Lecture 1: Shifted symplectic structures on moduli of local systems

Tony Pantev

University of Pennsylvania

Moduli Spaces, Representation Theory, and Quantization
June 23-27, 2019 RIMS

Outline

- joint with Dima Arinkin and Bertrand Toën
- Understand symplectic structures along the fibers of sheaves of derived stacks over a topological space.
- Apply to the geometry of the moduli of tame or wild local systems on a smooth variety over $\mathbb{C}$:
  - construct (shifted) Poisson structures;
  - describe their symplectic leaves.

Recall: unramified character varieties

- X - smooth projective curve/$\mathbb{C}$,
- G - a complex reductive group,
- $M_G(X)$ - the coarse moduli space of representations $\rho : \pi_1(X, x) \to G$.

Classical story:

- The smooth part $M_G^{\mathrm{sm}}(X)$ of $M_G(X)$ admits an algebraic symplectic structure;
- There are explicit descriptions:
  - **cohomological** construction in deformation theory Goldman, Karshon, Weinstein, . . . ;

Recall: unramified character varieties

- X - smooth projective curve/$\mathbb{C}$,
- G - a complex reductive group,
- $M_G(X)$ - the coarse moduli space of representations $\rho : \pi_1(X, x) \to G$.

Classical story:

- The smooth part $M_G^{\mathrm{sm}}(X)$ of $M_G(X)$ admits an algebraic symplectic structure;
- There are explicit descriptions:
  - **cohomological** construction in deformation theory Goldman, Karshon, Weinstein, . . . ;

non-degeneracy = Poincaré duality

Recall: unramified character varieties

- X - smooth projective curve$/\mathbb{C}$,
- G - a complex reductive group,
- $M_G(X)$ - the coarse moduli space of representations $\rho : \pi_1(X, x) \to G$.

Classical story:

- The smooth part $M_G^{\mathrm{sm}}(X)$ of $M_G(X)$ admits an algebraic symplectic structure;
- There are explicit descriptions:
  - **cohomological** construction in deformation theory Goldman, Karshon, Weinstein, . . . ;
  - **quasi-Hamiltonian reduction** construction Alekseev-Malkin-Meinrenken;

Recall: tame character varieties

- X - a smooth quasi-projective curve/$\mathbb{C}$.

Classical story: Fock-Rosly, Goldman, Guruprasad-Rajan, Guruprasad-Huebschmann-Jeffrey-Weinstein, ...

- $M_G^{\mathrm{sm}}(X)$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X)$ are the moduli spaces of ρ with fixed monodromy at infinity.
- There are **cohomological** and **quasi-Hamiltonian** descriptions of symplectic leaves.

Recall: tame character varieties

- X - a smooth quasi-projective curve/$\mathbb{C}$.

Classical story: Fock-Rosly, Goldman, Guruprasad-Rajan, Guruprasad-Huebschmann-Jeffrey-Weinstein, ...

- $M_G^{\mathrm{sm}}(X)$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X)$ are the moduli spaces of ρ with fixed monodromy at infinity.
- There are **cohomological** and **quasi-Hamiltonian** descriptions of symplectic leaves.

non-degeneracy = Lefschetz duality

Recall: tame character varieties

- X - a smooth quasi-projective curve$/\mathbb{C}$.

Classical story: Fock-Rosly, Goldman, Guruprasad-Rajan, Guruprasad-Huebschmann-Jeffrey-Weinstein, ...

- $M_G^{\mathrm{sm}}(X)$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X)$ are the moduli spaces of ρ with fixed monodromy at infinity.
- There are **cohomological** and **quasi-Hamiltonian** descriptions of symplectic leaves.

Note: Symplectic leaves = **tame character varieties**.

Recall: wild character varieties

- $\mathfrak{X}$ is a smooth projective curve$/\mathbb{C}$, $X = \mathfrak{X} - \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$;
- $\mathscr{I} = \{\mathscr{I}_1, \ldots, \mathscr{I}_k\}$, $\mathscr{I}_i$ irregular type at $\boldsymbol{x}_i$;
- $M_G(X, \mathscr{I})$ - the moduli of representations of $\pi_1(X)$ equipped with Stokes data of type $\mathscr{I}$.

Classical story: Boalch, Boalch-Yamakawa

- $M_G^{\mathrm{sm}}(X, \mathscr{I})$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G^{\mathrm{sm}}(X, \mathscr{I})$ are the moduli spaces of ρ with fixed formal monodromy at infinity.
- There is a **quasi-Hamiltonian** descriptions of symplectic leaves.

Goal

Remark:

- The ramified setting is often better behaved. Generic choices of local monodromies (in the tame case) or of irregular types and formal monodromies (in the wild case) ensure that $M_G^{\mathrm{sm}} = M_G$
- The **cohomological** description and the linear reason for non-degeneracy of the symplectic structure on the wild character varieties is not immediately clear.

Goal

- Construct Poisson structures everywhere on $M_G(X)$ and $M_G(X, \mathscr{I})$ including the singular points.
- Describe their symplectic leaves.
- Find **cohomological** and **quasi-Hamiltonian** descriptions of the symplectic form at the singular points.
- Extend the whole story to higher dimensional smooth varieties X.

Setup

Natural approach: Resolve the singularities of M_G in a minimal way so that the Poisson and symplectic structures extend to the resolution.

Lucky break: $M_G(X)$ and $M_G(X, \mathscr{I})$ admit natural resolutions which are again moduli spaces.

Note: These resolutions/refinements of M_G are not schemes but rather are derived algebraic stacks whicih are locally finitely presentable and in particular have perfect tangent complexes.

Refinements (i)

Unramified+tame cases

The moduli $M_G(X)$ can be refined to the derived stack

$$Loc_G(X) = \mathrm{Map}_{\mathrm{dSt}}(X, BG)$$

parametrizing G-local systems on X.

Key point:

- Any non-degenerate $\kappa \in (\mathrm{Sym}^2\,\mathfrak{g}^\vee)^G$ corresponds to a 2-shifted symplectic structure ω_κ on the Artin stack BG
- ω_κ induces 0-shifted symplectic or Poisson structures on $Loc_G(X)$ in the unramified or tame case respectively.

Refinements (ii)

Wild case:

The moduli $M_G(X, \mathscr{I})$ can be refined to the derived stack

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}}\right)$$

parametrizing Stokes filtered G-local systems of irregular type $\mathscr{I}$.

Refinements (ii)

Wild case:

The moduli $M_G(X, \mathscr{I})$ can be refined to the derived stack

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}}\right)$$

parametrizing Stokes filtered G-local systems of irregular type $\mathscr{I}$.

Note:

- $\mathrm{DMS}_{G,\mathscr{I}}$ denotes the Deligne-Malgrange-Stokes sheaf of Artin stacks on X classifying Stokes data of type $\mathscr{I}$.
- $\widehat{\mathfrak{X}}$ denotes the real oriented blow up of $\mathfrak{X}$ in the x_i.

Refinements (ii)

Wild case:

The moduli $M_G(X, \mathscr{I})$ can be refined to the derived stack

$$Loc_G(X, \mathscr{I}) = \Gamma\left(\widehat{\mathfrak{X}}, \mathrm{DMS}_{G,\mathscr{I}}\right)$$

parametrizing Stokes filtered G-local systems of irregular type $\mathscr{I}$.

Key point: $\mathrm{DMS}_{G,\mathscr{I}}$ is equipped with a natural 0-shifted relative symplectic structure which induces a Poisson structure on $Loc_G(X, \mathscr{I})$.

Moduli of local systems (i)

X - finite CW complex;
G - an affine reductive group over $\mathbb{C}$.

Main object of study: The derived moduli stack $Loc_G(X)$ of

Moduli of local systems (i)

X - finite CW complex;
G - an affine reductive group over $\mathbb{C}$.

Main object of study: The derived moduli stack $Loc_G(X)$ of G-local systems on X

locally constant principal G-bundles on X

Moduli of local systems (i)

X - finite CW complex;
G - an affine reductive group over $\mathbb{C}$.

Main object of study: The derived moduli stack $Loc_G(X)$ of G-local systems on X

Moduli of local systems (ii)

Properties:

- $Loc_G(X)$ is a derived Artin stack over $\mathbb{C}$.
- $t_0 Loc_G(X)$ depends only on the fundamental group of X:

$$t_0 Loc_G(X) = \mathcal{M}_G(X) = [R_G(\pi_1(X,x))/G]$$

$R_G(\pi_1(X,x))$ is the **character scheme** of X: the affine $\mathbb{C}$-scheme representing the functor

$$R_G(\pi_1(X,x)) : \quad \mathrm{commalg}_{\mathbb{C}} \longrightarrow \mathrm{Sets}, \\ A \longrightarrow \mathrm{Hom}_{\mathrm{grp}}(\pi_1(X,x), G(A)).$$

Moduli of local systems (iii)

Properties:

- $\mathcal{M}_G(X) = t_0 Loc_G(X)$ has a course moduli space which is the affine GIT quotient $M_G(X) = R_G(X)/\!/G$, and

$$M_G(X)(\mathbb{C}) = \begin{pmatrix} \text{conjugacy classes of } \rho : \pi_1(X,x) \to G \\ \text{with } \overline{\text{im}(\rho)}\text{-reductive} \end{pmatrix}$$

- In general the derived structure on $Loc_G(X)$ depends on the full homotopy type of X.

Shifted symplectic structures

Recall: [P-Toën-Vaquié-Vezzosi] ([PTVV])

- If F is derived Artin locally f.p. over $\mathbb{C}$ we have a **complex of closed p-forms** $\mathcal{A}^{p,cl}(F)$ on F.
- When $F = \mathbf{Spec}\, A$, then $\mathcal{A}^{p,cl}(F)$ corresponds to the module $\mathrm{tot}^{\prod}(F^p(A)[p])$.
- An n-cocycle ω in the complex $\mathcal{A}^{2,cl}(F)$ is a **closed n-shifted 2-form**.
- ω is an **n-shifted symplectic structure** if the contraction $\omega^\flat : \mathbb{T}_F \longrightarrow \mathbb{L}_F[n]$ with the induced element in $H^n(F, \bigwedge^2 \mathbb{L})$ is a quasi-iso.

Structures on maps

Let $f : F \to F'$ be a morphism in $\mathbf{dSt}_{\mathbb{C}}$, then

- An $(n-1)$-shifted **isotropic structure** on f is a pair (ω, h), where ω is an n-shifted symplectic structure on F', and h is a homotopy between $f^*(\omega)$ and 0 inside the complex $\mathcal{A}^{2,cl}(F)$.
- An isotropic structure (ω, h) is **Lagrangian** if the induced morphism $h^\flat : \mathbb{T}_f \xrightarrow{\sim} \mathbb{L}_F[n-1]$ is a quasi-isomorphism.

Note: An $(n-1)$-shifted Lagrangian structure $(0, h)$ on $f : F \to \operatorname{Spec} \mathbb{C}$ is simply an $(n-1)$-shifted symplectic structure on F.

Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the **stacky** (positive degrees) and the **derived** (negative degrees) parts of $\mathbb{L}_X$.
- If $G/\mathbb{C}$ is reductive any non-degenerate $\kappa \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$ gives rise to a canonical 2-shifted symplectic form ω_κ on BG whose underlying 2-shifted 2-form is

$$\mathbb{C} \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = \mathrm{Sym}^2 \mathfrak{g}^\vee$$

  given by κ.
- The ***n*-shifted cotangent bundle** $T^\vee X[n] := \mathrm{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Shifted symplectic structures: examples (ii)

Theorem: [PTVV] Let (F, ω) be n-shifted symplectic and let X be a derived stack equipped with an $\mathcal{O}$-orientation of dimension d. If $Map_{\mathbf{dSt}}(X, F)$ is a locally f.p. derived Artin stack, then it carries a natural $(n-d)$-shifted symplectic structure.

Remark:

0) Analogue of Alexandrov-Kontsevich-Schwarz-Zaboronsky result in QFT.

Shifted symplectic structures: examples (ii)

Theorem: **[PTVV]** Let (F, ω) be n-shifted symplectic and let X be a derived stack equipped with an $\mathcal{O}$-orientation of dimension d. If $Map_{\mathbf{dSt}}(X, F)$ is a locally f.p. derived Artin stack, then it carries a natural $(n-d)$-shifted symplectic structure.

Remark:

1) An d-dimensional compact Calabi-Yau X has an $\mathcal{O}$-orientation of dimension d (Serre duality).
2) A compact oriented topological d-manifold has an $\mathcal{O}$-orientation of dimension d (Poincaré duality).

Shifted symplectic structures: examples (iii)

Theorem: **[PTVV]** Let (F, ω) be an n-shifted symplectic derived Artin stack, and $L_i \to F$, $i = 1, 2$ be maps of derived stacks equipped with Lagrangian structures. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.

Remark: Many standard constructions in symplectic geometry are special cases of these two theorems.

Structures on $Loc_G(X)$ (i)

$(X, \partial X)$ - compact oriented topological manifold of dim $= d$
G - a reductive algebraic group over $\mathbb{C}$.

Theorem:

(a) [PTVV] If $\partial X = \varnothing$, then the derived stack $Loc_G(X)$ has a $(2-d)$-shifted symplectic structure which is canonical up to a choice of a non-degenerate element in $(\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$

(b) [Calaque] The restriction map $Loc_G(X) \longrightarrow Loc_G(\partial X)$ carries a canonical $(2-d)$-shifted Lagrangian structure for the $3-d = 2-(d-1)$-shifted symplectic structure on the target.

Structures on $Loc_G(X)$ (ii)

Note: When X is a Riemann surface with boundary we recover the symplectic structures on moduli of G-local systems on X with prescribed monodromies at infinity.

Structures on $Loc_G(X)$ (ii)

Example: Suppose $(X, \partial X)$ is an oriented surface with boundary. Then

- ∂X is a disjoint union of oriented circles, and so $Loc_G(\partial X) \simeq \prod [G/G]$.
- The stack $Loc_G(S^1) = [G/G]$ carries a canonical 1-shifted symplectic structure.
- For any $\lambda \in G$ with centralizer G_λ, the inclusion of the conjugacy class $\mathbf{C}_\lambda \subset G$ of λ gives a canonical Lagrangian structure on the map $BG_\lambda \simeq [\mathbf{C}_\lambda/G] \hookrightarrow [G/G]$.

Structures on $Loc_G(X)$ (iii)

Assigning elements $\lambda_i \in G$ to each boundary component of X, we get two 0-shifted Lagrangian morphisms

$$\begin{array}{ccc} \prod BG_{\lambda_i} & & Loc_G(X). \\ & \searrow \quad \swarrow & \\ & \prod [G/G] & \end{array}$$

By **[PTVV]** the fiber product of these two maps has a canonical 0-shifted symplectic structure. This fiber product, is the derived stack

$$Loc_G(X, \{\lambda_i\})$$

of G-local systems on X whose local monodromies at infinity belong to the conjugacy classes $\{\mathbf{C}_{\lambda_i}\}$.

Shifted Poisson structures (i)

Recall: [Calaque-P-Toën-Vaquié-Vezzosi] ([CPTVV])

- For F a derived Artin stack$/\mathbb{C}$, can form the dg Lie algebra of ***n*-shifted polyvector fields** $\Gamma(F, \mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_F[-n-1]))[n+1]$.
- An ***n*-shifted Poisson structure** on F is a morphism in the ∞-category of graded dg-Lie algebras

$$p : \mathbb{C}[-1](2) \longrightarrow \Gamma(F, \mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_F[-n-1]))[n+1],$$

where $\mathbb{C}[-1](2)$ is the graded dg Lie algebra which is $\mathbb{C}$ placed in homological degree 1 and grading degree 2, equipped with the zero Lie bracket.

Shifted Poisson structures (ii)

Remark: [Melani-Safronov,Costello-Rozenblyum,Nuiten] Shifted Poisson structures can always be described in terms of shifted symplectic groupoids (Weinstein program).

Shifted Poisson structures (ii)

Theorem: [Costello-Rozenblyum] If F is a derived Artin stack the space of n-shifted Poisson structure on F is weakly equivalent to the space of equivalence classes of n-shifted Lagrangian maps $F \to F'$ to formal derived stacks F'.

Note: $[F \to F'] \sim [F \to F'']$ if there exists an n-shifted Lagrangian map $F \to G$ and a commutative diagram

$$\begin{array}{ccc} & \nearrow & F' \\ & & \uparrow a \\ F & \longrightarrow & G \\ & & \downarrow b \\ & \searrow & F'' \end{array}$$

with a and b formally étale and compatible with the Lagrangian structures.

Shifted Poisson structures (iii)

Example: For a compact oriented d-dimensional manifold X with boundary ∂X, the restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X)$$

is Lagrangian **[Calaque]** and so can be viewed as a $(2-d)$-shifted Poisson structure on $Loc_G(X)$.

Symplectic leaves (i)

Classically a Poisson structure on a smooth variety induces a foliation of the variety by symplectic leaves.

For an n-shifted Poisson structure on a derived stack F given by a Lagrangian map $f : F \to F'$, the symplectic leaves are the appropriately interpreted fibers of f.

Definition: A **generalized symplectic leaf** of F is a derived stack of the form $F \times_{F'} \Lambda$ for any n-shifted Lagrangian morphism $\Lambda \to F'$

Note: By **[PTVV]** a generalized symplectic leaf carries a canonical n-shifted symplectic structure.

Symplectic leaves (ii)

Example: X - a compact oriented surface with boundary. The restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X) = \prod [G/G]$$

carries a 0-shifted Lagrangian structure and thus corresponds to a 0-shifted Poisson structure on $Loc_G(X)$.

$Loc_G(X, \{\lambda_i\})$ - the derived moduli stack of G-local systems on X with fixed monodromies at infinity - is a generalized symplectic leaf in $Loc_G(X)$.

Betti spaces - theorems (i)

The **boundary of a topological space** Y is the pro-homotopy type $\partial Y := \lim\limits_{K \subset Y} (Y - K) \in \mathrm{Pro}(\mathrm{SSets})$.

Betti spaces - theorems (i)

The **boundary of a topological space** Y is the pro-homotopy type $\partial Y := \lim_{K \subset Y} (Y - K) \in \mathrm{Pro}(\mathrm{SSets})$.

taken in the ∞-category SSets of homotopy types and over the opposite category of compact subsets $K \subset Y$

Betti spaces - theorems (i)

The **boundary of a topological space** Y is the pro-homotopy type $\partial Y := \lim_{K \subset Y} (Y - K) \in \mathrm{Pro}(\mathrm{SSets})$.

Note: The pro-object ∂Y is in general not constant and can be extremely complicated. However if $X = Z(\mathbb{C})$ for a smooth n-dimensional complex algebraic variety Z, we have:

Proposition: The pro-object ∂X is equivalent to a constant pro-object in SSets which has the homotopy type of a compact oriented topological manifold of dimension $2n - 1$.

Remark: ∂X has the homotopy type of the boundary of the real oriented blowup of a good compactification of Z along its normal crossing boundary.

Betti spaces - theorems (ii)

Suppose $X = Z(\mathbb{C})$ for a smooth n-dimensional complex algebraic variety Z, then

Claim: The canonical map $\partial X \longrightarrow X$ induces a restriction morphism of derived locally f.p. Artin stacks

$$r : Loc_G(X) \longrightarrow Loc_G(\partial X).$$

which is equipped with a canonical $(2-2n)$-shifted Lagrangian structure with respect to the canonical shifted symplectic structure on $Loc_G(\partial X)$.

In particular r can be viewed as a $(2-2n)$-shifted Poisson structure on $Loc_G(X)$.

Symplectic leaves - smooth D (i)

Assume Z admits a smooth compactification $Z \subset \mathfrak{Z}$ with $D = \mathfrak{Z} - Z = \coprod_i D_i$ a smooth divisor. Then

- $\partial X =\sim$ (oriented circle bundle over D) classified by elements $\alpha_i \subset H^2(D_i, \mathbb{Z})$, $\alpha_i = c_1(N_{D_i/\mathfrak{Z}})$.
- Given $\lambda_i \in G$ with centralizer G_{λ_i}, the group S^1 acts on $BG_{\lambda i}$ (via λ_i) and naturally on $[G/G]$ so that the Lagrangian structure on the map $BG_{\lambda i} \to [G/G]$ is S^1-equivariant.
- Twisting by α_i gives a 1-shifted Lagrangian morphism

$$(\dagger_i) \qquad {}_{\alpha_i}\widetilde{BG}_{\lambda_i} \longrightarrow {}_{\alpha_i}\widetilde{[G/G]}$$

of locally constant families of derived Artin stacks over D_i.

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

$$Loc_G(\partial_i X) = \mathrm{Map}\left(\partial_i X, BG\right) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{[G/G]}\right);$$

$$Loc_{G_{\lambda i},\alpha_i}(D_i) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{BG_{\lambda i}}\right)$$

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

$$Loc_G(\partial_i X) = \mathrm{Map}\left(\partial_i X, BG\right) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{[G/G]}\right);$$

$$Loc_{G_{\lambda i},\alpha_i}(D_i) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{BG_{\lambda i}}\right)$$

G local systems on the component $\partial_i X$ of ∂X mapping tp D_i

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

$$Loc_G(\partial_i X) = \mathrm{Map}\left(\partial_i X, BG\right) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{[G/G]}\right);$$

$$Loc_{G_{\lambda i},\alpha_i}(D_i) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{BG_{\lambda i}}\right)$$

$G_{\lambda i}$ local systems on D_i twisted by α_i

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

$$Loc_G(\partial_i X) = \mathrm{Map}(\partial_i X, BG) = \Gamma\left(D_{i,\,\alpha_i}\widetilde{[G/G]}\right);$$

$$Loc_{G_{\lambda i},\alpha_i}(D_i) = \Gamma\left(D_{i,\,\alpha_i}\widetilde{BG_{\lambda i}}\right)$$

Since D_i is a compact topological manifold endowed with a canonical orientation the map $(\dagger_i)$ induces a $(3-2n)$-shifted Lagrangian morphism of derived Artin stacks

$$r_i : Loc_{G_{\lambda i},\alpha_i}(D_i) \longrightarrow Loc_G(\partial_i X).$$

Symplectic leaves - smooth D (iii)

Combining all r_i we get a $(3-2n)$-shifted Lagrangian morphism

$$r = \prod_i r_i : \prod_i Loc_{G_{\lambda i},\alpha_i}(D_i) \longrightarrow \prod_i Loc_G(\partial_i X) = Loc_G(\partial X).$$

By **[PTVV]** the fiber product of derived stacks

$$Loc_G(X, \{\lambda_i\}) := \left(\prod_i Loc_{G_{\lambda i},\alpha_i}(D_i)\right) \underset{Loc_G(\partial X)}{\times} Loc_G(X)$$

has a canonical $(2-2n)$-shifted symplectic structure.

Symplectic leaves - smooth D (iv)

By construction

- $Loc_G(X, \{\lambda_i\})$ is the derived stack of G-local systems on X whose local monodromy around D_i is fixed to be in the conjugacy class $\mathbf{C}_{\lambda_i}$ of λ_i.
- The natural map

$$Loc_G(X, \{\lambda_i\}) \longrightarrow Loc_G(X)$$

realizes $Loc_G(X, \{\lambda_i\})$ as a generalized symplectic leaf of the $(2-2n)$-shifted Poisson structure on $Loc_G(X)$.

Symplectic leaves - two components (i)

Assume $D = \mathfrak{Z} - Z = D_1 \cup D_2$ has two smooth irreducible components meeting transversally at a smooth D_{12}. Then

$$\partial X \simeq \partial_1 X \bigsqcup_{\partial_{12} X} \partial_2 X.$$

where $\partial_i X$ is an oriented circle bundle over $D_i^o = D_i - D_{12}$, and $\partial_{12} X$ is an oriented $S^1 \times S^1$-bundle over D_{12}.

Note: Each $\partial_i X$ has the homotopy type of an oriented compact manifold of dimension $2n - 1$ with boundary canonically equivalent to $\partial_{12} X$.

Symplectic leaves - two components (ii)

Theorem: [P-Töen]

(i) For a commuting pair of elements $(\lambda_1, \lambda_2) \in G \times G$ the map

$$Loc_G(\partial_1 X, \lambda_1) \underset{Loc_G(\partial_{12} X)}{\times} Loc_G(\partial_2 X, \lambda_2) \longrightarrow Loc_G(\partial X) \times Loc_G(\partial_{12} X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural Lagrangian structure.

(ii) If moreover the pair (λ_1, λ_2) is strict then the derived Artin stack

$$Loc_G(X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural $(2 - 2n)$-shifted symplectic structure which is a symplectic leaf of $Loc_G(X)$.

Derived moduli of local systems (i)

The derived stack of G local systems can be viewed as an ∞-functor

$$
\begin{aligned}
Loc_G(X): \quad \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0} &\longrightarrow \mathrm{SSets} \\
A &\longrightarrow \mathrm{Map}\,(S(X), BG(A))
\end{aligned}
$$

Derived moduli of local systems (i)

The derived stack of G local systems can be viewed as an ∞-functor

$$
\begin{aligned}
Loc_G(X) : \quad \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0} &\longrightarrow \mathrm{SSets} \\
A &\longmapsto \mathrm{Map}\,(S(X), BG(A))
\end{aligned}
$$

singular simplices in X

Derived moduli of local systems (i)

The derived stack of G local systems can be viewed as an ∞-functor

$$\begin{aligned} Loc_G(X) : \quad \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0} & \longrightarrow \mathrm{SSets} \\ A & \longrightarrow \mathrm{Map}\left(S(X), BG(A)\right) \end{aligned}$$

simplicial set of
A-points of BG

Derived moduli of local systems (i)

The derived stack of G local systems can be viewed as an ∞-functor

$$
\begin{aligned}
Loc_G(X): \quad \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0} &\longrightarrow \mathrm{SSets} \\
A &\longmapsto \mathrm{Map}\left(S(X), BG(A)\right)
\end{aligned}
$$

Note: $Loc_G(X)$ admits a nice quotient presentation.

Derived moduli of local systems (ii)

Choose $\mathbf{\Gamma_\bullet}$ - a free simplicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.

Derived moduli of local systems (ii)

Choose $\mathbf{\Gamma}_\bullet$ - a free simplicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.

Note: $B\mathbf{\Gamma}_\bullet$ is a simplicial free resolution of the pointed homotopy type (X, x).

Derived moduli of local systems (ii)

Choose $\mathbf{\Gamma}_\bullet$ - a free simplicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.

Then:

- $R_G(\mathbf{\Gamma}_\bullet)$ is a cosimplicial affine $\mathbb{C}$-scheme;
- $\Gamma(R_G(\mathbf{\Gamma}_\bullet), \mathcal{O})$ is a commuttative simplicial $\mathbb{C}$-algebra.

Passing to normalized chains gives a $\mathscr{A}_G(X) \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$ which up to quasi-isomorphism is independent of the choice of the resolution $\mathbf{\Gamma}_\bullet$.

Derived moduli of local systems (iii)

The conjugation action of G on $R(\Gamma_\bullet)$ gives an action of G on the cdga $\mathscr{A}_G(X)$ and hence on the derived affine scheme $\mathbf{Spec}\,\mathscr{A}_G(X)$. The quotient stack

$$\boxed{Loc_G(X) = [\mathbf{Spec}\,\mathscr{A}_G(X)/\,G]}$$

is the **derived stack of G-local systems on X**.

Back

p-forms

$A \in \mathrm{cdga}_{\mathbb{C}}$, $\quad X = \mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement. Then:

$\oplus_{p\geqslant 0} \bigwedge_A^p \mathbb{L}_A = \oplus_{p\geqslant 0}\Omega_{QA}^p$ - a fourth quadrant bicomplex with
vertical differential $d : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p,i+1}$ induced by d_{QA}, and
horizontal differential $d_{DR} : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p+1,i}$ given by the de Rham differential.

Hodge filtration: $F^q(A) := \oplus_{p>q}\Omega_{QA}^p$: still a fourth quadrant bicomplex.

(shifted) closed p-forms

Motivation: If X is a smooth scheme$/\mathbb{C}$, then $\Omega_X^{p,cl} \cong \left(\Omega_X^{\geqslant p}[p], d_{DR}\right)$. Use $(\Omega_X^{\geqslant p}[p], d_{DR})$ as a model for closed p forms in general.

Definition:

- **complex of closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\prod}(F^p(A))[p]$
- **complex of n-shifted closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A;n) := \mathrm{tot}^{\prod}(F^p(A))[n+p]$
- **Hodge tower:** $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

(shifted) closed p-forms (ii)

Explicitly an n-shifted closed p-form ω on $X = \mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i \geqslant 0}, \qquad \omega_i \in \Omega_A^{p+i, n-i}$$

satisfying

$$d_{DR}\omega_i = -d\omega_{i+1}.$$

Note: The collection $\{\omega_i\}_{i \geqslant 1}$ is the **key** closing ω.

p-forms and closed p-forms

Note:

- The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbf{Spec}\, A$ is exactly Illusie's derived de Rham complex of A.
- There is an underlying p-form map

$$\mathbf{A}^{p,cl}(A; n) \to \bigwedge^{p} \mathbb{L}_{A/k}[n]$$

inducing

$$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \bigwedge^{p} \mathbb{L}_{A/k}).$$

- The homotopy fiber of the underlying p-form map can be very complicated (complex of **keys**): being closed is **not** a property but rather a list of coherent data.

Functoriality and gluing:

Globally we have:

- the *n*-shifted *p*-forms ∞-functor $\mathcal{A}^p(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\Omega^p_{QA}[n]\,|$, and
- the *n*-shifted closed *p*-forms ∞-functor $\mathcal{A}^{p,cl}(-;n) : \mathrm{cdga}_{\mathbb{C}} \to \mathrm{SSets} : A \mapsto |\,\mathbf{A}^{p,cl}(A)[n]\,|$.

Note: $\mathcal{A}^p(-;n)$ and $\mathcal{A}^{p,cl}(-;n)$ are **derived stacks** for the étale topology. underlying *p*-form map (of derived stacks)

$$\mathcal{A}^{p,cl}(-;n) \to \mathcal{A}^p(-;n)$$

Notation: $|-|$ denotes $Map_{\mathbb{C}-dgMod}(\mathbb{C},-) = \mathbf{DK}\tau^{\leqslant 0}(-)$ i.e. Dold-Kan applied to the $\leqslant 0$-truncation.

global forms and closed forms (i)

Fix a derived Artin stack X (locally of finite presentation $/\mathbb{C}$)

Definition:

- $\mathcal{A}^p(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-))$ - **space of p-forms** on X;
- $\mathcal{A}^{p,cl}(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-))$ - **space of closed p-forms** on X;
- n-shifted versions : $\mathcal{A}^p(X; n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-; n))$ and $\mathcal{A}^{p,cl}(X; n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,cl}(-; n))$
- an n-shifted (respectively closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X; n)$ (respectively in $\pi_0\mathcal{A}^{p,cl}(X; n)$)

global forms and closed forms (ii)

Note:

1) If X is a smooth scheme there are no negatively shifted forms.
2) When $X = \mathbf{Spec}\, A$ then there are no positively shifted forms.
3) For general X shifted forms may exist for any $n \in \mathbb{Z}$.

global forms and closed forms (ii)

- **underlying p-form** map (of simplicial sets)

$$\mathcal{A}^{p,cl}(X;n) \to \mathcal{A}^{p}(X;n)$$

- not a monomorphism for general X, its homotopy fiber at a given p-form ω_0 is the space of **keys** of ω_0.
- If X is a smooth and proper scheme then this map is indeed a mono (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.

global forms and closed forms (ii)

Theorem (PTVV): for X derived Artin, then forms satisfy smooth descent:

$$\mathcal{A}^p(X;n) \simeq \mathrm{Map}_{\mathrm{L}_{\mathrm{qcoh}}(X)}(\mathcal{O}_X, (\bigwedge^p \mathbb{L}_X)[n]).$$

In particular: an n-shifted p-form on X is an element in $H^n(X, \bigwedge^p \mathbb{L}_X)$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= |NC(A)(p)[n+p]|$$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left|\prod_{i\geqslant 0}\left(\Omega_A^{p+1}[n-i], d + d_{DR}\right)\right|$$

$$= \left|\mathrm{tot}^{\Pi}(F^p(A))[n]\right|$$

$$= |NC(A)(p)[n+p]|$$

negative cyclic complex of weight p

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= \left| NC(A)(p)[n+p] \right|$$

Hence

$$\boxed{\pi_0 \mathcal{A}^{p,cl}(X;n) = HC_-^{n-p}(A)(p).}$$

Back

Derived critical loci

If $f \in H^0(Y, \mathcal{O})$ is a function on a smooth variety Y, then its derived critical locus $\mathbb{R}Crit(f)$ is defined as the fiber product

$$\begin{array}{ccc} \mathbb{R}Crit(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle df} \\ Y & \xrightarrow[0]{} & T^{\vee}Y \end{array}$$

and is thus canonically (-1)-shifted symplectic.

Variant: If $f \in H^n(Y, \mathcal{O}) = \mathbb{H}^0(Y, \mathcal{O}[n])$ is an n-shifted function on a smooth variety Y, then its derived critical locus $\mathbb{R}Crit(f)$ is canonically $(n-1)$-shifted symplectic

Hamiltonian reduction (i)

Note: If G is a linear algebraic group$/\mathbb{C}$, then $[\mathfrak{g}^\vee/G] = T^\vee_{BG}[1]$ and hence is 1-shifted symplectic.

Suppose

- (M, ω) - complex algebraic symplectic manifold;
- $G \times M \to M$ - a Hamiltonian action of G;
- $\mu : M \to \mathfrak{g}^\vee$ - a G-equivariant moment map.

Then ω corresponds to a 0-shifted Lagrangian structure on the map $\mu : [M/G] \to [\mathfrak{g}^\vee/G]$.

Hamiltonian reduction (ii)

Similarly, given a coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^\vee$, the **Kirillov-Kostant-Souriau** symplectic structure $\omega_{\mathbb{O}}$ on $\mathbb{O}$ corresponds to a 0-shifted Lagrangian structure on the inclusion $[\mathbb{O}/G] \hookrightarrow [\mathfrak{g}^\vee/G]$.

The **derived Hamiltonian reduction** $[R\mu^{-1}(\mathbb{O})/G]$ is defined to be the Lagrangian intersection

$$\begin{array}{ccc} [R\mu^{-1}(\mathbb{O})/G] & \longrightarrow & [M/G] \\ \downarrow & & \downarrow \\ [\mathbb{O}/G] & \longrightarrow & [\mathfrak{g}^\vee/G] \end{array}$$

and is therefore canonically 0-shifted symplectic.

Quasi-Hamiltonian reduction (i)

Note: If G is a reductive linear algebraic group$/\mathbb{C}$, then $[G/G] = Map_{\mathbf{dSt}}(S^1, BG)$ and hence is 1-shifted symplectic.

Suppose

- (M, ω) - complex algebraic symplectic manifold;
- $G \times M \to M$ - a quasi-Hamiltonian action of G;
- $\mu : M \to G$ - a G-equivariant group valued moment map.

Then ω corresponds to a 0-shifted Lagrangian structure on the map $\mu : [M/G] \to [G/G]$.

Quasi-Hamiltonian reduction (ii)

Given a conjugacy class $\mathbf{C} \subset G$, the inclusion $[\mathbf{C}/G] \hookrightarrow [G/G]$ carries a canonical 0-shifted Lagrangian structure.

The **derived quasi-Hamiltonian reduction** $[R\mu^{-1}(\mathbf{C})/G]$ is defined to be the Lagrangian intersection

$$\begin{array}{ccc} [R\mu^{-1}(\mathbf{C})/G] & \longrightarrow & [M/G] \\ \downarrow & & \downarrow \\ [\mathbf{C}/G] & \longrightarrow & [G/G] \end{array}$$

and is therefore canonically 0-shifted symplectic.

Back

Orientations and structures (i)

Key observation: Lagrangian structures on a map between moduli of local systems exist always in the presence of relative orientations.

Orientations and structures (i)

$f : Y \to X$ - a continuous map between finite CW complexes; $C^{\bullet}(Y, X)$ - the cone of the pull-back map $f^{*}C^{\bullet}(X) \to C^{\bullet}(Y)$ on singular cochains with coefficients in $\mathbb{C}$.

An **orientation of dimension d on f** is a morphism of complexes or : $C^{\bullet}(Y, X) \longrightarrow \mathbb{C}[1-d]$, which is non-degenerate in the sense that the pairing

$$C^{\bullet}(X) \otimes C^{\bullet}(X, Y) \longrightarrow \mathbb{C}[1-d]$$

given by the composition of or with the cup product on $C(X)$ is non-degenerate on cohomology and induces a quasi-isomorphism $C^{\bullet}(Y, X) \simeq C^{\bullet}(X)^{\vee}[1-d]$.

Orientations and structures (ii)

$f : Y \to X$ - continuous map of CW complexes equipped with a relative orientation of dimension d.
G - a reductive algebraic group over $\mathbb{C}$.

Theorem: [Calaque,Brav-Dyckerhoff] The pullback map on the derived stacks of local systems

$$f^* : Loc_G(X) \longrightarrow Loc_G(Y)$$

carries a $(2-d)$-shifted Lagrangian structure which is canonical up to a choice of a non-degenerate element in $\mathrm{Sym}^2(\mathfrak{g}^\vee)^G$.

Back

Poisson bivectors

For a G-local system $\rho \in Loc_G(X)$ we have

- $\mathbb{T}_{Loc_G(X),\rho} = H^\bullet(X, ad(\rho))[1]$
- the bivector p underlying the $(2-d)$-shifted Poisson structure on $Loc_G(X)$ is given by

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\;p\;} & (H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, ad(\rho))[1])[d-2] \\ & \searrow^{\text{PD}} & \uparrow \\ & & H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, \partial X; ad(\rho))[d-2] \end{array}$$

Back

Obstructions - smooth D (i)

Caution: $Loc_{G_{\lambda i},\alpha_i}(D_i)$ may be empty. Indeed:

- $Loc_{G_{\lambda i},\alpha_i}(D_i)(\mathbb{C})$ is the groupoid of G-local systems on $\partial_i X$ whose local monodromy around D_i is conjugate to λ_i.
- A $G_{\lambda i}/\mathrm{Z}(G_{\lambda i})$-local system on D_i determines a class in $H^2(D_i, \mathrm{Z}(G_{\lambda i}))$, which is the obstruction to lifting it to a $G_{\lambda i}$-local system.
- For $Loc_{G_{\lambda i},\alpha_i}(D_i)(\mathbb{C})$ to be non-empty one needs to have a $G_{\lambda i}/\mathrm{Z}(G_{\lambda i})$-local system on D_i whose obstruction class matches with the image of α_i under the map $H^2(D_i, \mathbb{Z}) \to H^2(D_i, \mathrm{Z}(G_{\lambda i}))$ given by $\lambda_i : \mathbb{Z} \to \mathrm{Z}(G_{\lambda i})$.

Obstructions - smooth D (ii)

Example: If $G/\mathbb{C}$ is semisimple, and λ_i is a regular semi-simple element, then Z_i is a maximal torus in G and hence the image of α_i in $H^2(D_i, Z_i)$ is zero. If λ_i is of infinte order, this forces α_i to be a torsion class in $H^2(D_i, \mathbb{Z})$.

Back

Obstructions - two components (i)

Definition: A pair of commuting elements $(\lambda_1, \lambda_2) \in G \times G$ is called **strict** if the morphism

$$BG_{\{\lambda_1,\lambda_2\}} \longrightarrow [G_{\lambda 1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$$

is Lagrangian (for its canonical isotropic structure).

Here $G * G \subset G \times G$ is the commuting variety, and $G_{\{\lambda_1,\lambda_2\}}$ is the centralizer of the pair (λ_1, λ_2).

Note: Strictness is a group theoretic property.

Obstructions - two components (ii)

Proposition: Let (λ_1, λ_2) be a commuting pair of elements in G, and $u := \mathrm{Id} - \mathrm{ad}(\lambda_1)$ and $v := \mathrm{Id} - \mathrm{ad}(\lambda_2)$ be the corresponding endormorphisms of $\mathfrak{g}$. Then the pair (λ_1, λ_2) is strict if and only u is strict with respect to the kernel of v, i.e. if and only if

$$\mathrm{Im}(v_{|\ker(u)}) = \mathrm{Im}(v) \cap \ker(u).$$

Note: Strictness is symmetric by definition so equivalently (λ_1, λ_2) is strict if and only if v is strict with respect to the kernel of u.

Obstructions - two components (iii)

Corollary:

- If at least one of the λ_i is semi-simple then the pair (λ_1, λ_2) is strict.
- If (u, v) form a principal nilpotent pair [Ginzburg], then the pair (λ_1, λ_2) is strict.

Caution: Strictness is a non-trivial condition: if λ is any non-trivial unipotent element in G, then the pair (λ, λ) is not strict. In this case u is a non-zero nilpotent endomorphism of $\mathfrak{g}$ and thus $\ker(u) \cap \mathrm{Im}(u) \neq 0$, but $\mathrm{Im}(u_{|\ker(u)}) = 0)$.

Back