z$ is open.
- $U \subset X$ be the complementary open to Z.
- Define $\mathring{F}_\alpha := \left(\imath^* \jmath_* \left(F_{|U}\right)\right)_{|X_\alpha}$, and let $\mathsf{cosp}_\alpha : F_\alpha \to \mathring{F}_\alpha$ be the restriction of cosp_Z to X_α.

Note:

- $\mathring{F}_\alpha$ and cosp_α depend only on α and not on Z.
- $\mathring{F}_\alpha$ is the sheaf of nearby (co) cycles of F along X_α, and cosp_α is the integral of cospecialization maps over nearby points.

Sheaves on a stratified space (iii)

Note: As with gluing, nearby cycles, and integrated cospecializations make sense for constructible sheaves with values in any category C that admits finite limits.

Let $\mathbf{I}$ be the $\boxed{\infty\text{-category of exit paths}}$ for the stratification on X. Then

- Under our assumptions $\mathbf{I}$ is finitely presentable.
- $\mathrm{Sh}^{\mathrm{str}}(X) = \mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.
- For any category C with finite products we define

$$\mathrm{Sh}^{\mathrm{str}}(X, C) = \mathrm{Fun}(\mathbf{I}, C),$$

  i.e. C-valued constructible sheaves on X are $\mathbf{I}$-shaped diagrams in C.

Sheaves on a stratified space (iii)

Notation: Fix $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then:

- Every $x \in X$ gives an object $\boldsymbol{\iota}(x) \in \mathbf{I}$. The value $F(\boldsymbol{\iota}(x)) \in C$ is called the **stalk of F at x** and is denoted by F_x.
- $\Gamma(X, F) := \lim_{\boldsymbol{\sigma} \in \mathbf{I}} F_{\boldsymbol{\sigma}} \in C$ is the **global sections** object of F. We have a natural **evaluation map** $\mathrm{ev}_x : \Gamma(X, F) \to F_x$ for every $x \in X$.
- For $\alpha \in I$ the fundamental groupoid $\Pi_1(X_\alpha)$ of the stratum X_α embeds in $\mathbf{I}$ as the full subcategory $\mathbf{I}_\alpha \subset \mathbf{I}$ spanned by $\boldsymbol{\iota}(x)$ for all $x \in X_\alpha$. We define the **value of F on X_α** as $F_\alpha := F_{|\mathbf{I}_\alpha} \in \mathrm{Sh}^{\mathrm{str}}(X_\alpha, C)$.

Sheaves on a stratified space (iv)

Notation: Fix $\alpha \in I$, and $F \in \mathrm{Sh}^{\mathrm{str}}(X, C)$. Then consider:

- $\mathbf{I}_{\alpha}^{\geqslant} = \{a' \to a \mid a' \in \mathbf{I}, a \in \mathbf{I}_{\alpha}\}$
- $\mathbf{I}_{\alpha}^{>} \subset \mathbf{I}_{\alpha}^{\geqslant}$ - the full subcategory consisting of all $a' \to a$ which are **not** isomorphisms.
- $\mathring{F}_{\alpha} \in \mathrm{Sh}^{\mathrm{str}}(X_{\alpha}, C)$ - the right Kan extension of $F_{|\mathbf{I}_{\alpha}^{>}}$ along the natural functor $\mathbf{I}_{\alpha}^{>} \to \mathbf{I}_{\alpha}$.
- $\mathrm{cosp}_{\alpha} : F_{\alpha} \to \mathring{F}_{\alpha}$ - the map guaranteed by the universal property of the right Kan extension.

Terminology: $\mathring{F}_{\alpha}$ is the **sheaf of nearby cycles** of F at X_{α}, and cosp_{α} is the (integrated) **cospecialization map**.

Sheaves on a stratified space (v)

Given a nice stratified space X with strata labeled by a poset I, and an exit path category $\mathbf{I}$, and
$E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{Vect}_{\mathbb{C}})$,
$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) = \mathrm{Fun}(\mathbf{I}, \mathrm{dSt}_{\mathbb{C}})$, we get

- a **complex** $\mathbb{A}_X^{p,cl}(\mathscr{F})_E := \mathbb{A}_{\mathbf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}$ and a **space** $\mathbf{A}_X^{p,cl}(\mathscr{F})_E := \mathbf{A}_{\mathbf{I}}^{p,cl}(\mathscr{F}_\bullet)_{E_\bullet}$ **of global closed E-valued p-forms on $\mathscr{F}$**;
- a natural **global sections map** of complexes

$$\Gamma : \mathbb{A}_X^{p,cl}(\mathscr{F})_E \to \mathcal{A}^{p,cl}(\Gamma(X, \mathscr{F})) \otimes \Gamma(X, E).$$

Sheaves on a stratified space (vi)

Let $\mathscr{F}$ be a constructible sheaf of locally f.p. derived Artin stacks. For any point $z \in \Gamma(X, \mathscr{F})$ the relative tangent complex $\mathbb{T}_{\mathscr{F},z} \in \mathsf{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ is a constructible complex of vector spaces on X.

Given a point $z \in \Gamma(X, \mathscr{F})$, any closed form $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F})_E$ defines a map of constructible complexes on X

$$\omega_z^\flat : \bigwedge^p \mathbb{T}_{\mathscr{F},z} \to E.$$

Definition: The induced map $\Gamma(\omega_z^\flat)$ on global sections is the **value of ω at z**.

Sheaves on a stratified space (vi)

Note: All of this readily sheafifies:

- For reasonable open sets $U \subset X$ the assignment

$$U \longrightarrow \mathbb{A}_U^{p,cl}(\mathscr{F}_{|U})_{E_{|U}}$$

  is a presheaf of complexes. Its sheafification $\underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})_E$ is a sheaf of complexes on X equipped with a natural map

$$\Gamma : \Gamma(X, \underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})_E) \to \mathcal{A}^{p,cl}(\Gamma(X, \mathscr{F})) \otimes \Gamma(X, E).$$

- In terms of diagrams $\underline{\boldsymbol{\mathcal{A}}}_X^{p,cl}(\mathscr{F})_E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$ is constructed as the right Kan extension of $\boldsymbol{\mathcal{A}}_{\mathbf{I}}^{p,cl} : \mathbf{I}^{tw} \to \mathrm{Vect}_{\mathbb{C}}$ along the functor $t : \mathbf{I}^{tw} \to \mathbf{I}$.

Non-degeneracy (v)

Setup:

- X - nicely stratified space with equidimensional strata.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ is a constructible sheaf of locally f.p. derived Artin stacks (or just locally formally representable derived stacks).
- $E = K_X[n] \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$, where $n \in \mathbb{Z}$ and K_X is the Verdier's dualizing complex of X.

Note: The cosheaf notion of shifted closed forms and shifted symplectic forms has an equivalent reformulation in this setting.

Non-degeneracy (vi)

Definition:

(a) The **complex and space of relative n-shifted closed p-forms on $\mathscr{F}$** are defined to be

$$\mathbb{A}_X^{p,cl}(\mathscr{F}, n) := \mathbb{A}_X^{p,cl}(\mathscr{F})_{K_X[n]},$$
$$\mathbf{A}_X^{p,cl}(\mathscr{F}, n) := \mathbf{A}_X^{p,cl}(\mathscr{F})_{K_X[n]}.$$

(b) A closed relative n-shifted 2-form $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F}, n)$ is **symplectic** if for every $U \subset X$ and any point $z \in \Gamma(U, \mathscr{F})$, the map

$$\omega_z^\flat : \ \mathbb{T}_{\mathscr{F}_{|U},z} \longrightarrow \mathbb{L}_{\mathscr{F}_{|U},z} \otimes K_U[n]$$

is a quasi-isomorphism.

Non-degeneracy and nearby cycles (i)

Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$, $E = K_X[n]$, $\alpha \in I$.

- $\mathscr{F}_\alpha$ and $\mathring{\mathscr{F}}_\alpha$ are local systems of derived stacks on X_α, i.e. are derived stacks equipped with an action of $\Pi_1(X_\alpha) = \mathbf{I}_\alpha$.
- The fiber of $\mathrm{cosp}_\alpha^E : E_\alpha \to \mathring{E}_\alpha$ is equal to the !-restriction of $K_X[n]$ to X_α and so we have an exact triangle

$$(*) \qquad E_\alpha \xrightarrow{\mathrm{cosp}_\alpha^E} \mathring{E}_\alpha \longrightarrow \mathbb{C}\left[n + 1 + \dim X_\alpha\right].$$

Note: When $\alpha \in I$ is maximal we have $\mathring{E}_\alpha = 0$ and so $E_\alpha = \mathbb{C}\left[n + \dim X_\alpha\right]$.

Non-degeneracy and nearby cycles (ii)

Let $\omega \in \mathbf{A}_X^{2,cl}(\mathscr{F}, n) = \mathbf{A}_X^{2,cl}(\mathscr{F})_E$ be a relative n-shifted closed 2-form on $\mathscr{F}$. Then ω induces

- ω_α a relative closed E_α-valued 2-form on $\mathscr{F}_\alpha$, i.e. $\omega_\alpha \in \mathbf{A}_{X_\alpha}^{2,cl}(\mathscr{F}_\alpha)_{E_\alpha}$.
- $\mathring{\omega}_\alpha$ a closed $\mathring{E}_\alpha$-valued 2-form on $\mathring{\mathscr{F}}_\alpha$, i.e. $\mathring{\omega}_\alpha \in \mathbf{A}_{X_\alpha}^{2,cl}(\mathring{\mathscr{F}}_\alpha)_{\mathring{E}_\alpha}$.
- $\overline{\omega}_\alpha$ a closed $(n+1)$-shifted 2-form on $\mathring{\mathscr{F}}_\alpha$, i.e.

$$\overline{\omega}_\alpha \in \mathbf{A}_{X_\alpha}^{2,cl}(\mathring{\mathscr{F}}_\alpha, n+1) = \mathbf{A}_{X_\alpha}^{2,cl}(\mathring{\mathscr{F}}_\alpha)_{\mathbb{C}[n+1+\dim X_\alpha]}.$$

Non-degeneracy and nearby cycles (iii)

Note:

- $\overline{\omega}_\alpha$ is the pushout of $\mathring{\omega}_\alpha$ by the map

$$\mathring{E}_\alpha \to \mathbb{C}\left[n + 1 + \dim X_\alpha\right].$$

- Viewing $\mathring{\mathscr{F}}_\alpha$ as a constant derived Artin stack equipped with a $\Pi_1(X_\alpha)$ action, then we can view $\overline{\omega}_\alpha$ equivalently as an **absolute** $(n + 1 + \dim X_\alpha)$-shifted 2-form, i.e.

$$\overline{\omega}_\alpha \in \mathcal{A}^{2,cl}(\mathring{\mathscr{F}}_\alpha, n + 1 + \dim X_\alpha).$$

Non-degeneracy and nearby cycles (iv)

Key observation: Consider the $\Pi_1(X_\alpha)$-equivariant cospecialization map

$$\mathrm{cosp}_\alpha^{\mathscr{F}} : \mathscr{F}_\alpha \to \mathring{\mathscr{F}}_\alpha$$

of derived Artin stacks.
The exact triangle

$$E_\alpha \xrightarrow{\mathrm{cosp}_\alpha^E} \mathring{E}_\alpha \longrightarrow \mathbb{C}\left[n+1+\dim X_\alpha\right].$$

yields a natural path h_α between $\mathrm{cosp}_\alpha^{\mathscr{F}*}(\overline{\omega}_\alpha)$ and 0 in the space $\mathbf{A}^{2,cl}_{X_\alpha}(\mathscr{F}_\alpha, n+1)$ or equivalently a path between $\mathrm{cosp}_\alpha^{\mathscr{F}*}(\overline{\omega}_\alpha)$ and 0 in the space $\mathcal{A}^{2,cl}(\mathscr{F}_\alpha, n+1+\dim X_\alpha)$.

Non-degeneracy and nearby cycles (v)

Theorem: [Arinkin-P-Toën] Suppose

$\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of derived Artin stacks, locally of f.p.;

$\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F}, n)$ - a relative closed n-shifted 2-form on $\mathscr{F}$.

Then ω is symplectic if and only if for every $\alpha \in I$ the following two conditions hold:

(a) $\overline{\omega}_\alpha$ is symplectic.

(b) $\mathrm{cosp}^{\mathscr{F}}_\alpha : (\mathscr{F}_\alpha, h_\alpha) \to (\mathring{\mathscr{F}}_\alpha, \overline{\omega}_\alpha)$ is Lagrangian.

Claim: [Arinkin-P-Toën] Fix $\beta \in I$ and assume that **(a)** and **(b)** hold for all $\alpha > \beta$. Then $\overline{\omega}_\beta$ is a shifted symplectic form on $\mathring{\mathscr{F}}_\beta$.

Push-forwards (iv)

Pushing forward also makes sense in this setting. Suppose

- $f : X \to Y$ is a stratified map of nicely stratified spaces;
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
- $E \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{Vect}_{\mathbb{C}})$;

We have push-forwards

$$f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{dSt}_{\mathbb{C}}) \quad \text{and} \quad f_*E \in \mathrm{Sh}^{\mathrm{str}}(Y, \mathrm{Vect}_{\mathbb{C}})$$

computed via the right Kan extensions along the functor between exit path categories induced from f.

The pushforward of $\omega \in \mathbf{A}_X^{p,cl}(\mathscr{F})_E$ then is a closed relative form:

$$\boxed{f_*\omega \in \mathbf{A}_Y^{p,cl}(f_*\mathscr{F})_{f_*E}.}$$

Push-forwards (v)

Theorem: [Arinkin-P-Toën] Suppose that $f : X \to Y$ is proper and that $\omega \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n]}$ is symplectic. Then the pushforward

$$\operatorname{tr} f_*\omega \in \mathbf{A}^{2,cl}_Y(f_*\mathscr{F})_{K_Y[n]}$$

is symplectic as well.

Remark:

- Here $\operatorname{tr} : f_* K_X \to K_Y$ denotes the canonical trace map.
- Most of the standard constructions of shifted symplectic structures arise as special cases of the above theorem.

Deligne-Malgrange-Stokes sheaves (i)

Suppose X is a smooth surface underlying a quasi-projective complex algebraic curve and stratified with the following strata:

- a connected open stratum X_{in};
- a (not necessarily connected) open stratum X_{out};
- arcs X_e, $e \in E$;
- endpoints X_v, $v \in V$.

We require that the strata satisfy the following conditions:

(1) exactly two arcs meet at each enpoint;

(2) each arc separates X_{in} and X_{out};

(3) X_{out} retracts onto its boundary.

Deligne-Malgrange-Stokes sheaves (ii)

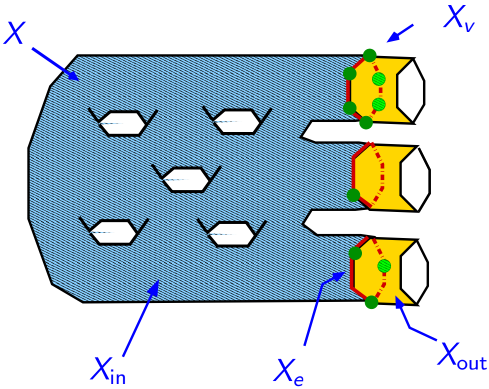

Figure: Stratified surface

Deligne-Malgrange-Stokes sheaves (iii)

Setup: Consider $\mathscr{F} \in \mathsf{Sh}^{\mathsf{str}}(X, \mathsf{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks satisfying:

$\boxed{\text{codim} = 0}$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

Deligne-Malgrange-Stokes sheaves (iii)

Setup: Consider $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - a constructible sheaf of stacks satisfying:

$\boxed{\text{codim} = 0}$ Locally on X_{in} and on X_{out}, $\mathscr{F}$ is isomorphic to BG for some reductive G.

typically the groups will be different on different connected components of strata

Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 1}$ For each point $x \in X_e$, $e \in E$ the specialization of strata

$$X_{\text{in}} \rightsquigarrow X_e \leftsquigarrow X_{\text{out}}$$

leads to a cospecialization diagram of stacks:

$$\mathscr{F}_{\text{in}} \longleftarrow \mathscr{F}_e \longrightarrow \mathscr{F}_{\text{out}} \ .$$

We require that this diagram must be isomorphic to

$$BG \longleftarrow BP \longrightarrow BL \ ,$$

where G is a reductive group, $P \subset G$ is a parabolic subgroup, and $P \to L$ is the Levi quotient.

Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 2}$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata

$$\begin{tikzcd} & X_{e_1} \arrow[d, squiggly] & \\ X_{\text{in}} \arrow[ur, squiggly] \arrow[r, squiggly] \arrow[dr, squiggly] & X_v & X_{\text{out}} \arrow[ul, squiggly] \arrow[l, squiggly] \arrow[dl, squiggly] \\ & X_{e_2} \arrow[u, squiggly] & \end{tikzcd}$$

Deligne-Malgrange-Stokes sheaves (iv)

$\boxed{\text{codim} = 2}$ In a neighborhood of a point $\{b\} = X_v$, $v \in V$, the specialization of strata leads to a diagram of stacks

$$\begin{tikzcd}
& \mathscr{F}_{e_1} \arrow[dl] \arrow[dr] & \\
\mathscr{F}_{\text{in}} & \mathscr{F}_v \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathscr{F}_{\text{out}} \\
& \mathscr{F}_{e_2} \arrow[ul] \arrow[ur] &
\end{tikzcd},$$

Deligne-Malgrange-Stokes sheaves (v)

$\boxed{\text{codim} = 2}$ We require that

$$\begin{tikzcd} & \mathscr{F}_{e_1} \arrow[dl] \arrow[dr] & \\ \mathscr{F}_{\text{in}} & \mathscr{F}_v \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathscr{F}_{\text{out}} \\ & \mathscr{F}_{e_2} \arrow[ul] \arrow[ur] & \end{tikzcd} \cong \begin{tikzcd} & BP_1 \arrow[dl] \arrow[dr] & \\ BG & B(P_1 \cap P_2) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & BL \\ & BP_2 \arrow[ul] \arrow[ur] & \end{tikzcd},$$

where G is a reductive group, $P_1, P_2 \subset G$ are parabolic subgroups that admit a common Levi, and $P_{1,2} \to L$ are the Levi quotients.

Deligne-Malgrange-Stokes sheaves (vi)

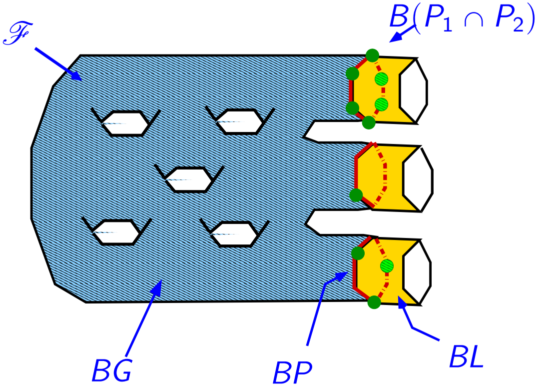

Figure: DMS sheaf

Deligne-Malgrange-Stokes sheaves (vii)

Theorem: **[Arinkin-P-Toën]** Let $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ be a DMS sheaf. Then

(1) The restriction map $\mathbf{A}^{2,cl}(\mathscr{F})_{K_X} \to \mathbf{A}^{2,cl}(\mathscr{F}_{\mathrm{in}})_{K_{X_{\mathrm{in}}}}$, is a homotopy equivalence.

(2) The extension ω of a form ω_{in} is non-degenerate if and only if ω_{in} is non-degenerate.

Deligne-Malgrange-Stokes sheaves (ix)

Remarks:

- $\mathscr{F}_{\mathrm{in}}$ is a local system of stacks with fiber BG. Hence $\mathbb{T}_{\mathscr{F}_{\mathrm{in}}}$ is a local system of complexes on X_{in} with fiber $\mathfrak{g}[1]$.
- In particular $\begin{pmatrix}\text{a } \Pi_1(X_{\mathrm{in}}) \times G\text{-invariant} \\ \text{symmetric pairing } \kappa \text{ on } \mathfrak{g}\end{pmatrix}$

$$\Updownarrow$$

$$\begin{pmatrix}\text{a } K_{X_{\mathrm{in}}}\text{-valued relative symplectic} \\ \text{form } \omega_{\mathrm{in}} = \omega_\kappa \text{ on } \mathscr{F}_{\mathrm{in}}\end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix}\text{a } K_X\text{-valued relative symplectic} \\ \text{form } \omega \text{ on } \mathscr{F}.\end{pmatrix}$$

- When $\mathscr{F}_{\mathrm{in}}$ is **constant**, the form κ exists automatically since G is assumed to be reductive.

Deligne-Malgrange-Stokes sheaves (ix)

Suppose

- $\mathfrak{X}$ is a smooth projective curve/$\mathbb{C}$;
- $X = \mathfrak{X} - \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$;
- $\mathscr{I} = \{\mathscr{I}_1, \ldots, \mathscr{I}_k\}$, $\mathscr{I}_i$ an irregular type at $\boldsymbol{x}_i$.

Then:

• $\mathscr{I}$ can be recorded equivalently in $\mathrm{DMS}_{G,\mathscr{I}} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$;
• $\mathrm{DMS}_{G,\mathscr{I}}$ classifies Stokes data on $\mathfrak{X}$ of irregular type $\mathscr{I}$, in the sense that

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}|\widehat{\mathfrak{X}}}\right).$$

Here $\widehat{\mathfrak{X}} \subset X$ denotes the real oriented blow-up of $\mathfrak{X}$ at the points x_i.

Deligne-Malgrange-Stokes sheaves (x)

Note:

- The sheaf $\mathsf{DMS}_{G,\mathscr{I}}$ tautologically satisfies the codim $= 0, 1, 2$ properties.
- Since the underlying G-local systems are untwisted $\mathsf{DMS}_{G,\mathscr{I}}$ comes with a canonical relative symplectic form which depends only on a choice of a non-degenerate $\kappa \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$.

Because of this we introduce the following:

Definition: A **DMS sheaf** on X is a sheaf $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ that satisfies the codim $= 0, 1, 2$ properties and admits a relative K_X-valued symplectic structure.

Deligne-Malgrange-Stokes sheaves (xi)

Suppose

$\mathscr{F}$ is a DMS sheaf of stacks euipped with a K_X-valued relative symplectic form ω.

$f : X \to (0,1]$ is the stratified map which collapses $X - X_{\mathrm{out}}$ to 1 and projects each cylinder component of X_{out} onto its ruling $(0,1)$.

Then f is a proper stratified map and

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\, f_*\omega$ is a relative $K_{(0,1]}$-valued symplectic structure on $f_*\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$.

Deligne-Malgrange-Stokes sheaves (xii)

Hence

- tr $f_*\omega$ defines a 1-shifted symplectic structure on $\Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}})$.
- tr $f_*\omega$ defines a 0-shifted Lagrangian structure on the cospecialization map

$$\Gamma(X - X_{\text{out}}, \mathscr{F}) \to \Gamma(X_{\text{out}}, \mathscr{F}_{\text{out}}).$$

Deligne-Malgrange-Stokes sheaves (xiii)

In the special case when $\mathscr{F} = \mathrm{DMS}_{G,\mathscr{I}}$ we get

$$\begin{aligned} Loc_G(X, \mathscr{I}) &= \Gamma(X - X_{\mathrm{out}}, \mathscr{F}) \\ Loc_{\widetilde{L}}(\partial X) &= \Gamma(X_{\mathrm{out}}, \mathscr{F}_{\mathrm{out}}) \end{aligned}$$

where $\widetilde{L}$ is the local system of Levi subgroups on X_{out} for which $\mathscr{F}_{\mathrm{out}} = B\widetilde{L}$.

Moreover in this setting the map

$$r_{\mathscr{I}} : Loc_G(X, \mathscr{I}) \to Loc_{\widetilde{L}}(\partial X)$$

assigns to each Stokes filtered local system its formal monodromy at ∞.

Deligne-Malgrange-Stokes sheaves (xiv)

Since $\widetilde{L}$ is a locally constant sheaf we again have that fixing a flat section $\lambda \in \Gamma(\partial X, \widetilde{L})$ gives a Lagrangian map

$$\prod_{i=1}^{k} BG_{\lambda_i} \to \prod_{i=1}^{k} [G/G] = Loc_{\widetilde{L}}(\partial X).$$

The intersection

$$Loc_G(X, \mathscr{I}; \lambda)$$

of this Lagrangian with the Lagrangian map $r_{\mathscr{I}}$ is the moduli of Stokes data of type $\mathscr{I}$ having a fixed formal monodromy at ∞ and is therefore symplectic.

Constant sheaves (i)

Setup:

- X - a topological space = a stratified space with a single stratum.
- $F \in \mathrm{dSt}_{\mathbb{C}}$ - a derived Artin stack, locally of finite presentation.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ - the constant sheaf on X with fiber F.
- $\omega \in \mathcal{A}^{2,cl}(F, n)$ - an (absolute) n-shifted symplectic form on F.

Constant sheaves (ii)

Note:

- ω corresponds to a relative $\mathbb{C}_X[n]$-valued closed 2-form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{\mathbb{C}[n]}$ on $\mathscr{F}$.
- If X is an oriented manifold of pure dimension d, then $K_X = \mathbb{C}_X[d]$ and so $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n-d]}$.

$$\begin{pmatrix} \omega \text{ is non-degenerate as} \\ \text{an } n\text{-shifted absolute} \\ \text{form on } F \end{pmatrix} \Leftrightarrow \begin{pmatrix} \omega_X \text{ is non-degenerate} \\ \text{as a } K_X[n-d]\text{-valued} \\ \text{relative form on } \mathscr{F}. \end{pmatrix}$$

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega_X))$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X, F)$.

Constant sheaves (ii)

Note:

- ω corresponds to a relative $\mathbb{C}_X[n]$-valued closed 2-form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{\mathbb{C}[n]}$ on $\mathscr{F}$.
- If X is an oriented manifold of pure dimension d, then $K_X = \mathbb{C}_X[d]$ and so $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X[n-d]}$.

$$\begin{pmatrix}\omega \text{ is non-degenerate as} \\ \text{an } n\text{-shifted absolute} \\ \text{form on } F\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\omega_X \text{ is non-degenerate} \\ \text{as a } K_X[n-d]\text{-valued} \\ \text{relative form on } \mathscr{F}.\end{pmatrix}$$

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega_X))$ is an $(n-d)$-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = Map_{\mathrm{dSt}_{\mathbb{C}}}(X, F)$.
This is the mapping stack theorem from **[PTVV]**.

Lagrangian maps

Setup:

- $X = (0, 1]$ is the half-open interval stratified by the strata $X_{\mathrm{in}} = (0, 1)$ and $X_1 = \{1\}$, labeled by $I = \{\mathrm{in}, 1\}$ with $1 < \mathrm{in}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and one cospecialization map $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Note: $K_X =$ (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[1]$ from X_{in} to X). Thus a relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- an n-shifted Lagrangian structure on $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\mathrm{in}} = (0,1)$, $X_0 = \{0\}$, and $X_1 = \{1\}$, where $I = \{0, 1, \mathrm{in}\}$ with $0 < \mathrm{in}$, $1 < \mathrm{in}$, while 0 and 1 are incomparable.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Note: $K_X = $ (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[1]$ from X_{in} to X). A relative n-shifted symplectic structure on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and

$\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\mathrm{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\mathrm{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

A relative n-shifted symplectic structure ω on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\mathrm{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\mathrm{in}}$ and

$\mathscr{F}_1 \to \mathscr{F}_{\mathrm{in}}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is an n-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = \mathscr{F}_0 \times^h_{\mathscr{F}_{\mathrm{in}}} \mathscr{F}_1$.

Lagrangian intersections

Setup:

- $X = [0,1]$ stratified by $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}}) \Longrightarrow$ specified by $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_{\text{in}} \in \mathrm{dSt}_{\mathbb{C}}$, and two cospecialization maps $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and $\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

A relative n-shifted symplectic structure ω on $\mathscr{F}$ consists of:

- an $(n+1)$-shifted symplectic structure on $\mathscr{F}_{\text{in}}$;
- n-shifted Lagrangian structures on $\mathscr{F}_0 \to \mathscr{F}_{\text{in}}$ and

$\mathscr{F}_1 \to \mathscr{F}_{\text{in}}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega))$ is an n-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = \mathscr{F}_0 \times^h_{\mathscr{F}_{\text{in}}} \mathscr{F}_1$.

This is the Lagrangian intersection theorem from **[PTVV]**.

Hamiltonian reduction (iii)

Setup:

- $X = [0,1]$ with strata $X_{\text{in}} = (0,1)$, $X_0 = \{0\}$, $X_1 = \{1\}$.
- G - a linear algebraic group$/\mathbb{C}$, and
  $\mathbb{O} \subset \mathfrak{g}^\vee$ - a coadjoint orbit.
- (M, ω) - an algebraic symplectic manifold equipped with a Hamiltonian G-action.
- $\mu : M \to \mathfrak{g}^\vee$ - a G-equivariant moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbb{O}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = [\mathfrak{g}^\vee/G] = T^\vee_{BG}[1]$$

and maps $[\mathbb{O}/G] \hookrightarrow [\mathfrak{g}^\vee/G]$ and $\mu : [M/G] \to [\mathfrak{g}^\vee/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega)\right)$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

Hamiltonian reduction (iv)

Note: The **Kirillov-Kostant-Souriau** form on $\mathfrak{g}^\vee$ induces a relative 0-shifted symplectic structure ω on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega)\right)$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbb{O})/G]$.

This is the **Marsden-Weinstein** Hamiltonian reduction theorem.

Quasi-Hamiltonian reduction (iii)

Setup:

- $X = S^2$ with strata $X_{\text{in}} = S^2 - \{S, N\}$, $X_0 = \{S\}$, $X_1 = \{N\}$.
- G - a complex reductive group, $\mathbf{C} \subset G$ - a conjugacy class.
- (M, ω) - an algebraic symplectic manifold equipped with a quasi-Hamiltonian G-action.
- $\mu : M \to G$ - a G-equivariant group valued moment map.
- $\mathscr{F} \in \mathrm{Sh}^{\mathrm{str}}(X, \mathrm{dSt}_{\mathbb{C}})$ given by

$$\mathscr{F}_0 = [\mathbf{C}/G], \quad \mathscr{F}_1 = [M/G], \quad \mathscr{F}_{\text{in}} = BG$$

and maps $[\mathbf{C}/G] \hookrightarrow [G/G]$ and $\mu : [M/G] \to [G/G]$.

Quasi-Hamiltonian reduction (iv)

Note:

- K_X = (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[2]$ from X_{in} to X).
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}_X^{2,cl}(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\left(\Gamma(\omega_X)\right)$ is a 0-shifted symplectic structure on $\Gamma(X, \mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

Back

Quasi-Hamiltonian reduction (iv)

Note:

- $K_X = $ (extension by zero of $\mathbb{C}_{X_{\mathrm{in}}}[2]$ from X_{in} to X).
- The standard 2-shifted symplectic form ω_κ on BG extends to a natural relative 0-shifted symplectic form $\omega_X \in \mathbf{A}^{2,cl}_X(\mathscr{F})_{K_X}$ on $\mathscr{F}$.

Pushforward theorem $\Longrightarrow$ $\mathrm{tr}\,(\Gamma(\omega_X))$ is a 0-shifted symplectic structure on $\Gamma(X,\mathscr{F}) = [R\mu^{-1}(\mathbf{C})/G]$.

This is the **Alexeev-Malkin-Meinrenken** quasi-Hamiltonian reduction theorem.

Back

Sheaves, cosheaves, and functors (i)

X - locally compact Hausdorff space;
$\mathscr{U}(X)$ - the poset of opens in X;
$\mathscr{C}$ - an ∞-category which admits all small limits.

Definition: A $\mathscr{C}$-valued **sheaf** on X is a functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ satisfying the **sheaf condition**: for every open cover $\{U_\alpha\}$ of an open set U the natural map

$$F(U) \longrightarrow \varprojlim_{V} F(V)$$

is an equivalence in $\mathscr{C}$. Here the limit is taken over all open subsets $V \subset U$ which are contained in one of the U_α.

Sheaves, cosheaves, and functors (ii)

X - locally compact Hausdorff space;
$\mathscr{U}(X)$ - the poset of opens in X;
$\mathscr{C}$ - an ∞-category which admits all small colimits.

Definition: A $\mathscr{C}$-valued **cosheaf** on X is a functor $F : \mathscr{U}(X)^{op} \to \mathscr{C}$ satisfying the **cosheaf condition**: for every open cover $\{U_\alpha\}$ of an open set U the natural map

$$\underset{\overrightarrow{V}}{\mathrm{colim}}\, F(V) \to F(U)$$

is an equivalence in $\mathscr{C}$. The colimit is taken over all opens $V \subset U$ which are contained in one of the U_α.

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Representability conditions (i)

Let $\mathscr{F} \in \mathbf{dSt}(X)$ be a sheaf of derived stacks. We can consider $\mathscr{F}$ as a functor

$$\mathscr{F}(-) : \mathsf{cdga}_{\mathbb{C}} \longrightarrow \mathsf{St}(X)$$

from cdga to stacks (of homotopy types) over X.

Note: We can describe the condition that $\mathscr{F}$ is **locally formally representable** over X in terms of the functor $\mathscr{F}(-)$.

Representability conditions (ii)

$\mathscr{F}$ is locally formally representable if and only if for any cartesian diagram of local Artinian cdga

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$$

where $H^0(A') \to H^0(A)$ is surjective, the corresponding diagram

$$\begin{tikzcd} \mathscr{F}(B') \arrow[r] \arrow[d] & \mathscr{F}(B) \arrow[d] \\ \mathscr{F}(A') \arrow[r] & \mathscr{F}(A) \end{tikzcd}$$

is cartesian in $\mathrm{St}(X)$.

Back

Exit paths (i)

Setup:

- X be a stratified space with strata labeled by a poset I.
- $|\Delta^n| = \left\{ (t_0, \ldots, t_n) \in [0,1]^{\times n} \;\middle|\; \sum_{i=0}^{1} = 1 \right\}$ is the standard simplex.

Definition: The **simplicial set of exit paths** of X is the simplicial subset $\mathbf{I} \subset \mathrm{Sing}(X)$ consisting of those simplices $\sigma : |\Delta^n| \to X$ that satisfy the condition

$$(*) \quad \left(\begin{array}{l} \text{There exists a chain of elements } \alpha_0 < \alpha_1 < \cdots < \alpha_n \in I \\ \text{so that for every point } (t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in |\Delta^n| \\ \text{with } t_i \neq 0 \text{ we have that } \sigma(t_0, t_1, \ldots, t_i, 0, \ldots, 0) \in X_{\alpha_i}. \end{array} \right)$$

Exit paths (ii)

Theorem: [Lurie]

(a) If the stratification on X is conical, then $\mathbf{I}$ is an ∞-category.

(b) Let X be a paracompact topological space which is locally of a singular shape, and is equipped with a conical I-startification. Then the ∞-category of I-constructible sheaves of spaces on X is equivalent to the ∞-category $\mathrm{Fun}(\mathbf{I}, \mathrm{SSets})$.

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