Painlevé τ-function and Topological Recursion

Kohei Iwaki (Nagoya University)

Moduli Spaces, Representation Theory and Quantization

June 27 2019 @RIMS Kyoto

Based on arXiv:1902.06439

Background / Motivation

- **Topological recursion** ([Eynard-Orantin 07], [Chekhov-Eynard-Orantin 06]):

  spectral curve —| topological recursion (TR) |$\rightsquigarrow W_{g,n}(z_1,\ldots,z_n),\ F_g$

  Outputs are analogue of correlators and free energy of matrix models.

- It is related to **integrability**.

  Example: $\underset{(q_0 = \sqrt{-t/6})}{y^2 = 4(x-q_0)^2(x+2q_0)} \quad \overset{\text{TR}}{\rightsquigarrow} \quad Z(t;\hbar) := \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g(t)\right)$

Theorem ([Brézin-Kazakov 90], ..., [Eynard-Orantin 07])

TR partion function $Z(t;\hbar)$ = ***τ*-function** for the **Painlevé I equation**

$$(P_{\mathrm{I}}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t.$$

Namely, the following formal power series satisfies (P_{I}):

$$q(t;\hbar) := -\hbar^2 \frac{d^2}{dt^2} \log Z(t;\hbar) = q_0(t) + \hbar^2 q_2(t) + \hbar^4 q_4(t) + \cdots .$$

(c.f., [I-Marchal-Saenz 18] for all six Painlevé equations.)

Background / Motivation (Cont.)

- The previous solution is "perturbative" one ("0-parameter solution").
- General solution ("2-parameter solution") is known in several expressions:
  - [Takano 89], [Aoki et.al. 96], [Anicet et.al. 12],...
  - [Gamayun-Iorgov-Lisovyy 12] gave the Painlevé VI τ-function (with $\hbar = 1$):

$$\tau_{P_{\rm VI}}(t;\nu,\rho) = \sum_{k \in \mathbb{Z}} e^{2\pi i k \rho}\, C(\nu + k)\, t^{(\nu+k)^2 - \theta_0^2 - \theta_t^2}\, B(t;\nu + k)$$

    where $B(t, \nu)$ = 4-point Virasoro conformal block with $c = 1$,
    = Nekrasov partition function with $\varepsilon_1 + \varepsilon_2 = 0$.
  - [Bonelli-Lisovyy-Maryoshi-Sciarappa-Tanzini 15] proposed a generalization of GIL formula for irregular Painlevé equations via Argyres-Douglas theory.
- **Question**: **Can we construct such a 2-parameter solution from TR ?**

  ([Eyanrd-Mariño 08], [Borot-Eynard 12]: "non-perturbative partition function")

Main Results

Main Theorem ([I 19])

Let $W_{g,n}$ and F_g be the TR correlators / free energy of the spectral curve

$$y^2 = 4x^3 + 2tx + u(t,\nu) \qquad \Big(\nu := \oint_A y\,dx \;:\; t\text{-independent}\Big).$$

(i) The discrete Fourier transform of the TR partition function

$$\tau(t,\nu,\rho;\hbar) := \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\nu+k\hbar;\hbar)$$

gives a 2-parameter family of formal τ-function for (P_{I}).

(ii) Another Fourier series

$$\Psi_{\pm}(x,t,\nu,\rho;\hbar) := \frac{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\nu+k\hbar;\hbar)\,\psi_{\pm}(x,t,\nu+k\hbar;\hbar)}{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\hbar;\nu+k\hbar)}$$

$$\left(\psi_{\pm}(x,t,\nu;\hbar) = \exp\left(\sum_{g,n} \frac{(\pm\hbar)^{2g-2+n}}{n!} \int^{z(x)} \cdots \int^{z(x)} W_{g,n}(z_1,\ldots,z_n)\right)\right)$$

gives a solution of the **isomonodromy system** associated with (P_{I}).

Isomondomy System associated with Painlevé I

- **Fact** (c.f., [Okamoto 80, Jimbo-Miwa-Ueno 81, Jimbo-Miwa 81]):

  $(P_I) \Leftrightarrow$ compatibility condition ($[L, M] = 0$) of the system of linear PDEs

$$(L_{\mathrm{I}}) \quad : \quad L\Psi := \left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar}{x-q}\left(\hbar\frac{\partial}{\partial x} - p\right) - (4x^3 + 2tx + 2H)\right]\Psi = 0$$

$$(D_{\mathrm{I}}) \ : \ M\Psi := \left[\hbar\frac{\partial}{\partial t} - \frac{1}{2(x-q)}\left(\hbar\frac{\partial}{\partial x} - p\right)\right]\Psi = 0$$

  where $\quad p := \hbar\dfrac{dq}{dt} \quad$ and $\quad H := \dfrac{p^2}{2} - 2q^3 - tq$

  (Remark: The previous spectral curve is a “naive” classical limit of (L_{I}).)

- **Stokes multipliers** for $L\Psi = 0$ around $x = \infty$ is **independent of** t.
  $\rightsquigarrow$ Stokes data are first integrals of (P_{I}). (**Integrability** of Painlevé equations).

- Assuming a “**Borel summability conjecture**” etc., we will give a conjectural answer to the **direct Monodromy problem** (i.e., computation of Stokes data) via the **exact WKB method** (or **spectral networks**).

Topological Recursion

Spectral Curve

Definition

A **spectral curve** is a triplet (Σ, x, y), where

- Σ : compact Riemann surface with a prescribed A and B cycles in $H_1(\Sigma; \mathbb{Z})$.
- x, y : meromorphic functions on Σ.

such that dx and dy never vanish simultaneously.

- Example 1 (Airy curve) :

$$\Sigma = \mathbb{P}^1, \quad x(z) = z^2, \quad y(z) = z. \qquad (y^2 - x = 0)$$

- Example 2 (Elliptic curve) :

$$\Sigma = \mathbb{C}/\Lambda, \quad x(z) = \wp(z), \quad y(z) = \wp'(z). \qquad (y^2 = 4x^3 - g_2 x - g_3)$$

where $\Lambda = \mathbb{Z}\omega_A + \mathbb{Z}\omega_B$ and

$$\wp(z) = \wp(z; \omega_A, \omega_B) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Eynard-Orantin Correlators

Definition [Eynard-Orantin 07] ([Chekhov-Eynard-Orantin 06])

To a given spectral curve (Σ, x, y), define

$\{W_{g,n}(z_1, \ldots, z_n)\}_{g\geq 0, n\geq 1}$: a sequence of meromorphic multi-differentials on Σ

by the following recursion relation (called **topological recursion**):

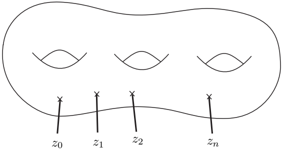
$$W_{0,1}(z) := y(z)dx(z), \quad W_{0,2}(z_1, z_2) := \text{Bergman bi-differential}$$

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) := \sum_{a\,:\,\text{ramification point}} \operatorname*{Res}_{z=a} K_a(z_0, z) \Bigg(W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g,\ I_1 \sqcup I_2 = \{1,\ldots,n\}, \\ \text{except for } (g_i=0\ \&\ I_i=\emptyset)}} W_{g_1,1+|I_1|}(z, z_{I_1})\, W_{g_2,1+|I_2|}(\bar{z}, z_{I_2}) \Bigg).$$

- $W_{0,2}(z_1, z_2) = \dfrac{dz_1 dz_2}{(z_1 - z_2)^2}$ (for $\Sigma = \mathbb{P}^1$), and $\left(\wp(z_1 - z_2) + \dfrac{\eta_A}{\omega_A}\right) dz_1 dz_2$ (for elliptic curve).
- Ramification points are zeros of dx (we assume that they are simple).
- $\bar{z}$ is the local conjugation of z near a ramification point.
- $K_a(z_0, z) := \dfrac{1}{2(y(z) - y(\bar{z}))\, dx(z)} \displaystyle\int_{w=\bar{z}}^{w=z} W_{0,2}(z_0, w)$ is the "recursion kernel".

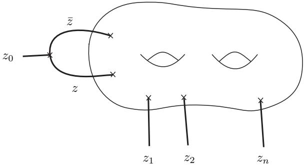
Diagrammatic Expression of Topological Recursion

"$W_{g,n}(z_1, \ldots, z_n)$ $\longleftrightarrow$ genus g Riemann surface with n marked points"

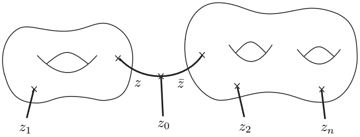


$$W_{g,n+1}(z_0, z_1, \ldots, z_n)$$

"Degeneration" of
Riemann surfaces



$$K_a(z_0, z) W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n)$$



$$K_a(z_0, z) W_{g_1,1+|I_1|}(z, z_{I_1}) W_{g_2,1+|I_2|}(\bar{z}, z_{I_2})$$

Free Energy and Partition Function

Definition [Eynard-Orantin 07] ([Chekhov-Eynard-Orantin 06])

- For $g \geq 2$, define **g-th free energy** F_g of the spectral curve by

$$F_g := \frac{1}{2-2g} \sum_{a\ :\ \text{ramification points}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \qquad \left(\Phi(z) := \int^z y(z) dx(z) \right)$$

  (F_0 and F_1 are also defined but in a different manner.)

- **Free energy** F and **partition function** Z of the spectral curve are defined by

$$F := \sum_{g=0}^{\infty} \hbar^{2g-2} F_g, \qquad Z := \exp(F) = \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g \right)$$

Properties [Eynard-Orantin 07]

- $W_{g,n}(z_1,\ldots,z_n)$: holomorphic (as a differential of each z_i) on $\Sigma \setminus R$.
- $W_{g,n}(\cdots,z_i,\cdots,z_j,\cdots) = W_{g,n}(\cdots,z_j,\cdots,z_i,\cdots)$.
- $W_{g,n}$ is normalized along A-cycles $A_1,\ldots,A_{g(\Sigma)}$:

$$\oint_{z_1\in A_j} W_{g,n}(z_1,\ldots,z_n) = 0 \quad \text{except for } (g,n)=(0,1)$$

- $W_{g,n}$ satisfies **differentiation formulas** (with respect to moduli parameters). For example, the differentiation with respect to

$$\nu_j := \frac{1}{2\pi i}\oint_{A_j} W_{0,1}(z) \quad (j=1,\ldots,g(\Sigma))$$

is given by

$$\frac{\partial}{\partial \nu_j} W_{g,n}(z(x_1),\ldots,z(x_n)) = \oint_{x_{n+1}\in B_j} W_{g,n+1}(z(x_1),\ldots,z(x_n),z(x_{n+1}))$$

$$\frac{\partial}{\partial \nu_j} F_g = \oint_{z\in B_j} W_{g,1}(z)$$

TR and Various Geometric Invariants

- Airy curve $(\mathbb{P}^1, x(z) = z^2, y(z) = z)$
  $\rightsquigarrow$ Gromov-Witten invariants for point:

$$W^{\mathrm{Airy}}_{g,n}(z_1, \ldots, z_n) = \frac{1}{2^{2g-2+n}} \sum_{d_1,\ldots,d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{z_i^{2d_i}} dz_i$$

- Landau-Ginzburg mirror of $\mathbb{P}^1$ $(\mathbb{C}^*, x(z) = z + z^{-1}, y(z) = \log z)$
  $\rightsquigarrow$ Gromov-Witten invariants for $\mathbb{P}^1$.
  [Norbury-Scott 14], [Dunin-Barkowski et.al 13], [Fang et.al 16].
- Bouchard-Klemm-Mariño-Pasquetti conjecture on open Gromov-Witten invariants for toric CY3. [Bouchard et.al 08], [Eynard-Orantin 13]
- KdV τ-function [Kontsevich 92], [Eynard-Orantin 07],
- Painlevé τ-functions (corresponding to “perturbative solution”)
  [Borot-Eynard 09, I-Saenz 15, I-Marchal-Saenz 17].
- $\cdots$

Topological Recursion and WKB : Quantum Curve

$$(\mathbb{P}^1, x(z) = z^2, y(z) = z) \quad : \quad \text{Airy curve } (y^2 = x)$$

$$W_{0,1}^{\text{Airy}}(z_1) = y(z)dx(z) = 2z_1^2 dz_1, \quad W_{0,2}^{\text{Airy}}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

$$W_{0,3}^{\text{Airy}}(z_1, z_2, z_3) = -\frac{dz_1 dz_2 dz_3}{2z_1^2 z_2^2 z_3^2}, \quad W_{1,1}^{\text{Airy}}(z_1) = -\frac{dz_1}{16 z_1^4}, \quad \ldots$$

Theorem [Gukov-Sułkowski 12, Zhou 12, ...]

The formal series

$$\psi(x;\hbar) := \exp\left(\sum_{g \geq 0, n \geq 1} \frac{\hbar^{2g-2+n}}{n!} \int_{\infty}^{z(x)} \cdots \int_{\infty}^{z(x)} W_{g,n}^{\text{Airy}}(z_1, \ldots, z_n) \right)$$

is a WKB formal solution of the Airy equation

$$\left(\hbar^2 \frac{d^2}{dx^2} - x \right) \psi(x;\hbar) = 0$$

(Precisely speaking, we need to regularize the term corresponding to $(g, n) = (0, 2)$ since $W_{0,2}(z_1, z_2)$ has singularity along $z_1 = z_2$.)

Main Result :

2-parameter τ-function

of Painlevé I

A Family of Genus 1 Spectral Curves

Consider a family of elliptic curves

$$y^2 = 4x^3 + 2tx + u(t, \nu)$$

(with a prescribed A-cycle and B-cycle such that $\mathrm{Im}(\omega_B/\omega_A) > 0$) satisfying

$$\nu := \frac{1}{2\pi i} \oint_A y dx \quad \text{is \textbf{independent of} } t.$$

- The condition requires

$$\frac{\partial u}{\partial t} = 2\frac{\eta_A}{\omega_A} \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{4\pi i}{\omega_A}$$

- Regard this as a spectral curve

$$\Sigma = \mathbb{C}/\Lambda, \quad x(z) = \wp(z), \quad y(z) = \wp'(z)$$

  by the Weierstrass $\wp$-function. $\rightsquigarrow W_{g,n}(z_1, \ldots, z_n)$ and F_g by TR.

Lemma (cf. [Eynard-Orantin 07])

$$\frac{\partial F_0}{\partial t} = \frac{1}{2}u, \quad \frac{\partial F_0}{\partial \nu} = \oint_B y\, dx, \quad \frac{\partial^2 F_0}{\partial \nu^2} = 2\pi i\, \frac{\omega_B}{\omega_A}$$

(i.e., F_0 = **Seiberg-Witten prepotential**.)

Key Facts (Quantum Curve and Formal Monodromy)

Key Lemma 1

The WKB-type formal series

$$\psi_{\pm}(x,t,\nu;\hbar) := \exp\left(\sum_{g\geq 0, n\geq 1} \frac{(\pm\hbar)^{2g-2+n}}{n!} \int_0^{z(x)} \cdots \int_0^{z(x)} W_{g,n}(z_1,\ldots,z_n) \right)$$

satisfies

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \left(4x^3 + 2tx + 2\hbar^2 \frac{\partial}{\partial t} F(t,\nu;\hbar) \right) \right] \psi_{\pm}(x,t,\nu;\hbar) = 0$$

(Remark: The above PDE is a quantization of $y^2 = 4x^3 + 2tx + u(t,\nu)$.)

Key Lemma 2

Formal monodoromy (term-wise analytic continuation) along A and B-cycle:

$$\psi_{\pm}(x,t,\nu;\hbar) \mapsto \begin{cases} e^{\pm 2\pi i \nu/\hbar}\, \psi_{\pm}(x,t,\nu;\hbar) & \text{along } A\text{-cycle} \\ \dfrac{Z(t,\nu\pm\hbar;\hbar)}{Z(t,\nu;\hbar)}\, \psi_{\pm}(x,t,\nu\pm\hbar;\hbar) & \text{along } B\text{-cycle} \end{cases}$$

Here $Z(t,\nu;\hbar) = \exp(F(t,\nu;\hbar)) = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g(t,\nu)\right)$ is the TR partition function.

Proof of Key Lemma 2

Using the **differentiation formulas** for $W_{g,n}$ and F_g, we have

Term-wise analytic continuation of $\psi_\pm(x,t,\nu;\hbar)$ along the B-cycle

$$= \exp\left(\sum_{g\ge 0,\, n\ge 1} \frac{(\pm\hbar)^{2g-2+n}}{n!} \int_0^{z(x)+\omega_B} \cdots \int_0^{z(x)+\omega_B} W_{g,n}(z_1',\dots,z_n') \right)$$

$$= \exp\left(\sum_{g\ge 0,\, n\ge 1} \frac{(\pm\hbar)^{2g-2+n}}{n!} \sum_{\ell=0}^{n} \binom{n}{\ell} \underbrace{\oint_B \cdots \oint_B}_{\ell} \underbrace{\int_0^{z(x)} \cdots \int_0^{z(x)}}_{n-\ell} W_{g,n}(z_1',\dots,z_n') \right)$$

$$= \exp\left(\sum_{\substack{g,\ell_1,\ell_2\ge 0 \\ \ell_1+\ell_2\ge 1}} \frac{(\pm\hbar)^{2g-2+\ell_1+\ell_2}}{\ell_1!\cdot\ell_2!} \frac{\partial^{\ell_1}}{\partial\nu^{\ell_1}} \underbrace{\int_0^{z(x)} \cdots \int_0^{z(x)}}_{\ell_2} W_{g,\ell_2}(z_1',\dots,z_{\ell_2}') \right)$$

$$= \exp\left(\sum_{\ell_1\ge 1} \frac{(\pm\hbar)^{\ell_1}}{\ell_1!} \frac{\partial^{\ell_1}}{\partial\nu^{\ell_1}} \sum_{g\ge 0} \hbar^{2g-2} F_g(t,\nu) \right)$$

$$\times \exp\left(\sum_{\ell_1\ge 0} \frac{(\pm\hbar)^{\ell_1}}{\ell_1!} \frac{\partial^{\ell_1}}{\partial\nu^{\ell_1}} \sum_{g\ge 0,\, \ell_2\ge 1} \frac{(\pm\hbar)^{2g-2+\ell_2}}{\ell_2!} \int_0^{z(x)} \cdots \int_0^{z(x)} W_{g,\ell_2}(z_1',\dots,z_{\ell_2}') \right)$$

$$= \frac{Z(t,\nu\pm\hbar;\hbar)}{Z(t,\nu;\hbar)} \psi_\pm(x,t,\nu\pm\hbar;\hbar).$$

Main Theorem

The formal monodromy relations for $\psi_\pm$ imply that the Fourier series

$$\tilde{\Psi}_\pm(x,t,\nu,\rho;\hbar) := \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar}\, Z(t,\nu+k\hbar;\hbar)\, \psi_\pm(x,t,\nu+k\hbar;\hbar)$$

has t**-independent (and diagonal) formal monodromy**:

$$\tilde{\Psi}_\pm(x,t,\nu,\rho;\hbar) \mapsto \begin{cases} e^{\pm 2\pi i\nu/\hbar}\, \tilde{\Psi}_\pm(x,t,\nu,\rho;\hbar) & \text{along } A\text{-cycle} \\ e^{\mp 2\pi i\rho/\hbar}\, \tilde{\Psi}_\pm(x,t,\nu,\rho;\hbar) & \text{along } B\text{-cycle} \end{cases}$$

Theorem ([I 19])

The formal series

$$\Psi_\pm(x,t,\nu,\rho;\hbar) := \frac{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar}\, Z(t,\nu+k\hbar;\hbar)\, \psi_\pm(x,t,\nu+k\hbar;\hbar)}{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar}\, Z(t,\nu+k\hbar;\hbar)}$$

is a formal solution of the isomonodoromy system (L_{I}) and (D_{I}) associated with (P_{I}). Here H, q, p in the isomonodromy system are given by

$$H = \hbar^2 \frac{d}{dt} \log\left(\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t,\nu+k\hbar;\hbar) \right), \quad q = -\hbar \frac{dH}{dt}, \quad p = \hbar \frac{dq}{dt}.$$

Main Theorem (cont)

Theorem ([I 19])

The formal series

$$\tau_{P_{\mathrm{I}}}(t,\nu,\rho;\hbar) := \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar}\, Z(t,\nu+k\hbar;\hbar)$$

is a 2-parameter formal **τ-function** for (P_{I}).

Remark : This is the **non-perturbative partition function** of [Eynard-Mariño 08] and [Borot-Eynard 12]. They observed that the above Fourier series can be expressed as a formal power series of $\hbar$ whose coefficients are described by θ-functions (and their derivatives) :

$$\tau_{P_{\mathrm{I}}} = Z(\nu)\cdot\left[\theta(z,\tau) + \hbar\left(\frac{1}{6}\frac{\partial^3 F_0}{\partial\nu^2}\theta'''(z,\tau) + \frac{\partial F_1}{\partial\nu}\theta'(z,\tau)\right) + \cdots\right]_{z=\frac{\phi(t)+\rho}{\hbar},\ \tau=\frac{\omega_B}{\omega_A}}$$

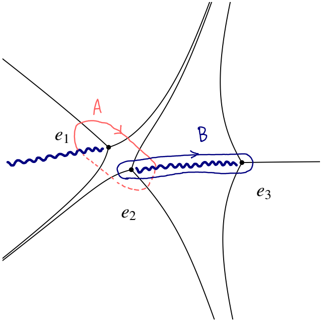
where $\quad \theta(z,\tau) = \sum_{k\in\mathbb{Z}} e^{2\pi i k z + \pi i k^2\tau} \quad$ and $\quad \phi(t) = \frac{1}{2\pi i}\oint_B y\,dx = \frac{1}{2\pi i}\frac{\partial F_0}{\partial\nu}.$

$\rightsquigarrow$ This recovers the **Boutroux's elliptic asymptotic** ([Boutroux 1913]).

Direct Monodromy Problem (Exact WKB Approach)

Stokes Graph and Borel Summability Conjecture

Define the **Stokes graph** by $\mathrm{Im} \int_{e_i}^{x} \sqrt{4x^3 + 2tx + u(t,\nu)}\, dx = 0 \quad (i = 1, 2, 3)$.



(Remark : Stokes graph = **spectral network** defined by $\left(4x^3 + 2tx + u(t,\nu)\right) dx^{\otimes 2}$)

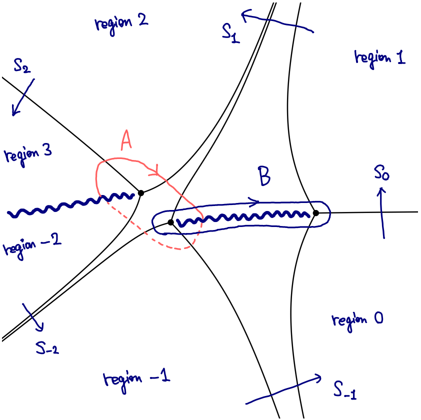
Conjecture

The WKB solution $\psi_\pm(x,t,\nu;\hbar)$ of the PDE

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \left(4x^3 + 2tx + 2\hbar^2 \frac{\partial}{\partial t} F(t,\nu;\hbar)\right)\right] \psi_\pm(x,t,\nu;\hbar) = 0$$

constructed in Key Lemma 1 is **Borel summable** as $\hbar$-formal power series on each complement of Stokes graph (if there is no saddle connection).

Stokes Multipliers of $(L_{\rm I})$ Around $x = \infty$



$$(\Psi_+^{(\ell)}, \Psi_-^{(\ell)}) = \begin{cases} (\Psi_+^{(\ell+1)}, \Psi_-^{(\ell+1)}) \cdot \begin{pmatrix} 1 & 0 \\ s_\ell & 1 \end{pmatrix} & \text{for } \ell = 0, \pm 2, \\ (\Psi_+^{(\ell+1)}, \Psi_-^{(\ell+1)}) \cdot \begin{pmatrix} 1 & s_\ell \\ 0 & 1 \end{pmatrix} & \text{for } \ell = \pm 1, \end{cases}$$

s_ℓ = **Stokes multiplier** of $(L_{\rm I})$

Borel sum on the region ℓ ↓

$$\Psi_\pm^{(\ell)} = \frac{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t, \nu + k\hbar; \hbar) \cdot \psi_\pm^{(\ell)}(x, t, \nu + k\hbar; \hbar)}{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t, \nu + k\hbar; \hbar)}$$

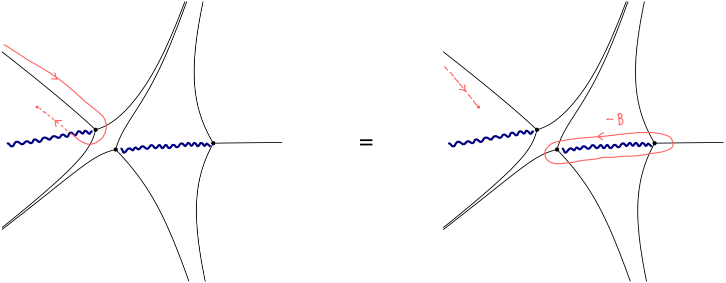
(We also assume the convergence of the Fourier series.)

Computation of s_2

- **Voros connection formula** (**path-lifting rule**):
  Single-valuedness of Borel sum around branch points

$$\Rightarrow \quad \psi_+^{(2)}(x,t,\nu;\hbar) = \psi_+^{(3)}(x,t,\nu;\hbar) + \tilde{\psi}_-^{(3)}(x,t,\nu;\hbar)$$

where $\tilde{\psi}_-$ is the formal analytic continuation of ψ_+ along a “detoured” path.



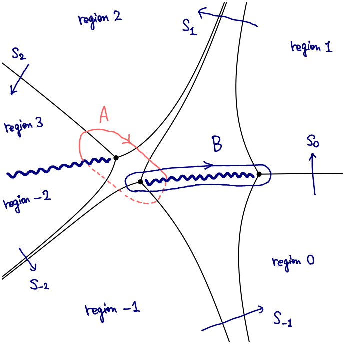
- Deforming the detoured path, we have

$$\tilde{\psi}_-^{(3)}(x,t,\nu;\hbar) = i\,\frac{Z(t,\nu-\hbar;\hbar)}{Z(t,\nu;\hbar)}\,\psi_-(x,t,\nu-\hbar;\hbar)$$

- Taking the discrete Fourier transform, we have

$$\Psi_+^{(2)} = \Psi_+^{(3)} + i\,e^{2\pi i\rho/\hbar}\,\Psi_-^{(3)} \qquad \text{i.e.,} \qquad s_2 = i\,e^{2\pi i\rho/\hbar}.$$

List of Stokes multipliers



$$\begin{cases} s_{-2} = i\,(e^{-2\pi i\rho/\hbar} - e^{2\pi i(\nu-\rho)/\hbar}), \\ s_{-1} = i\,(-e^{-2\pi i(\nu-\rho)/\hbar} + e^{-2\pi i\nu/\hbar}), \\ s_0 \;\; = i\,e^{2\pi i\nu/\hbar}, \\ s_1 \;\; = i\,(e^{-2\pi i\nu/\hbar} - e^{-2\pi i(\nu+\rho)/\hbar} + e^{-2\pi i\rho/\hbar}), \\ s_2 \;\; = i\,e^{2\pi i\rho/\hbar}. \end{cases}$$

- **Observation 1**: All s_ℓ is **independent of** t.
- **Observation 2**: They satisfies the **consistency condition** (i.e., defining equation of wild character variety or A_2-cluster algebra):

$$\begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s_{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_{-2} & 1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{i.e.,} \quad 1 + s_{\ell-1}\,s_\ell + i\,s_{\ell+2} = 0 \quad (s_{\ell+5} = s_\ell)$$

Problems and Questions

- Generalization to other Painlevé equations ?
  - Higher order: [Gavrylenko-Iorgov-Lisovyy 18], [Marchal-Orantin 19],...
  - q-analogues: [Bershtein-Shchechkin 16], [Bonelli-Grassi-Tanzini 17],...
- Justification of computation of Stokes data ?
  (Borel summability and resurgence, non-linear Stokes phenomenon.)
- Closed and combinatorial expression of the τ-function ?
  (In terms of Barnes G-function ?)
- Relation to irregular conformal blocks ? (c.f., [Nagoya 15–18],...)
- Relation to Nakajima-Yoshioka Blow-up equation ?
  (c.f., [Bershtein-Shchechkin 15-19], ...)

$$\text{For Painlevé I} \; : \; \hbar^4 D_t^4 \, \tau_{P_{\mathrm{I}}} \cdot \tau_{P_{\mathrm{I}}} + 2t \, \tau_{P_{\mathrm{I}}} \cdot \tau_{P_{\mathrm{I}}} = 0$$

- New proof of the Nekrasov conjecture ([Nekrasov-Okounkov] etc.) ?
- Relation to cluster algebras, Bridgeland stability, wall-crossing formulas,....?
  (c.f., [Chekhov-Mazzocco-Rubtsov 15], ...)

Thank you for your attention !