Confluence of Singular Points of Fuchsian Equations & Deformation of Star-Shaped Quiver Varieties

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Moduli spaces, Representation theory and Quantization

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0. Intro

Confluence of Gauss Hypergeometric Equation

$$z(1-z)\frac{d^2}{dz^2}y + (\gamma - (\alpha+\beta+1)z)\frac{d}{dz}y - \alpha\beta y = 0$$

Gauss hypergeometric equation

reg. sing. at $x = 0, 1, \infty$

$$z = \varepsilon\zeta, \quad \beta = \frac{1}{\varepsilon}$$

$$\zeta(1-\varepsilon\zeta)\frac{d^2}{d\zeta^2}y + (\gamma - (\alpha+1+\varepsilon)\zeta)\frac{d}{d\zeta}y - \alpha y$$

reg. sing. $x = 1$ $x = \infty$

$$\varepsilon \to 0$$

irreg. sing. $x = \infty$

$$\zeta\frac{d^2}{d\zeta^2}y + (\gamma - \alpha)\frac{d}{d\zeta}y - \alpha y = 0$$

Confluent Gauss HG equation

reg. sing. at $x = 0$, **irreg. sing. at** $x = \infty$

Isomonodromic deformation & Weyl group symmetry

$$\frac{d^2}{dz^2}y+\left\{\frac{1-\kappa_0}{z}+\frac{1-\kappa_1}{z-1}+\frac{1-\theta}{z-t}-\frac{1}{z-\lambda}\right\}\frac{d}{dz}y+\left\{\frac{\kappa}{z(z-1)}+\frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}-\frac{t(t-1)H}{z(z-1)(z-t)}\right\}y=0$$

$$t(t-1)H=\lambda(\lambda-1)(\lambda-t)\mu^2-\left\{\kappa_0(\lambda-1)(\lambda-t)+\kappa_1\lambda(\lambda-t)+(\theta-1)\lambda(\lambda-1)\right\}\mu+\kappa(\lambda-t)$$

reg. sing. at $x=0,1,t\,\infty$

$x=\lambda$ **is an apparent sing.**

μ **is an accessory parameter**

Let us move t **so that the monodromy is preserved**

↓

$$\frac{d\lambda}{dt}=\frac{\partial H}{\partial \mu},\qquad \frac{d\mu}{dt}=-\frac{\partial H}{\partial \lambda}$$

Painlevé VI equation

Set $t \to (1-t), \quad \lambda \to (1-\lambda),$ **then**

$$t(t-1)H = \lambda(\lambda-1)(\lambda-t)\mu^2 - \left\{\kappa_0(\lambda-1)(\lambda-t) + \kappa_1\lambda(\lambda-t) + (\theta-1)\lambda(\lambda-1)\right\}\mu + \kappa(\lambda-t)$$

$$t(t-1)(-\tilde{H}) = \lambda(\lambda-1)(\lambda-t)\mu^2 - \left\{(-\kappa_1)(\lambda-1)(\lambda-t) + (-\kappa_0)\lambda(\lambda-t) + (-(\theta-1))\lambda(\lambda-1)\right\}\mu + (-\kappa)(\lambda-t)$$

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda} \quad \xrightarrow{t \to (1-t), \quad \lambda \to (1-\lambda)} \quad \frac{d\lambda}{dt} = \frac{\partial \tilde{H}}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial \tilde{H}}{\partial \lambda}$$

This transformation gives another Painlevé VI

These transformations generate $D_4^{(1)}$ **Weyl group (Okamoto)**

Confluence & Weyl groups

$$\frac{d^2}{dz^2}y+\left\{\frac{1-\kappa_0}{z}+\frac{1-\kappa_1}{z-1}+\frac{1-\theta}{z-t}-\frac{1}{z-\lambda}\right\}\frac{d}{dz}y+\left\{\frac{\kappa}{z(z-1)}+\frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}-\frac{t(t-1)H}{z(z-1)(z-t)}\right\}y=0$$

set $t\to 1+\varepsilon t,\ \kappa_1\to\eta\varepsilon^{-1},\ \theta\to-\eta\varepsilon^{-1}$

and $\varepsilon\to 0$

$$\frac{d^2}{dz^2}y+\left\{\frac{1-\kappa_0}{z}+\frac{\eta t}{(z-1)^2}+\frac{1-\theta}{z-1}-\frac{1}{z-\lambda}\right\}\frac{d}{dz}y+\left\{\frac{\kappa}{z(z-1)}+\frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}-\frac{tH_V}{z(z-1)^2}\right\}y=0$$

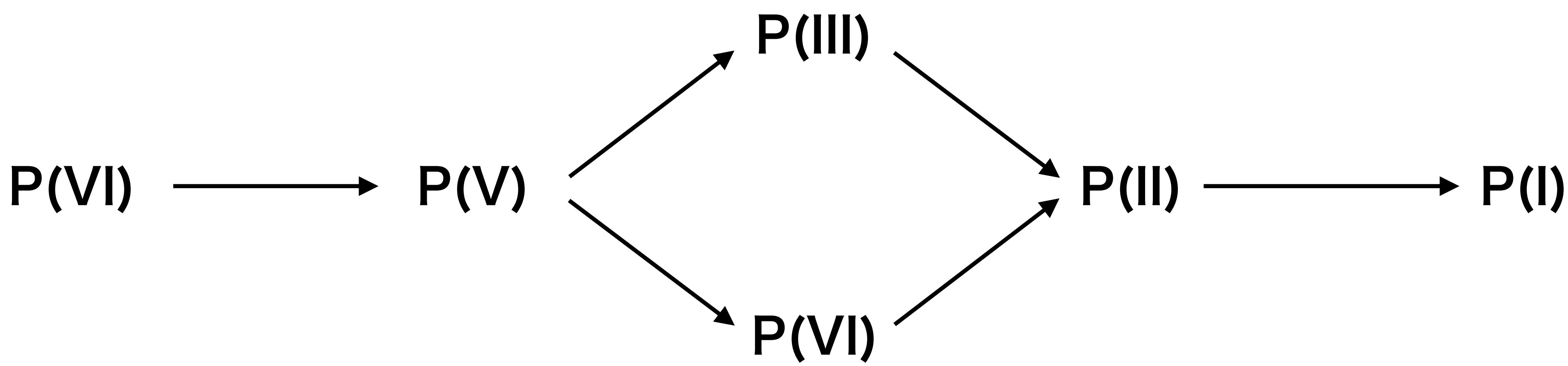
$$t^2H_V=\lambda(\lambda-1)^2\mu^2-\left\{\kappa_0(\lambda-1)^2+\theta\lambda(\lambda-1)-\eta t\lambda\right\}\mu+\kappa(\lambda-1)$$

$$\frac{d\lambda}{dt}=\frac{\partial H}{\partial\mu},\qquad \frac{d\mu}{dt}=-\frac{\partial H}{\partial\lambda}\qquad \xrightarrow{\textbf{Confluence}}\qquad \frac{d\lambda}{dt}=\frac{\partial H_V}{\partial\mu},\qquad \frac{d\mu}{dt}=-\frac{\partial H_V}{\partial\lambda}$$

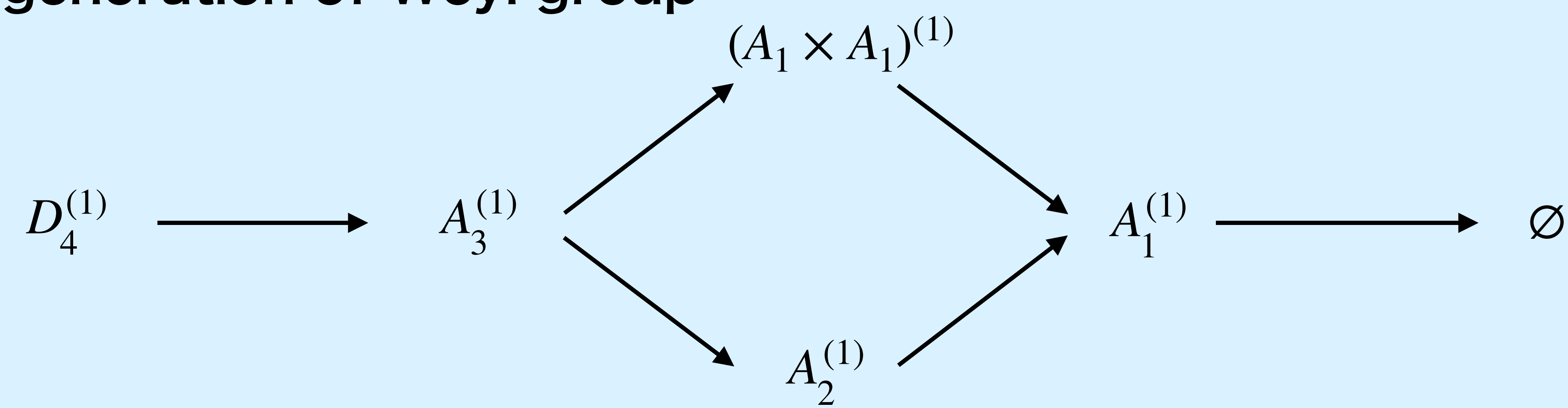
Painlevé VI $\qquad$ **Painlevé V**

$D_4^{(1)}$ $\xrightarrow{\textbf{Weyl group}}$ $A_3^{(1)}$

Degeneration scheme of Painlevé equations



Degeneration of Weyl group



We would like to explain this table of Weyl groups via moduli spaces of meromorphic connections

Previous isomonodromic deformation is equivalent to that of the following ODE.

$$\frac{d}{dz}Y = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right) Y$$

$$A_0 \sim \begin{pmatrix} 0 & 0 \\ 0 & \kappa_0 \end{pmatrix} \qquad A_1 \sim \begin{pmatrix} 0 & 0 \\ 0 & \kappa_1 \end{pmatrix} \qquad A_t \sim \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix} \qquad A_\infty := -\left(A_0 + A_1 + A_t\right) = \begin{pmatrix} \chi & 0 \\ 0 & \chi + \kappa_\infty \end{pmatrix}$$

(μ, λ): **phase space** ⟷ **Accessory parameters of ODE (moduli)**

$$H = \frac{A_t^{11}}{\lambda - t} + \frac{A_0^{11} + A_t^{11} - \kappa_0\theta}{t} + \frac{A_1^{11} + A_t^{11} - \kappa_1\theta}{t-1} + \mathrm{tr}\left(A_t\left(\frac{A_0}{t} + \frac{A_1}{t-1}\right)\right)$$

λ: **zero of** $\dfrac{A_0^{12}}{z} + \dfrac{A_t^{12}}{z-1} + \dfrac{A_t^{12}}{z-t}$

Painlevé equation is defined over the moduli space of the ODE

Moduli spaces of ODEs and Quiver varieties

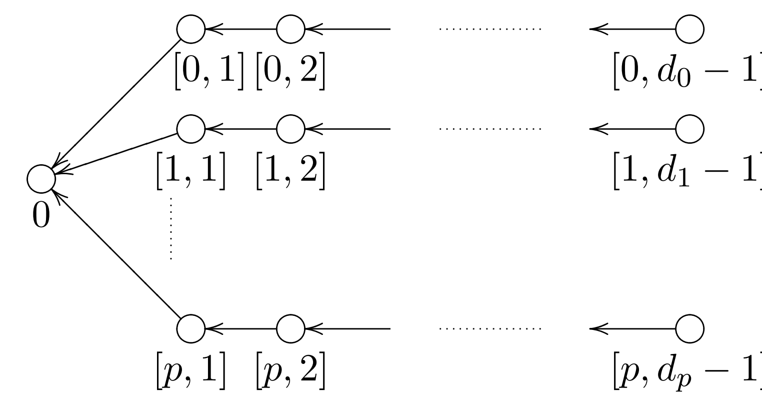
$\mathscr{C}_0, \ldots, \mathscr{C}_p$: **conjucacy classes in** $M(n, \mathbb{C})$

$$\mathscr{M}(\mathscr{C}) := \left\{ \frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z - a_i} Y \;\middle|\; \underset{A_i \in \mathscr{C}_i}{(A_i)\text{: irreducible}} \right\} \Big/ \mathrm{GL}(n, \mathbb{C}) \qquad \left(A_0 := -\sum_{i=1}^{p} A_i \right)$$

Thm (Crawley-Boevey)

$\exists Q = (Q_{vartex}, Q_{edge})$: **star-shaped quiver**

$\exists \alpha \in \mathbb{Z}^{Q_v}, \quad \exists \lambda \in \mathbb{C}^{Q_v}$



$$\mathscr{M}(\mathscr{C}) \cong \mathfrak{M}_\lambda^{reg}(Q, \alpha)$$

$$\mathfrak{M}_\lambda^{reg}(Q, \alpha) := \left\{ x \in \mu_\alpha^{-1}(\lambda) \mid x\text{: irreducible} \right\} / \mathbf{G}(\alpha)$$

$$\mu_\alpha\colon \mathrm{Rep}(\bar{Q}, \alpha) \to \prod_{a \in Q_v} M(\alpha_a, \mathbb{C}) \qquad \mu_\alpha(x) := \sum_{\substack{\rho \in Q_e \\ t(\rho) = a}} x_\rho x_{\rho*} - \sum_{\substack{\rho \in Q_e \\ s(\rho) = a}} x_{\rho*} x_\rho$$

$\bar{Q}$: **double of** Q

$$\mathbf{G}(\alpha) := \prod_{a \in Q_v} \mathrm{GL}(\alpha_a, \mathbb{C})$$

This picture is valid even for irregular singular case

$$\mathcal{M}_V := \left\{ \frac{d}{dz}Y = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + B_\infty\right) Y \text{ irred.} \;\middle|\; \begin{array}{l} A_0 \sim \begin{pmatrix} 0 & 0 \\ 0 & \theta_0 \end{pmatrix}, \quad A_1 \sim \begin{pmatrix} 0 & 0 \\ 0 & \theta_1 \end{pmatrix} \\ -B_\infty z^2 + A_\infty z^{-1} \sim \begin{pmatrix} 0 & 0 \\ 0 & -t \end{pmatrix} z^{-2} + \begin{pmatrix} \theta_1^\infty & 0 \\ & 0 \; \theta_2^\infty \end{pmatrix} \end{array} \right\} \Big/ \sim$$

$\mathcal{M}_V \cong$ **phase space of Painlevé V**

$\mathcal{M}_V \cong \mathfrak{M}_{\lambda_V}(Q_V, \alpha_V)$ Q_V: **Type** $A_3^{(1)}$ **(Boalch)**

$\mathcal{M}_{IV} \cong \mathfrak{M}_{\lambda_{IV}}(Q_V, \alpha_{IV})$ Q_{IV}: **Type** $A_2^{(1)}$ **(Boalch)**

$\mathcal{M}_{III} \hookrightarrow \mathfrak{M}_{\lambda_{III}}(Q_{III}, \alpha_{III})$ Q_{III}: **Type** $A_3^{(1)}$ **(H.)** $(A_1 \times A_1)^{(1)} \hookrightarrow A_3^{(1)}$

$\mathcal{M}_{II} \cong \mathfrak{M}_{\lambda_{II}}(Q_V, \alpha_{II})$ Q_{II}: **Type** $A_1^{(1)}$ **(H.-Yamakawa)**

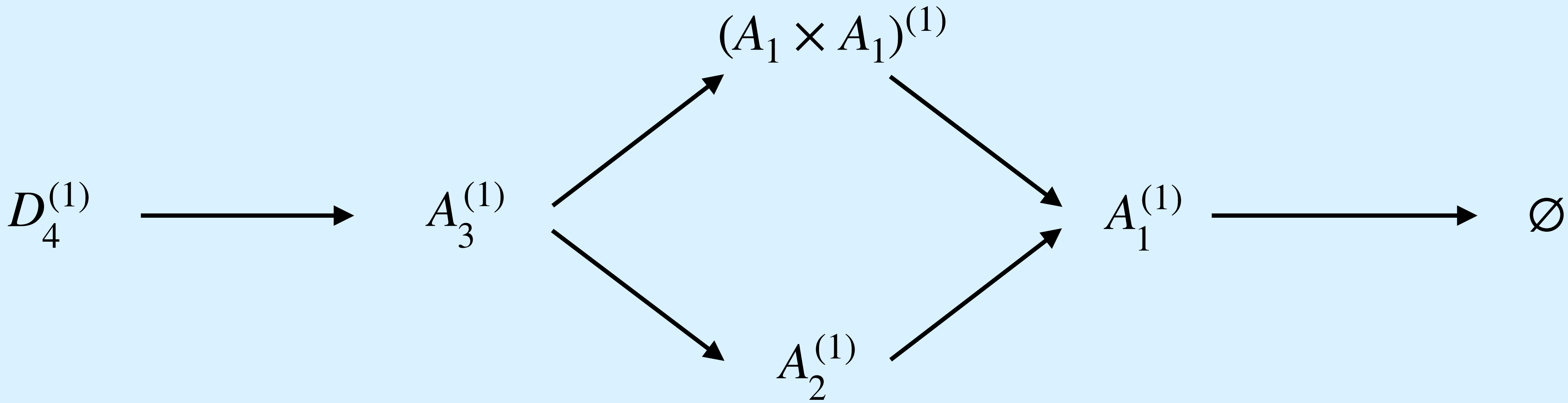
α_*: **indivisible null root of** $Q_*, (* = II, \ldots, VI)$

Weyl groups of Painlevé equations come from that of quiver varieties.
(Haraoka-Filipuk, Boalch, Yamakawa)

Q. Where do the Weyl groups of Painlevé equations come from ?

A. From the Weyl groups of quiver varieties (moduli spaces) via Isomonodromic deformation !

Degeneration of Weyl group of Painlevé equations



Want to understand this degeneration sheme as
a deformation of moduli spaces (quiver varieties) via
confluence of singular points

Main Theorem

Thm (precise statement will be given later)

$\mathscr{M}(\mathscr{C}) \cong \mathfrak{M}^{reg}_{\lambda}(Q, \alpha) \neq \emptyset$: a moduli space of Fuchsian equations

$\exists M$: a complex manifold $\quad\quad \exists \mathbb{D} \subset \mathbb{C}^N$: a small polydisc

$\exists \pi : M \to \mathbb{D}$ surjective holomorphic map satisfying the following

Every fiber $\pi^{-1}(c)$ is equidimentional with $\dim \mathscr{M}(\mathscr{C})$

For generic $c \in \mathbb{D}$, $\pi^{-1}(c) \cong \mathfrak{M}^{reg}_{\lambda(c)}(Q, \alpha)$

For every $c \in \mathbb{D}$, $\pi^{-1}(c) \cong$ a moduli space of differential equations

For every $c \in \mathbb{D}$, $\pi^{-1}(c) \exists \hookrightarrow \mathfrak{M}^{reg}_{\lambda(c)}(Q(c), \alpha(c))$: open dense embedding

($Q(c)$, $\alpha(c)$ are piecewise constant)

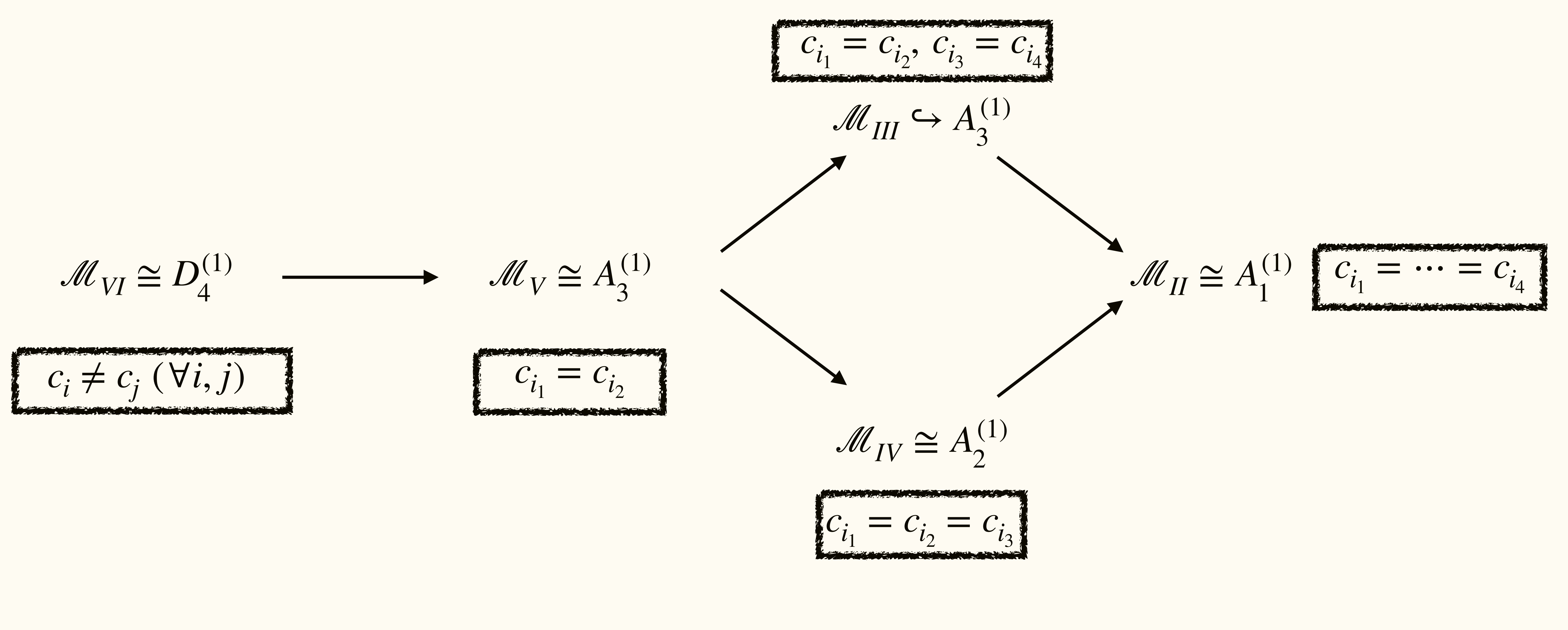
Rem.

cf. Inaba for 1-parameter deformation of moduli spaces of meromorphic connections on parabolic bundles over Riemann surfaces

Painlevé case

$\pi \colon M \to \mathbb{D} \subset \mathbb{C}^4 = \{(c_1, \ldots, c_4)\}$

$\pi^{-1}(c)$ **are isomorphic to moduli spaces in the following way**



1.Spectral types

Spectral types of Fuchsian equations

$\mathscr{C}$: a conjugacy class in $M(n,\mathbb{C})$ $\quad f_{\mathscr{C}}(t)=\prod_{i=1}^{d}(t-\xi_i)$: minimal polynomial of $\mathscr{C}$

The spectral type $(m_1,\dots,m_d)$ of $\mathscr{C}$ is defined by

$$m_1 := n - \mathrm{rank}(A-\xi_1)$$

$$m_j := \mathrm{rank}\prod_{i=1}^{j-1}(A-\xi_i) - \mathrm{rank}\prod_{i=1}^{j}(A-\xi_i), \qquad A\in\mathscr{C}$$

Then the pair $(\xi_1,\dots,\xi_d)$ and $(m_1,\dots,m_d)$ is invariant of $\mathscr{C}$, namely

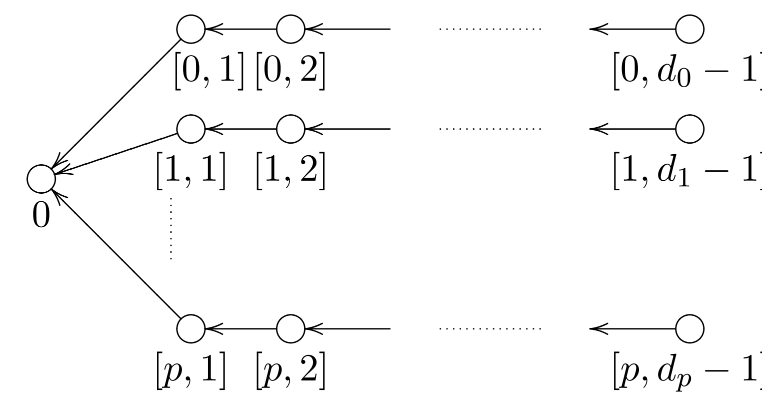
$$B\in\mathscr{C} \Leftrightarrow \begin{cases} n-\mathrm{rank}(B-\xi_1)=m_1 \\ \mathrm{rank}\prod_{i=1}^{j-1}(B-\xi_i)-\mathrm{rank}\prod_{i=1}^{j}(B-\xi_i)=m_j \end{cases}$$

$\mathscr{C}_0, \ldots, \mathscr{C}_p$: **conjucacy classes in** $M(n, \mathbb{C})$

$$\mathscr{M}(\mathscr{C}) := \left\{ \frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z-a_i} Y \;\middle|\; \underset{A_i \in \mathscr{C}_i}{(A_i)\text{: irreducible}} \right\} \Big/ \mathrm{GL}(n, \mathbb{C}) \qquad \left(A_0 := -\sum_{i=1}^{p} A_i \right)$$

$\mathscr{C}_j$ **define spectral types** $(\xi_{[j,1]}, \ldots, \xi_{[j,d_j]})$, $(m_{[j,1]}, \ldots, m_{[j,d_j]})$

Then Crawley-Boevey showed that



$$\mathscr{M}(\mathscr{C}) \cong \mathfrak{M}_\lambda(Q, \alpha)$$

Here

$$\begin{cases} \alpha_0 := n \\ \alpha_{[i,j]} := n - \sum_{k=1}^{j} m_{[i,k]} \end{cases} \qquad \begin{cases} \lambda_0 := -\sum_{i=0}^{p} \xi_{[i,1]} \\ \lambda_{[i,j]} := \xi_{[i,j]} - \xi_{[i,j+1]} \end{cases}$$

Irregular case (unframified only)

$$H = \mathrm{diag}(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \ldots, q_m(z^{-1})I_{n_m} + R_m z^{-1}) \qquad q_i(t) \in t^2\mathbb{C}[t] \qquad R_i \in M(n_i, \mathbb{C})$$

is called Hukuhara-Turrittin-Levelt normal form

Then the spectral type of

$$H = \frac{H_k}{z^k} + \frac{H_{k-1}}{z^{k-1}} + \cdots + \frac{H_1}{z}$$

is the collection of

$$H_i \to \overrightarrow{\xi_i} = (\xi_1^{(i)}, \ldots, \xi_{m^{(i)}}^{(i)}), \quad \overrightarrow{n}_i = (n_1^{(i)}, \ldots, n_{m^{(i)}}^{(i)})$$

with the monotone decreasing ordering

$$\overrightarrow{n}_k > \overrightarrow{n}_{k-1} > \cdots > \overrightarrow{n}_1$$

$>$: refinement of compositions of n

Unfolding of spectral type after Oshima

$$H = \frac{H_k}{z^k} + \frac{H_{k-1}}{z^{k-1}} + \cdots + \frac{H_1}{z}$$

Irregular normal form

$$H(c_1, c_2, \ldots, c_k) := \frac{H_k}{(z-c_1)(z-c_2)\cdots(z-c_k)} + \frac{H_{k-1}}{(z-c_1)(z-c_2)\cdots(z-c_{k-1})} + \cdots + \frac{H_1}{z-c_1}$$

(perturbation)

$$H(c_1, c_2, \ldots, c_k) = \sum_{i=1}^{k} \frac{A_i(\mathbf{c})}{z-c_i}$$

Fuchsian normal form

for generic $\mathbf{c} \in \mathbb{D}_\epsilon := \left\{ \mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{C}^k \mid |\mathbf{c}| < \epsilon \right\}$

where

$$A_i(\mathbf{c}) := \sum_{j=i}^{k} \frac{H_j}{\prod_{1 \le \nu \le j, \nu \ne i} (c_i - c_\nu)}$$

Unfolding of spectral type

$$H = \frac{H_k}{z^k} + \frac{H_{k-1}}{z^{k-1}} + \cdots + \frac{H_1}{z}$$

spectral type →

$$(\overrightarrow{\xi_1}, \overrightarrow{\xi_2} \ldots, \overrightarrow{\xi_k}), \quad (\overrightarrow{n_1} < \overrightarrow{n_2} < \cdots < \overrightarrow{n_k})$$

↓ perturbation

↓ forget $<$ — unfolding of spectral type

$$H(c_1, c_2, \ldots, c_k) = \sum_{i=1}^{k} \frac{A_i(\mathbf{c})}{z - c_i}$$

→

$\overrightarrow{\xi_i(\mathbf{c})}, \overrightarrow{n}_i$: spectral type of $A_i(\mathbf{c})$

Def. (confluent structure of a spectral type)

$(\overrightarrow{\xi_1}, \overrightarrow{\xi_2} \ldots, \overrightarrow{\xi_k}), \quad (\overrightarrow{n_1} < \overrightarrow{n_2} < \cdots < \overrightarrow{n_k})$: spectral data of a HTL normal form

↓ unfolding

$(\overrightarrow{\xi_1}(\mathbf{c}), \overrightarrow{n_1}), \ldots, (\overrightarrow{\xi_k}(\mathbf{c}), \overrightarrow{n_k})$: spectral data with a confluent structure

Example

$$H = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{z^3} + \frac{\begin{pmatrix} c & & & \\ & c & & \\ & & d & \\ & & & e \end{pmatrix}}{z^2} + \frac{\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix}}{z} \xrightarrow{\text{unfolding}} \frac{A_1(\mathbf{c})}{z - c_1} + \frac{A_2(\mathbf{c})}{z - c_2} + \frac{A_3(\mathbf{c})}{z - c_3}$$

(3,1) > (2,1,1) > (1,1,1,1)

$$A_1(\mathbf{c}) = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_1 - c_2)(c_1 - c_3)} + \frac{\begin{pmatrix} c & & & \\ & c & & \\ & & d & \\ & & & e \end{pmatrix}}{(c_1 - c_2)} + \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix}$$

(1,1,1,1)

$$A_2(\mathbf{c}) = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_2 - c_1)(c_2 - c_3)} + \frac{\begin{pmatrix} c & & & \\ & c & & \\ & & d & \\ & & & e \end{pmatrix}}{(c_2 - c_1)}$$

(2,1,1)

$$A_3(\mathbf{c}) = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_3 - c_1)(c_3 - c_2)}$$

(3,1)

$$\frac{A_1(\mathbf{c})}{z-c_1}+\frac{A_2(\mathbf{c})}{z-c_2}+\frac{A_3(\mathbf{c})}{z-c_3} \xrightarrow[(c_1=c_2)]{\text{specialize}} \frac{B_2^{(1)}(\mathbf{c})}{(z-c_1)^2}+\frac{B_1^{(1)}(\mathbf{c})}{z-c_1}+\frac{A_3'(\mathbf{c})}{z-c_3}$$

$(1,1,1,1)$, $(2,1,1)$, $(3,1)$ $\qquad\qquad$ $(2,1,1) > (1,1,1,1) \qquad (3,1)$

$$B_2^{(1)}(\mathbf{c}) = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_1-c_3)} + \begin{pmatrix} c & & & \\ & c & & \\ & & d & \\ & & & e \end{pmatrix} \qquad B_1^{(1)}(\mathbf{c}) = -\frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_1-c_3)^2} + \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix}$$

$(2,1,1) > (1,1,1,1)$

$$A_3'(\mathbf{c}) = \frac{\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix}}{(c_3-c_1)^2} \qquad (3,1)$$

holomorphic family of normal forms

$$\frac{A_1(\mathbf{c})}{z-c_1}+\frac{A_2(\mathbf{c})}{z-c_2}+\frac{A_3(\mathbf{c})}{z-c_3}$$

(1,1,1,1), (2,1,1), (3,1)

$(c_2 = c_3)$

(1,1,1,1), (3,1) > (2,1,1)

$$\frac{A_1'(\mathbf{c})}{z-c_1}+\frac{B_2^{(2)}(\mathbf{c})}{(z-c_2)^2}+\frac{B_1^{(2)}(\mathbf{c})}{z-c_2}$$

$$\frac{B_2^{(1)}(\mathbf{c})}{(z-c_1)^2}+\frac{B_1^{(1)}(\mathbf{c})}{z-c_1}+\frac{A_3'(\mathbf{c})}{z-c_3}$$

(2,1,1) > (1,1,1,1), (3,1)

$(c_1 = c_2)$

$$\frac{B_3(c)}{(z-c)^3}+\frac{B_2(c)}{(z-c)^2}+\frac{B_1(c)}{z-c}$$

(3,1) > (2,1,1) > (1,1,1,1)

$(c_1 = c_2 = c_3 = c)$

2. Deformation of moduli spaces of differential equations

Truncated orbit

$$H = \frac{H_k}{z^k} + \frac{H_{k-1}}{z^{k-1}} + \cdots + \frac{H_1}{z} \in M(n, z^{-k}\mathbb{C}[\![z]\!]/\mathbb{C}[\![z]\!])\colon \text{ HTL normal form}$$

$$\mathscr{O}_H := \mathrm{GL}(n, \mathbb{C}[\![z]\!]/z^{k+1}\mathbb{C}[\![z]\!]) \cdot H\colon \text{ truncated orbit of } H$$

Moduli space of differentia equations with unratified HTL normal forms

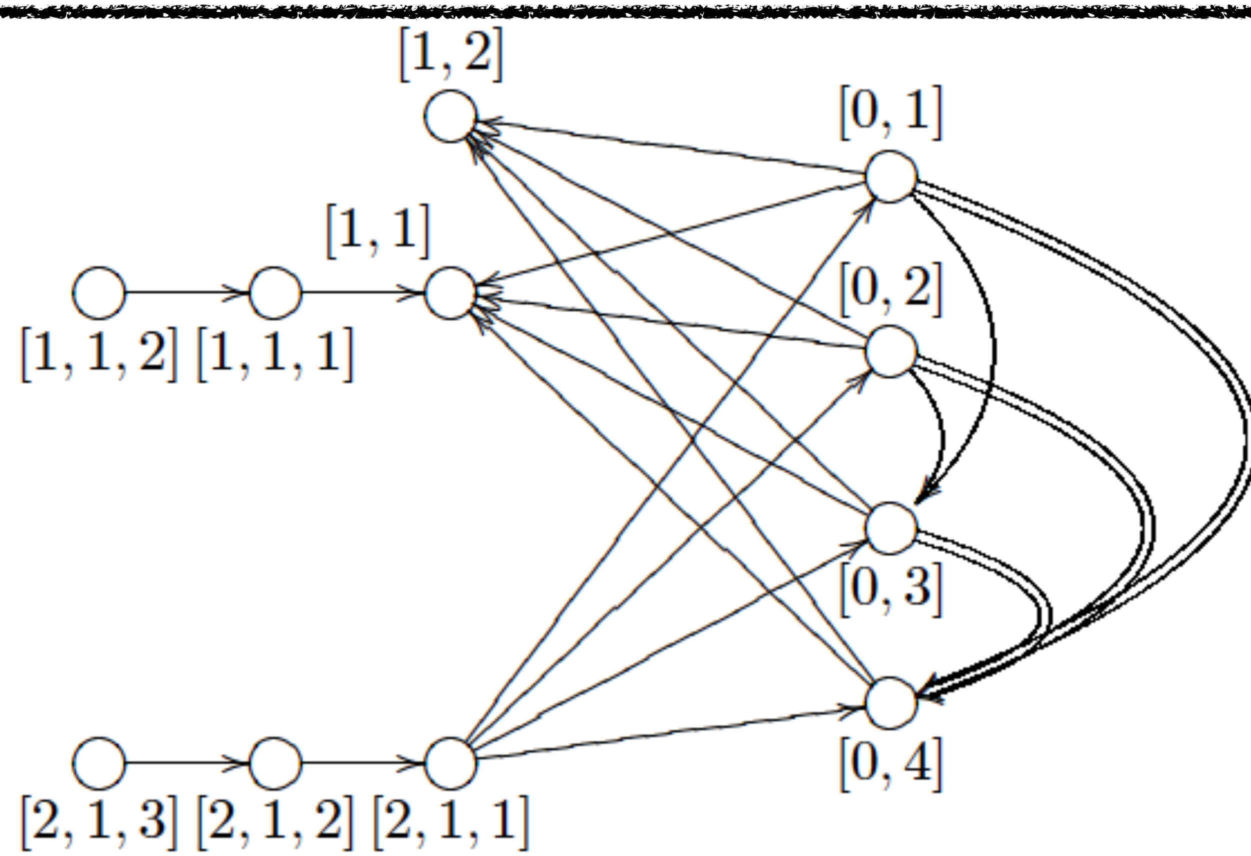
$$H_i \in M(n, z^{-k_i}\mathbb{C}[\![z]\!]/\mathbb{C}[\![z]\!])\colon \text{ HTL normal forms } i = 1,,2,\ldots,p$$

$$\mathscr{M}(\mathscr{H}) := \left\{ \frac{d}{dz}Y = \sum_{i=1}^{p}\sum_{j=1}^{k_i} \frac{A_j^{(i)}}{(z-a_i)^j}Y\colon \text{ irreducible} \;\middle|\; \sum_{i=1}^{p} A_1^{(i)} = 0, \sum_{j=1}^{k_i} \frac{A_j^{(i)}}{z^j} \in \mathscr{O}_{H_i} \right\} \Big/ \mathrm{GL}(n, \mathbb{C})$$

Embegging theorem (Crawley-Boevey, Boalch, Yamakawa, H-Yamakawa,H)

$\exists Q, \exists \alpha, \exists \lambda$

$\mathscr{M}(\mathscr{H}) \hookrightarrow \mathfrak{M}_{\lambda}^{reg}(Q, \alpha)$: embedding into an open dense subset



Quiver with the spectral type

(3,1) > (2,1,1) > (1,1,1,1) > (1,1,1,1)   (3,1) > (1,1,1,1)   (1,1,1,1)

Deformation of truncated orbits

$(\overrightarrow{\xi_1}(\mathbf{c}), \ldots, \overrightarrow{\xi_k}(\mathbf{c})), \quad (\overrightarrow{n_1}, \ldots, \overrightarrow{n_k})$: **spectral data with a confluent structure**

$$\downarrow$$

$$H(c_1, c_2, \ldots, c_k) = \frac{H_k}{(z-c_1)(z-c_2)\cdots(z-c_k)} + \frac{H_{k-1}}{(z-c_1)(z-c_2)\cdots(z-c_{k-1})} + \cdots + \frac{H_1}{z-c_1}$$

family of HTL normal forms

Prop (holomorphic family of truncated orbits)

$\exists \pi \colon \tilde{O}_{H(\mathbf{c})} \to \mathbb{D}_\epsilon \subset \mathbb{C}^k$: **holomorphic family**

$$\pi^{-1}(\mathbf{c}) = \mathcal{O}_{H(\mathbf{c})} \quad \forall \mathbf{c} \in \mathbb{D}_\epsilon$$

Example (construction of a family $\pi\colon \tilde{O}_{H(\mathbf{c})} \to \mathbb{D}_\epsilon \subset \mathbb{C}^k$)

$$H = \begin{pmatrix} t_1 I_{n_1} & \\ & t_2 I_{n_2} \end{pmatrix} z^{-2} + \begin{pmatrix} \alpha_1 I_{n_1} & \\ & \alpha_2 I_{n_2} \end{pmatrix} z^{-1} \qquad (n_1, n_2) > (n_1, n_2)$$

$$\begin{aligned} \mathcal{O}_H &= \mathrm{GL}(n, \mathbb{C}[\![z]\!]/z^2\mathbb{C}[\![z]\!]) \cdot H \\ &= \left(\mathrm{GL}(n, \mathbb{C}) \times \left\{ I_n + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} z \mid X \in M_{n_1 \times n_2}, Y \in M(n_2 \times n_1) \right\} \right) \cdot H \\ &= \mathrm{GL}(n, \mathbb{C}) \cdot \left\{ \begin{pmatrix} t_1 I_{n_1} & \\ & t_2 I_{n_2} \end{pmatrix} z^{-2} + \begin{pmatrix} \alpha_1 I_{n_1} & (t_2 - t_1)X \\ (t_1 - t_2)Y & \alpha_2 I_{n_2} \end{pmatrix} z^{-1} \right\} \end{aligned}$$

$$H(c_1, c_2) = \frac{H_2}{(z - c_1)(z - c_2)} + \frac{H_1}{z - c_1} = \frac{A_1(c)}{z - c_1} + \frac{A_2(c)}{z - c_2} \qquad c := c_1 - c_2$$

$$A_1(c) = \begin{pmatrix} (\frac{t_1}{c} + \alpha_1) I_{n_1} & \\ & (\frac{t_2}{c} + \alpha_2) I_{n_2} \end{pmatrix}, \quad A_2(c) = \begin{pmatrix} t_1 I_{n_1} & \\ & t_1 I_{n_2} \end{pmatrix}$$

For

$$(g, x(c), y(c)) \in \mathrm{GL}(n, \mathbb{C}) \times \left\{ \begin{pmatrix} I_{n_1} & cX \\ 0 & I_{n_2} \end{pmatrix} \,\middle|\, X \in M(n_1 \times n_2) \right\} \times \left\{ \begin{pmatrix} I_{n_1} & 0 \\ cY & I_{n_2} \end{pmatrix} \,\middle|\, Y \in M(n_2 \times n_1) \right\}$$

the action is defined by

$$(g, x(c), y(c)) \cdot H(c_1, c_2) := \frac{A_1'(c)}{z - c_1} + \frac{A_2'(c)}{z - c_2}$$

where

$$A_1'(c) := (gx(c)y(c))\, A_1(c)\, (gx(c)y(c))^{-1} \qquad A_2'(c) := g\, A_2(c)\, g^{-1}$$

Then

Orbit of $H(c_1, c_2) = \mathscr{C}_{A_1(c)} \times \mathscr{C}_{A_2(c)}$ (**if** $c = c_1 - c_2 \neq 0$)

Here we note that $\mathrm{Stab}(A_2(c)) = \left\{ \begin{pmatrix} G_1 & \\ & G_2 \end{pmatrix} \,\middle|\, G_i \in \mathrm{GL}(n_i, \mathbb{C}) \right\}$

and

$$\left\{ \begin{pmatrix} G_1 & \\ & G_2 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} I_{n_1} & cX \\ 0 & I_{n_2} \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} I_{n_1} & 0 \\ cY & I_{n_2} \end{pmatrix} \right\} \cong \mathrm{GL}(n, \mathbb{C})$$

Notice that

$$(g, x(c), y(c)) \cdot H(c_1, c_2) = \frac{A_1'(c)}{z - c_1} + \frac{A_2'(c)}{z - c_2} = \frac{A_2'(c)}{(z - c_1)(z - c_2)} + \frac{A_1'(c) - \frac{1}{c}A_2'(c)}{z - c_1}$$

and

$$A_1'(c) \equiv \begin{pmatrix} (\frac{t_1}{c} + \alpha_1)I_{n_1} & (t_2 - t_1)X \\ (t_1 - t_2)Y & (\frac{t_2}{c} + \alpha_2)I_{n_2} \end{pmatrix} \quad (\textbf{mod } c)$$

Thus we have

$$(g, x(c), y(c)) \cdot H(c_1, c_2) = g \cdot \left(\frac{\begin{pmatrix} t_1 I_{n_1} & \\ & t_2 I_{n_2} \end{pmatrix}}{(z - c_1)(z - c_2)} + \frac{\begin{pmatrix} \alpha_1 I_{n_1} & (t_2 - t_1)X \\ (t_1 - t_2)Y & \alpha_2 I_{n_2} \end{pmatrix} + cZ}{z - c_1} \right)$$

$(c_1 = c_2, \ (c = 0))$

↓

Orbit of $H(c_1, c_1) = \mathrm{GL}(n, \mathbb{C}) \cdot \left\{ \begin{pmatrix} t_1 I_{n_1} & \\ & t_2 I_{n_2} \end{pmatrix} (z - c_1)^{-2} + \begin{pmatrix} \alpha_1 I_{n_1} & (t_2 - t_1)X \\ (t_1 - t_2)Y & \alpha_2 I_{n_2} \end{pmatrix} (z - c_1)^{-1} \right\}$

$= \mathcal{O}_{H(c_1, c_1)}$

Main theorem

$(H_i(\mathbf{c}_i))_{\mathbf{c_i}\in\mathbb{D}^{(i)}_{\epsilon_i}}$: **holomorphic families of HTL normal forms,** $i = 1,2,\ldots,p$

$\pi_i\colon \tilde{O}_{H_i(\mathbf{c_i})} \to \mathbb{D}^{(i)}_{\epsilon_i}$: **hol. fam. fo orbits of** $H_i(\mathbf{c}_i), \quad i = 1,2,\ldots,p$

$$\widetilde{\mathscr{M}}(\mathscr{H}) := \left\{ \frac{d}{dz}Y = \sum_{i=0}^{p} \tilde{A}_i(\mathbf{c_i},\, z - a_i)Y \;\middle|\; \text{satisfying the following} \right\} \Big/ \mathrm{GL}(n,\mathbb{C})$$

1. $$\tilde{A}_i(\mathbf{c}_i, z) \in \tilde{\mathscr{O}}_{H_i(\mathbf{c}_i)}, \quad i = 1,2,\ldots,p$$

2. $\displaystyle\frac{d}{dz}Y = \sum_{i=0}^{p} \tilde{A}_i(\mathbf{c_i},\, z - a_i)Y$ **is irreducible at each** $\displaystyle(\mathbf{c}_i)_{i=1,2,\ldots,p} \in \prod_{i=1}^{p} \mathbb{D}^{(i)}_{\epsilon_i}$

3. $$\oint_{C_R} \sum_{i=0}^{p} \tilde{A}(\mathbf{c}_i, z - a_i)\, dz = 0$$

C_R: **circle centered at the origin with the radius** $\displaystyle R > \max_{i=1,2,\ldots,p} \{\, |a_i| + |\epsilon_i| \,\}$

Thm.

Define $\pi\colon \widetilde{M}(\mathscr{H}) \to \prod_{i=1}^{p} \mathbb{D}_{\epsilon_i}^{(i)}$ as before

Suppose there exists a generic fiber $\pi^{-1}(\mathbf{c}) \neq \emptyset$

Then we have the following

1. $\widetilde{M}(\mathscr{H})$ is a nonsingular complex manifold
2. π is surjective with equidimensional fibers
3. Every fiber $\pi^{-1}((\mathbf{c}_i)_{i=1,2,\ldots,p})$ is isomorphic to the moduli space $\mathscr{M}((H_i(\mathbf{c}_i))_{i=1,2,\ldots,p})$
4. For each $(\mathbf{c}_i) \in \prod_{i=1}^{p} \mathbb{D}_{\epsilon_i}^{(i)},$ there exists an open embedding

$$\pi^{-1}((\mathbf{c}_i)) \hookrightarrow \mathfrak{M}_{\lambda(\mathbf{c}_i)}(Q(\mathbf{c}_i, \alpha(\mathbf{c}_i)))$$