Localization in Equivariant Cohomology

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Abstract

Localization formulae for integrals reduce the problem of computing complicated integrals to finite summations. Well known examples include the complex residue formula and the Gauss-Bonnet-Hopf formula. This thesis gives a treatment of the equivariant localization formula due to Berline-Vergne and Atiyah-Bott, which says that, given a torus T acting smoothly on a compact oriented smooth manifold M, the integral of an equivariantly closed form over M is equal to a sum of integrals over the connected components of the fixed point set M^T. We give a review of the Borel construction of equivariant cohomology and its realisation via the Weil and Cartan models. We also discuss basic results such as the equivariant tubular neighbourhood theorem, fiber integration, characteristic classes and the Thom isomorphism, needed for the proof of the equivariant localization formula.

We assume the reader has basic knowledge of differential geometry, Lie theory and algebraic topology.

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Contents

Introduction

The two main topics of this thesis are equivariant cohomology and the equivariant localization formula.

Equivariant cohomology is a cohomology theory for topological spaces equipped with a group action. In this thesis we confine our study to compact smooth oriented manifolds M and de Rham cohomology, particularly of equivariant differential forms. To briefly describe what an equivariant differential form is, consider the action of a compact connected Lie group G on M (we want our manifolds to be compact and oriented to avoid complications with integration). An equivariant differential form is essentially a sum of monomials $\sum_I x^I \otimes \omega_I$, where $x^I \in S(\mathfrak{g}^*)$ are polynomials in the Lie algebra $\mathfrak{g}$, and the $\omega_I \in \Omega(M)$ are ordinary differential forms. Similar to the ordinary de Rham complex $(\Omega(M), d)$, the space of equivariant differential forms $\Omega_G(M)$ has an analogue of the exterior derivative called the *Cartan differential* d_G, which makes $(\Omega_G(M), d_G)$ into a cochain complex. The primary purpose of the Cartan complex is to provide a computational model of equivariant cohomology. However, we outline the theory of equivariant cohomology in the more general context of G^* algebras, which are essentially generalizations of the de Rham complex.

Equivariant localization is an important application of equivariant cohomology. In this thesis we focus on a central formula called the equivariant localization formula, which says the following:

Theorem. Let $T \subseteq G$ be a maximal torus. Suppose $\omega \in \Omega_T^k(M)$ $(k \geq d)$ is d_T compact. Then, for all $\xi \in \mathfrak{g}$,

$$\int_M \omega(\xi) = \sum_{X \subseteq M^T} \int_X \frac{i_X^* \omega(\xi)}{e^T(\mathcal{N}_X)(\xi)}$$

where the sum ranges over the connected components X of the fixed point set M^T, $i_X : X \to M$ denotes inclusion, and $e^T(\mathcal{N}_X)$ is the equivariant Euler class of the normal bundle of X.

The equivariant localization formula reduces the problem of integrating equivariant differential forms to a sum of potentially more tractable integrals. For instance, in the case that the fixed point set M^T is finite, the integral becomes a finite sum.

The structure of the thesis is as follows. In chapter one we discuss the Borel construction, a topological model of equivariant cohomology, and the Weil and Cartan models, which are algebraic models, and show the relationship between these models. Chapter two covers the theory needed to understand and prove the localization formula. We finish by considering some applications.

The appendix contains some supplementary material for the readers benefit. Appendix A contains some extra results in the theory of G^* modules that were omitted from the core thesis, along with some background on connections, curvatures and the notion of a pushforward in equivariant cohomology.

Appendix B reviews relevant notions in spectral sequences and double complexes needed to understand the abelianization theorem.

Appendix C is discusses maximal tori, representations of the circle group, and a statement of the Chevalley restriction theorem needed for proving the abelianization theorem.

Chapter 1

Equivariant Cohomology

1.1 Topological Equivariant Cohomology and the Borel Construction

1.1.1 Basic Notions

Definition 1.1.1. . *Let X be a topological space, and G be a Lie group that acts freely on M. We define the* ***equivariant cohomology*** *groups H_G^* of X by the cohomology groups (e.g. singular) of the orbit space $X\backslash G$, i.e.*

$$H_G^*(X) := H^*(X\backslash G) \tag{1.1}$$

A problem occurs when the action of G on X is *not* free. In this case the orbit space X/G may be ”pathological” in the sense that the topology of the orbit space results in loss of information about the group action when passing to cohomology. An example of such a scenario is $\mathbb{S}^1$ acting on $\mathbb{S}^2 \subset \mathbb{R}^3$ via rotation about the z-axis. The stabilizer groups G_p for p at the north and south poles of $\mathbb{S}^2$ are exactly $\mathbb{S}^1$. For all other $p \in \mathbb{S}^2$, $G_p = \{e\}$. The orbit space $\mathbb{S}^2/\mathbb{S}^1$ is isomorphic to the closed interval $[-1, 1]$, a contractible space, and so is trivial in cohomology.

If G does not act freely on X then we want to construct a space X^* that is equivariantly homotopy equivalent to X (hence equivalent in cohomology) on which it does act freely. Note that if we have a space E on which G acts freely, then clearly any point of $X \times E$ is fixed only by the identity, i.e. G acts freely on $X \times E$. If the space E on which G acts freely is contractible, $(X \times E)\backslash G$ and $X\backslash G$ are equivalent in cohomology. Thus, instead of looking for a space EG that is merely weakly contractible, it is advantageous to find a space E that is contractible. Assuming such a space can be found, its classifying space $BG := E/G$ forms a classifying bundle $E \to B$.

Definition 1.1.2. *Let X be a topological space and let G be a Lie group that does not necessarily act freely on X. Let E be a classifying space of G. Then we define the equivarient cohomology groups of X by*

$$H_G^*(X) := H^*((X \times E)\backslash G) \tag{1.2}$$

Example 1.1.3. *Consider the zero-dimensional manifold pt. We have*

$$H_G^*(pt) = H^*((pt \times EG)\backslash G) = H^*(EG\backslash G) = H^*(BG)$$

1.1.2 Existence of Classifying Bundles

From the theory of principal G-bundles we know that the definition of equivariant cohomology groups is independent of the choice of contractible space E on which G acts freely. The next theorem shows that such an E always exists for any compact Lie group.

Theorem 1.1.4. . *Any compact Lie group G has a classifying bundle $E \to B$.*

Sketch proof. Note that if G is a subgroup of the compact Lie group K, and there exists a contractible space E on which K acts freely, then the restriction of the K-action to G gives a free G-action. It is a well-known fact that any compact Lie group has an embedding e into a subgroup of $U(n)$ for some $n \in \mathbb{N}$. Therefore, if we can find a contractible space E and a free action μ of $U(n)$ on E, then $\mu \circ e$ gives a free action of G on E.

Consider the infinite dimensional space $V = L^2[0, \infty)$ of square integrable functions (in the Lebesgue measure) on $[0, \infty)$. Let E consist of all n-tuples of functions $\mathbf{v} = (v_1, ..., v_n),\ v_i \in V,\ < v_i, v_j >= \delta_{ij}$ in which, for each $A \in U(n)$, $U(n)$ acts on E by $A\mathbf{v} = \mathbf{w} = (w_1, ..., w_n),\ w_i = \sum_{j=1}^n a_i j v_j$. Let $E' \subset E$ be the set of n-tuples of functions who vanish on $[0, 1]$. It remains to prove that E is contractible in two steps. It suffices to show that there exists a deformation retract of E onto E' and that E' is contractible to a point within E. For a proof see ([5], Proposition 1.2.1, p.5).

1.2 Equivariant Cohomology in G^* Modules

In the previous section we gave a topological model for the equivariant cohomology groups $H_G^*(X)$ of a topological G-space X. We saw that in the case were G did not act freely on X, the groups $H^*(X/G)$ did not preserve information about the G-action due to the topology of the orbit space X/G. The solution was to define $H_G^*(X) = H^*(X \times_G E)$, where E is some contractible space on which G acts freely.

In this section we present an algebraic analogue of the Borel construction of equivariant cohomology for a manifold M defined in terms of the de Rham cohomology of $\Omega(M)$. One might think that an obvious candidate for these equivariant cohomology groups would be the de Rham cohomology of $\Omega(M) \times \Omega(E)$. However, it is unclear on how to define $\Omega(E)$, since it can be shown that any contractible space on which G acts freely is infinite-dimensional ([5], p.xiv). A suitable substitute for $\Omega(E)$ is a construction called the *Weil algebra*, which we will define later.

With this viewpoint, we can also see intuitively why the topological definition $H_G^*(M) = H^*(M/G)$ in the last section fails when the G-action is not free, since the quotient manifold theorem fails and M/G will not be a manifold in general, hence it is not clear how we define the de Rham cohomology $H(\Omega(M/G))$.

The underlying categories for equivariant cohomology in an algebraic setting are G^* algebras and G^* modules, which we describe below. For context, we first give a recap of some differential-geometric identities of the de Rham complex, and review basic definitions of superalgebras and supervector spaces.

1.2.1 The de Rham Complex

Definition. Let M be a smooth manifold and $\Omega(M)$ its de Rham complex. Let G be a Lie group acting on M with Lie algebra $\mathfrak{g}$. We have a representation (action) ρ of G on $\Omega(M)$ given, for each $g \in G$ and $\omega \in \Omega(M)$, by

$$\rho(g)\omega = (L_g^{-1})^*\omega \tag{1.3}$$

Definition. There is an action of $\mathfrak{g}$ on Ω given, for each $\xi \in \mathfrak{g}$, by the degree 0 derivation the Lie derivative $\mathcal{L}_\xi : \Omega^k(M) \to \Omega^k(M)$, which we take by default to be the Lie derivative with respect to the fundamental vector field $\xi^\#$ generated by ξ.

$$\mathcal{L}_\xi\omega = \frac{d}{dt}(\rho_{\exp(t\xi)}\omega)|_{t=0} = \frac{d}{dt}(L^*_{\exp(-t\xi)}\omega)|_{t=0} \tag{1.4}$$

$\mathcal{L}_\xi$ satisfies the Leibniz rule that: $\forall \mu, \nu \in \Omega(M)$,

$$\mathcal{L}_\xi(\mu\nu) = (L_\xi\mu)\nu + \mu(L_\xi\nu) \tag{1.5}$$

Another action is given by the derivation of degree -1 on $\Omega(M)$, called the **interior product**, $\iota_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$, which we similarly take by default to be the interior product by the fundamental vector field $\xi^\#$ generated by ξ. The interior product satisfies

$$\iota_\xi(\mu\nu) = (\iota_\xi\mu)\nu + (-1)^m\mu(\iota_\xi\nu)$$

$\forall \mu, \nu \in \Omega(M)$. Lastly there is the degree $+1$ derivation the **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$, satisfying

$$d(\mu\nu) = (d\mu)\nu + (-1)^m\mu(d\nu)$$

Definition. Let $\xi_1, ..., \xi_n$ be basis of $\mathfrak{g}$. We use the notation

$$\mathcal{L}_j := \mathcal{L}_{\xi_j} \quad \iota_j := \iota_{\xi_j} \tag{1.6}$$

The basis elements satisfy (making use of Einstein summation notation over Latin letters)

$$[\xi_i, \xi_j] = c_{ij}^k\xi_k \tag{1.7}$$

where the c_{ij}^k are called the **structure constants** of the Lie algebra $\mathfrak{g}$ with respect to the given basis. Note that since $[\xi_j, \xi_i] = -[\xi_i, \xi_j]$ we have

$$c_{ji}^k = -c_{ij}^k$$

We note that $\mathfrak{g}$ always has a basis since Lie groups are parallelizable.

Proposition 1.2.1 (Weil equations). *$\mathcal{L}_\xi, \iota_\xi, d$ satisfy*

$$\iota_i\iota_j + \iota_j\iota_i = 0 \tag{1.8}$$
$$\mathcal{L}_i\iota_j - \iota_j\mathcal{L}_i = c_{ij}^k\iota_k \tag{1.9}$$
$$\mathcal{L}_i\mathcal{L}_j - \mathcal{L}_j\mathcal{L}_i = c_{ij}^k\mathcal{L}_k \tag{1.10}$$
$$d\iota_i + \iota_j d = \mathcal{L}_i \tag{1.11}$$
$$d\mathcal{L}_i - \mathcal{L}_i d = 0 \tag{1.12}$$
$$d^2 = 0 \tag{1.13}$$

Further, $\iota, \mathcal{L}, d$ are G-equivariant in the sense that, for all $a \in G$, $\xi \in \mathfrak{g}$,

$$\rho_a \circ \mathcal{L}_\xi \circ \rho_a^{-1} = \mathcal{L}_{Ad_a\xi} \tag{1.14}$$
$$\rho_a \circ \iota_\xi \circ \rho_a^{-1} = \iota_{Ad_a\xi} \tag{1.15}$$
$$\rho_a \circ d \circ \rho_a^{-1} = d \tag{1.16}$$

1.2.2 Superalgebras

In this section we define the category of superalgebras, which are essentially graded algebras equipped with a bracket operation, and supervector spaces, which are graded vector spaces. From here we can describe G^* modules and G^* algebras, which are (respectively) superalgebras and supervector spaces equipped with representations of a Lie group G and its Lie algebra $\mathfrak{g}$ satisfying some additional properties.

Definition. A **supervector space** V is a vector space with $\mathbb{Z}_2$-gradation:

$$V = V_{\mathbf{0}} \oplus V_{\mathbf{1}} \tag{1.17}$$

Elements of V_0 are called **even** and elements of V_1 are called **even**. Generally V will have an additional structure that the $V_{\mathbf{0}}, V_{\mathbf{1}}$ parts can be decomposed further to give a $\mathbb{Z}$-gradation:

$$V = \bigoplus_{i\in\mathbb{Z}} V_i, \quad V_{\mathbf{0}} = \bigoplus_{j\in\mathbb{Z}} V_{2j} \quad V_{\mathbf{1}} = \bigoplus_{j\in\mathbb{Z}} V_{2j+1}$$

where we say elements of V_{2j} or V_{2j+1} are even or odd respectively. An element of V_i is said to be of **degree** i.

Definition. A **superalgebra** A is a supervector space together with a multiplicative operator $\cdot$ satisfying, $\forall i, j \in \{0, 1\}$

$$A_{\mathbf{i}} \cdot A_{\mathbf{j}} \subset A_{\mathbf{i+j}} \tag{1.18}$$

if A is $\mathbb{Z}_2$-graded, or similarly

$$A_i \cdot A_j \subset A_{i+j} \tag{1.19}$$

if A is $\mathbb{Z}$-graded.

Definition. Let A be a superalgebra. The **supercommutator** (or simply **commutator**) on A is a bracket $[\cdot,\cdot] : A_{\mathbf{i}} \times A_{\mathbf{j}} \to A_{\mathbf{i+j}}$ given, for any $L \in A_{\mathbf{i}}$ and $M \in A_{\mathbf{j}}$, by

$$[L, M] := LM - (-1)^{\mathbf{ij}} ML \tag{1.20}$$

in the case of a $\mathbb{Z}_2$-gradation, and analogously in the case of a $\mathbb{Z}$-gradation.

We can now define the concept of a *Lie superalgebra*, which we will see are ubiquitous in equivariant de Rham theory.

Definition. A ($\mathbb{Z}$-graded) **Lie superalgebra** is a $\mathbb{Z}$-graded supervector space

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \tag{1.21}$$

together with a bracket $[\cdot,\cdot] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, which is (super) anti-commutative in the sense that any $\xi \in \mathfrak{g}_i, \eta \in \mathfrak{g}_j$ satisfy the relation

$$[\xi, \eta] + (-1)^{ij}[v, u] = 0 \tag{1.22}$$

and the super Jacobi identity

$$[u, [v, w]] = [[u, v], w] + (-1)^{ij}[v, [u, w]] \tag{1.23}$$

Example. The most important example of a Lie superalgebra we will be using is the following. Given a Lie algebra $\mathfrak{g}$ with basis $\xi_1, ..., \xi_n$, define the Lie superalgebra

$$\tilde{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1.24}$$

where $\mathfrak{g}_{-1}$ is an n-dimensional vector space with basis $\iota_1, ..., \iota_n$, $\mathfrak{g}$ is an n-dimensional vector space with basis $\mathcal{L}_1, ..., \mathcal{L}_n$, and $\mathfrak{g}_1$ is a 1-dimensional vector space with basis d. Equations (1.8) – (1.13) can be easily re-written in terms of the bracket on $\tilde{\mathfrak{g}}$ by

$$[\iota_i, \iota_j] = 0 \tag{1.25}$$
$$[\mathcal{L}_i, \iota_j] = c_{ij}^k \iota_k \tag{1.26}$$
$$[\mathcal{L}_i, \mathcal{L}_j] = c_{ij}^k \mathcal{L}_k \tag{1.27}$$
$$[d, \iota_i] = \mathcal{L}_i \tag{1.28}$$
$$[d, \mathcal{L}_i] = 0 \tag{1.29}$$
$$[d, d] = 0 \tag{1.30}$$

1.2.3 G^* algebras and G^* modules

Definition 1.2.2. *. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A G^** ***algebra*** *is a commutative superalgebra A, together with a linear representation $\rho : G \to Aut(A)$, and actions of $\tilde{\mathfrak{g}}$ as (super)derivations of A, which satsify the following properties $\forall g \in G$ and $\xi \in \mathfrak{g}$:*

$$\frac{d}{dt}\rho(\exp(t\xi))|_{t=0} = \mathcal{L}_\xi \tag{1.31}$$

$$\rho(g)\mathcal{L}_\xi\rho(g^{-1}) = \mathcal{L}_{Ad_g\xi} \tag{1.32}$$

$$\rho(g)\iota_\xi\rho(g^{-1}) = \iota_{Ad_g\xi} \tag{1.33}$$

$$\rho(g)d\rho(g^{-1}) = d \tag{1.34}$$

A G^* ***module*** *is a supervector space A with a linear representation of G on A and a homomorphism $\tilde{\mathfrak{g}} \to End\ A$ satisfying Eqs.* (1.31) − (1.34)*.*

Note that a superalgebra can be viewed as a supervector space if the multiplicative structure is disregarded. In this way we can see a G^* algebra as a G^* module with a left multiplication. More specifically, a G^* algebra is a commutative superalgebra which is also a G^* module, with the additional condition that G acts as automorphisms and $\tilde{\mathfrak{g}}$ acts as superderivations. Further, it is clear that any tensor product of G^* algebras is a G^* algebra by defining the actions of G and $\mathfrak{g}$ component-wise in the obvious way. For each $g \in G$, when we say $\rho(g)$ is an automorphism, we mean it is an isomorphism that preserves the grading of A. That is, each $\rho(g)$ is a degree 0 morphism.

Given a smooth manifold M, its de Rham complex $\Omega(M) = \bigoplus_{k\in\mathbb{N}} \Omega^k(M)$ can be viewed as a G^* algebra, with the multiplication operation of the underlying superalgebra being the wedge product where, given any $\mu \in \Omega^i(M), \nu \in \Omega^j(M)$, we have

$$\mu \wedge \nu \in \Omega^{i+j}(M)$$

and the actions of G and $\widetilde{\mathfrak{g}}$ defined as in Section 1.2.1. Further, by the graded anti-symmetry of the wedge product, we have $\mu \wedge \nu - (-1)^{ij}\nu \wedge \mu$, hence $\Omega(M)$ is a commutative superalgebra. Therefore, given a smooth manifold M acted on by a Lie group G with Lie algebra $\mathfrak{g}$, the definition of $\tilde{\mathfrak{g}}$ is to say that the elements of $\mathfrak{g}_i$ act as derivations of degree i on the commutative superalgebra $A = \Omega(M)$ whenever we are given an action of G on M (since $\mathcal{L}_\xi$ is defined in terms of the action of G).

Definition 1.2.3. *On any superalgebra A we have $d^2 = 0$, so A is a cochain complex. We henceforth define the cohomology $H(A) = H(A, d)$ to be the de Rham cohomology of A with respect to the differential d.*

If $A = \Omega(M)$, $H^*(\Omega(M))$ is equivalent to the simplicial homology $H^*(M)$ by de Rham's theorem. The de Rham complex $(\Omega(M), d)$ over a field $\mathbb{F}$ (in our case $\mathbb{F} = \mathbb{C}$) is **acyclic**, i.e. satisfies

$$H^k(\Omega(M), d) = \begin{cases} \mathbb{F}, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{1.35}$$

1.2.4 Free Actions and Type (C) G^* Algebras

Definition 1.2.4. . *Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on a smooth manifold M. The action of G is said to be* ***locally free*** *if it satisfies one of the following to equivalent definitions*

1. *For each $\xi \in \mathfrak{g}$, the fundamental vector field $\xi^\#$ is nowhere vanishing, (i.e. $\xi^\#_p \neq 0\ \forall p \in M$.*

2. *For each $p \in M$, the stabilizer group $Stab(p)$ is discrete.*

Clearly free actions are locally free since they have trivial stabilizer groups. Note that in the case of a locally free action of a compact Lie group, the stabilizer groups being discrete means they are finite (i.e. there are only finitely many $g \in G$ that fix any $p \in M$).

Definition 1.2.5. . *Let G be a Lie group and A be a G^* algebra. Let $\mathfrak{g}$ be the Lie algebra of G with basis $\xi_1, ..., \xi_n$. We say A is of* ***type (C)*** *if there exist elements $\theta^1, ..., \theta^n \in A_1$ (which we will later see are given by connection forms) satisfying*

$$\iota_a \theta^b = \delta^b_a \tag{1.36}$$

and such that the subspace $C \subseteq A_1$ that they span is G-invariant.

We give an explicit formula for the θ^i in the case of a compact abelian Lie group G.

Theorem 1.2.6 ([2], Theorem 23.5, p.98)**.** *Let G be a a compact abelian group acting locally freely on a smooth manifold M. Then $\forall \xi \in \mathfrak{g}$, $\xi \neq 0$, there exists a G-invariant 1-form θ on M such that $\iota_\xi \theta = 1$.*

More explicitly, this form is given in terms of ξ by

$$\theta_p(X_p) = \frac{\langle \xi^\#_p, X_p \rangle}{\|\xi^\#_p\|} \tag{1.37}$$

$\forall X_p \in T_pM$, where $\langle \cdot, \cdot \rangle$ is some G-invariant Riemmanian metric on M and $\| \cdot \|$ its induced norm. To see that such a metric exists, given any Riemannian metric $\langle \cdot, \cdot \rangle'$ we can construct a new G-invariant metric $\langle \cdot, \cdot \rangle$ by integrating $\langle \cdot, \cdot \rangle'$ over G (this integral exists by compactness of G). More specifically, for each $p \in M$ and $X_p, Y_p \in T_pM$, define

$$\langle X_p, Y_p \rangle_p = \int_G \langle L_{g*} X_p, L_{g*} Y_p \rangle_p \, d\mu(g) \tag{1.38}$$

where μ is a left haar measure on G, which exists by Haar's theorem. An obvious corollary follows:

Corollary 1.2.7. . *Let G be a compact abelian group acting locally freely on a manifold M. Let $\mathfrak{g}$ be the Lie algebra of G with basis $\xi^1, ..., \xi^n$. There exists a daul basis $\theta^1, ...\theta^n$ of $\mathfrak{g}^*$ satisfying*

$$\iota_a \theta^b = \delta^b_a \tag{1.39}$$

with each θ^i given, for each $p \in M$ and $\chi \in \mathfrak{g}$, by

$$\iota_{\chi_p} \theta^i_p = \iota_{(\chi^\#)_p} \theta^i_p = \frac{\langle (\xi^\#_i)_p, (\chi^\#)_p \rangle}{\|(\xi^\#_i)_p\|} \tag{1.40}$$

or alternatively for each $p \in M$ and $X_p \in T_pM$, by

$$\theta^i_p(X_p) = \frac{\langle (\xi_i^\#)_p, X_p \rangle}{\|(\xi_i^\#)_p\|} \tag{1.41}$$

Note that the θ^i satisfy (1.39).

Remark 1.2.8. *We have shown that, in the case of a compact abelian Lie group G acting on a smooth manifold M, the de Rham complex is type (C).*

1.2.5 Equivariant Cohomology of G^* Algebras

In order to define equivariant cohomology on G^* algebras we first need to define the set of basic elements of a G^* algebra.

1.2.5.1 The Basic Subcomplex

Definition 1.2.9. . *Let $\pi : P \to M$ be a principal G-bundle. The space of differential forms on P that are pullbacks of forms on M*

$$\pi^*\Omega(M) \subset \Omega(P) \tag{1.42}$$

*are called **basic forms**.*

Definition 1.2.10. *Let G be a Lie group (with Lie algebra $\mathfrak{g}$) with a smooth left action on a smooth manifold M. A differential form $\omega \in \Omega(M)$ is said to be G-**invariant** if $(L_g^{-1})^*\omega = \omega \; \forall g \in G$.*

Theorem 1.2.11. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A differential form $\omega \in \Omega(M)$ is G-invariant if and only if $\mathcal{L}_\xi \omega = 0 \; \forall \xi \in \mathfrak{g}$.*

Proof. '$\underline{\implies}$'. Suppose $\omega \in \Omega(M)$ is G-invariant. Then

$$\mathcal{L}_\xi \omega = \frac{d}{dt}(L^*_{\exp(-t\xi)}\omega|_{t=0} = \frac{d}{dt}\omega|_{t=0} = 0$$

'$\underline{\impliedby}$'. Let $\omega \in \Omega(M)$ and suppose $\mathcal{L}_\xi \; \forall \xi \in \mathfrak{g}$. Given $p \in M$, define the map $h : \mathbb{R} \to \bigwedge T^*pM$ by $h_\xi(t) = (L^*_{\exp(-t\xi)}\omega)(p)$. We have $h_\xi(0) = \omega(p)$. We aim to show that $h_\xi(t)$ is constant. By assumption,

$$(\mathcal{L}_\xi \omega)(p) = \frac{d}{dt}(L^*_{\exp(-t\xi)}\omega)(p)|_{t=0} = \frac{d}{dt}h_\xi(t)|_{t=0} = 0$$

or equivalently $h'_\xi(0) = 0$. We note that, $\forall t, s \in \mathbb{R}$,

$$h_\xi(t+s) = (L^*_{\exp(-(t+s)\xi}\omega)(p) = (L^*_{\exp(-t\xi)}L^*_{\exp(-s\xi)}\omega)(p)$$

Hence, given $t \in \mathbb{R}$, we have

$$\begin{aligned} h'_\xi(t) = \frac{d}{ds}h(t+s) &= \frac{d}{ds}(L^*_{\exp(-t\xi)}L^*_{\exp(-s\xi)}\omega)(p)|_{s=0} \\ &= \frac{d}{ds}(L^*_{\exp(-s\xi)}\omega)(\exp(-t\xi) \cdot p)|_{s=0} \\ &= \mathcal{L}_\xi \omega(\exp(-t\xi) \cdot p) = 0 \end{aligned}$$

So $h_\xi(t) = h_\xi(0) = \omega(p)$. The exponential map restricts to a diffeomorphism onto some open neighbourhood U of $e \in G$ ([4], Proposition 20.8, p. 519). That is, for each $g \in U$ there exists $\xi \in \mathfrak{g}$ such that $g = \exp(\xi)$. But since G is connected, any neighbourhood of e generates G ([4], Proposition 7.14, p. 156). So U generates G. That is, for any $g \in G$ there exist $\xi_1, ..., \xi_n$ such that $g = \exp(\xi_1)... \exp(\xi_n)$. So

$$L_g^*\omega = L^*_{\exp(\xi_1)...\exp(\xi_n)}\omega = L^*_{\exp(\xi_1)}...L^*_{\exp(\xi_n)}\omega = h_{\xi_1}(-1)...h_{\xi_n}(-1)\omega = \omega$$

Thus ω is G-invariant. □

Remark 1.2.12. *On any principal G-bundle $\pi : P \to M$, $\pi \circ L_g = \pi\ \forall g \in G$. So basic forms are G-invariant. For let $\mu \in \Omega(M)$, $\omega \in \pi^*\Omega(M)$ be such that $\omega = \pi^*\mu$. Then $L_g^*\omega = L_g^*\pi^*\mu = (\pi \circ L_g)^*\mu = \pi^*\mu = \omega$.*

Definition 1.2.13. *Let $\pi : P \to M$ be a principal G-bundle. We say a form $\omega \in \Omega(P)$ is **horizontal** if at each point $p \in P$, ω vanishes whenever one of its arguments is vertical, i.e. if*

$$\iota_{X_p}\omega = 0\ \forall X_p \in \mathcal{V}_pP \tag{1.43}$$

Where $\mathcal{V}_pP$ is the vertical bundle of P at p.

The isomorphism $j_{p*} : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_pP$ (see Appendix A and A.2.5) gives an (often more useful) alternative definition of a horizontal form in terms of the basis $_1, ..., \xi_n$ of $\mathfrak{g}$. We say the form ω is horizontal if, for each $1 \leq i \leq n$,

$$\iota_i\omega = \iota_{\xi_i}\omega = \iota_{\xi_i^\#}\omega = 0 \tag{1.44}$$

where $\xi_i^\#$ is the fundamental vector field generated by ξ_i. Here ι_{ξ_i} can be expressed locally in terms of a basis $v_1, ..., v_n$ for T_pP by $\iota_{(\xi_i)_p} = \iota_{j_{p*}^{-1}(v_i)}$.

Proposition 1.2.14. *Basic forms are horizontal.*

Proof. Let $\mu \in \Omega(M), \omega \in \pi^*\Omega(M)$ be such that $\omega = \pi^*\mu$. Let $p \in P$, and let $X_p \in \mathcal{V}_p$ (i.e. π_*X_p) and $v_1, ..., v_{k-1} \in T_pP$. Then, using the properties of pullbacks and pushforwards,

$$\begin{aligned}(\iota_{X_p}\omega)(v_1, ..., v_{k-1}) = \omega(X_p, v_1, ..., v_{k-1}) &= \pi^*\mu(X_p, v_1, ..., v_{k-1}) \\ &= \mu(\pi_*X_p, \pi_*v_1, ..., \pi_*v_{k-1}) = 0\end{aligned}$$

□

We have shown that basic forms are horizontal and G-invariant. In fact, it can be shown that these conditions are equivalent.

Theorem 1.2.15 ([5], p.26)**.** *$\pi : P \to M$ be a principal G-bundle. A form $\omega \in \Omega(P)$ is basic if and only if it is horizontal and G-invariant.*

Definition 1.2.16. *. Let A be a G^* algebra. We denote the set of basic elements of A by A_{bas}.*

Proposition 1.2.17. *Let G be a connected Lie group. Any G^* algebra A satisfies $dA_{bas} \subseteq A_{bas}$.*

Proof. Let $\omega \in A_{\text{bas}}$. By (1.11), for all $\xi \in \mathfrak{g}$ we have

$$\begin{aligned} d\iota_\xi \omega + \iota_\xi d\omega &= \mathcal{L}_\xi \omega \\ \implies \iota_\xi d\omega &= \mathcal{L}_\xi \omega \text{ since } \omega \text{ is horizontal} \\ \implies \iota_\xi d\omega &= 0 \text{ since } \omega \text{ is } G\text{-invariant} \end{aligned}$$

So $d\omega$ is horizontal. Further, since d commutes with pullbacks, we have $(L_g^{-1})^* d\omega = d(L_g^{-1})^*\omega = d\omega$, so ω is G-invariant. Hence $d\omega$ is basic. □

We have shown that A_{bas} is closed under d, i.e. (A_{bas}, d) is a subcomplex (A, d). We denote the cohomology of (A_{bas}, d), called the **basic cohomology** of A, by either $H(A_{\text{bas}}, d)$ or $H_{\text{bas}}(A)$.

Corollary 1.2.18. . *We have shown that A_{bas} is a G^* subalgebra of A (alternative a G^* submodule if A is only a G^* module). Let $\phi : A \to B$ be a morphism of G^* modules. Then by* (A.4)*, we have $\phi(A_{bas}) \subset B_{bas}$, and so ϕ induces a linear map $\phi_* : H_{bas}(A) \to H_{bas}(B)$*

1.2.5.2 The Equivariant de Rham Theorem

We define equivariant cohomology groups in the setting of G^* algebras similar to how they are defined in the Borel construction. Let E be an acyclic, type (C) G^* algebra (we prove that such an algebra exists in Section 1.3.2). For any G^* algebra A, its equivariant cohomology ring $H_G(A)$ is the basic cohomology of $A \otimes E$:

$$H_G(A) := H_{\text{bas}}(A \otimes E) = H((A \otimes E)_{\text{bas}}, d) \tag{1.45}$$

We immediately see parallels with the Borel construction. The Borel equivariant cohomology is the *ordinary cohomology* of the quotient space of the *cartesian product* of a *topological space* with a *weakly contractible space* on which G acts *freely*. Here we have defined the equivariant cohomology groups as the *basic cohomology* of the *tensor product* of a *G^* algebra* with an *acyclic G^* algebra* that is type (C), which we have shown follows automatically when G is compact abelian acting locally freely.

De Rham's theorem asserts that, given a smooth manifold M, the de Rham cohomology of $\Omega(M)$ is equivalent to the singular cohomology of M. The equivalence of the Borel equivariant cohomology of M with the equivariant cohomology of $\Omega(M)$ as a G^* algebra is the content of the equivariant de Rham theorem.

Theorem 1.2.19 (Equivariant de Rham Theorem)**.** *Let G be a compact Lie group acting on a compact smooth manifold M. Then $H_G^*(M) = H_G(\Omega(M))$.*

Proof. See ([5], Theorem 2.5.1, p.28) or ([2], Chapter 6).

1.3 The Weil and Cartan Models

In the previous section we defined the equivariant cohomology of a G^* algebra A as the basic cohomology of $A \otimes E$, where E is an acylic type (C) G^* algebra. In the next section we show the existence of such a $G^* algebra$ in the same way how in the Borel construction we showed the existence of a contractible space E on which our group G acts freely in order to define the equivariant cohomology groups $H_G^*(X) = H^*(X \times_G E)$. This candidate is the *Weil algebra*. Unfortunately, the Weil algebra is not practical for computational purposes, and so later we will define a more useful construct called the *Cartan complex*.

1.3.1 Koszul Complex

We define the Weil algebra in terms of a more general structure called the Koszul algebra.

Definition 1.3.1. *Let V be an n-dimensional vector space over a field $\mathbb{F}$. Consider the exterior algebra $\bigwedge(V)$ of V and the symmetric algebra $S(V)$ of V consisting of tensor products of elements of V with even degree (but not in the usual sense). We grade $\bigwedge(V)$ in the usual way, but grade $S(V)$ by assigning to each element of $S^k(V)$ the degree $2k$. In other words, given a dual basis $\alpha^1, ..., \alpha^n$ of V, $S(V)$ is equivalent to $\mathbb{C}[\alpha^1, ..., \alpha^n]$. The product $K(V) = \bigwedge(V) \otimes S(V)$ is called the* ***Koszul algebra*** *of V.*

Given a basis $x_1, ..., x_n$ for V, the collection of elements

$$\theta^i := x_i \otimes 1,\; z^i := 1 \otimes x_i \tag{1.46}$$

gives a basis for $K(V) = \bigwedge(V) \otimes S(V)$. The ***Koszul complex*** *is the Koszul algebra is equipped with a derivation called the* ***Koszul differential****, denoted by d_K, defined on generators by*

$$d_K(\theta^i) = z^i,\; d_K(z^i) = 0 \tag{1.47}$$

It is clear that $d_K^2 = 0$ on generators, and hence on all of $K(V)$ using the property of d_K as a derivation on tensor products of elements in $K(V)$. This can be used to show the following useful result.

Proposition 1.3.2 ([5],p.33)**.** *The* ***Koszul complex*** *$(K(V), d_K)$ is acyclic.*

1.3.2 The Weil Algebra

Definition 1.3.3 (Weil Algebra)**.** *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. The* ***Weil Algebra*** *$W(\mathfrak{g})$, or W for short, is the Koszul algebra of $\mathfrak{g}^*$:*

$$W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \tag{1.48}$$

G acts canonically on $\mathfrak{g}$ by the adjoint representation and hence on $\mathfrak{g}^*$ be the coadjoint representation. Each coadjoint representation gives an automorphism on W. The aim of this section is to describe three superderivations on W analogous to $\mathcal{L}, \iota, d$ that make W into a G^* algebra. This requires some more exposition.

Let $\xi_1, ..., \xi_n$ be a basis for $\mathfrak{g}$, and $\alpha^1, ..., \alpha^n$ be the corresponding dual basis for $\mathfrak{g}^*$. There is a clear choice for the basis of $W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$, given by

$$\theta^i = \alpha^i \otimes 1, \quad z^i = 1 \otimes \alpha^i \tag{1.49}$$

and satisfying

$$d\theta^i = z^i, \quad dz^i = 0 \tag{1.50}$$

Proposition 1.3.4 ([5], p.24)**.** *Let G be a compact Lie group acting locally freely on a smooth manifold M. The forms α^i above satisfy*

$$\mathcal{L}_a \alpha^i = -c^i_{ab} \alpha^b \tag{1.51}$$

In particular, $\mathcal{L}_a \theta^i = -c^i_{ab} \theta^b$ and $\mathcal{L}_a z^i = -c^i_{ab} z^b$.

Theorem 1.3.5 ([5], Theorem 3.2.1, p.34-45)**.** *W is an acyclic type (C) G^* algebra.*

Proof. A basis $\xi_1, ...\xi_n$ for $\mathfrak{g}$ induces the generators θ^i, z^i of W given above. To make W into a G^* algebra, we need to define its Lie superalgebra $\widetilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by defining the actions of the derivations $d, \iota, \mathcal{L}$ on generators of W in order to satisfy the Weil equations (Eqs. (1.8)-(1.13)), then extending linearly to all of W.

We prescribe to the $\mathfrak{g}_1$ part the Koszul differential $d = d_K$, which is certainly G-equivariant. We assign to $\mathfrak{g}_0$ the ordinary Lie derivative $\mathcal{L}_a$, which is G-equivariant by Prop. 1.3.4. Next, we want to show that W is type (C) by establishing the θ^i as connection elements. In order for this to hold, the θ^i must satisfy Eq. (1.39), and so we *define* the action of ι_a on the θ^i to satisfy Eq. (1.39), i.e. $\iota_a\theta^i = \delta^i_a$. Clearly ι is also G-equivariant. In order to satisfy (1.11) $d\iota_a + \iota_a d = \mathcal{L}_a$, we must have

$$\iota_a z^i = \iota_a d\theta^i = (\iota_a d + d\iota_a)\theta^i = \mathcal{L}_a\theta^i = -c^i_{ab}\theta^b \tag{1.52}$$

It is then a matter of straightforward calculations to check that the remaining Weil equations hold. □

Definition 1.3.6. *Let A be a G^* algebra. Provided that we can construct the Weil algebra W, which we have shown is an acyclic type (C) G^* algebra, we can define the equivariant cohomology groups*

$$H_G(A) := H_{bas}(A \otimes W) = H((A \otimes W)_{bas}, d) \tag{1.53}$$

Theorem 1.3.7. $H_{bas}(W) = S(g^*)^G$

Proof. We first show $W = \bigwedge(\mathfrak{g}^*) \otimes W_{\text{hor}}$. Define $\mu^i := z^i + \frac{1}{2}c^i_{jk}\theta^j\theta^k$. μ^i is analogous to a curvature. Indeed, we will show that the μ^i, θ^i satisfy the 2nd structural equation and Bianchi identity. Further, note that $z^i = \mu^i - \frac{1}{2}c^i_{jk}\theta^j\theta^k$, so the θ^i, μ^i generate W. In particular,

$$W = \bigwedge(\theta^1, ..., \theta^n) \otimes \mathbb{C}[\mu^1, ..., \mu^n] = \bigwedge(g^*) \otimes S(\mathfrak{g}^*) \tag{1.54}$$

We have

$$\begin{aligned}
\iota_a\mu^i &= \iota_a z^i + \frac{1}{2}c^i_{jk}\iota_a(\theta^j\theta^k) \\
&= \iota_a z^i + \frac{1}{2}c^i_{jk}\delta^j_a\theta^k - \frac{1}{2}c^i_{jk}\theta^j\delta^k_a \\
&= \iota_a z^i + c^i_{a\ell}\theta^\ell = 0 \quad \text{(by Eq. (1.52))}
\end{aligned}$$

That is, the μ^b are horizontal. It is clear that an element of $\bigwedge(\theta^1, ..., \theta^n)$ cannot be horizontal by definition of the θ^i, so we have $W_{\text{hor}} = S(\mathfrak{g}^*)$ and, in particular, $W_{\text{bas}} = S(\mathfrak{g}^*)^G$.

The Jacobi identity, along with the observation that the element $c^i_{jk}\xi\theta^j \otimes \theta^k$ is exactly the image of the Lie bracket, can be used to show that (see [5], Theorem 3.2.2, p.35)

$$\mathcal{L}_a\mu^i = -c^i_{ab}\mu^b \tag{1.55}$$

From here we show that d acts trivially on W_{bas}. By definition of μ^i and Eq. (1.50), we have

$$d\theta^i = -\frac{1}{2}c^i_{jk}\theta^j\theta^k + \mu^i$$

From which it follows that

$$\begin{aligned}
d\mu^i &= \frac{1}{2}c^i_{jk}d\theta^j\theta^k - \frac{1}{2}c^i_{jk}\theta^j d\theta^k \\
&= \frac{1}{2}c^i_{jk}(-\frac{1}{2}c^j_{\ell k}\theta^\ell\theta^k + \mu^j)\theta^k - \frac{1}{2}c^i_{jk}\theta^j(-\frac{1}{2}c^k_{j\ell}\theta^j\theta^\ell + \mu^k) \\
&= \frac{1}{2}c^i_{jk}\mu^j\theta^k - \frac{1}{2}c^i_{jk}\theta^j\mu^k
\end{aligned}$$

The terms not involving μ cancel and we are left with

$$d\mu^i = -c^i_{jk}\theta^j\mu^k \tag{1.56}$$

Putting Eqs.(1.55) and (1.56) together gives $d\mu^i = (\theta^a\mathcal{L}_a)\mu^i$. Since the μ^i generate W_{hor} we have

$$dx = \theta^a\mathcal{L}_a x \quad \forall x \in W_{\text{hor}} \tag{1.57}$$

Moreover, if $x \in W_{\text{bas}} = (S(\mathfrak{g}^*)^G$, then $\mathcal{L}_a x = 0$. So $dx = 0$ for all $x \in W_{\text{bas}}$ and so $H(W_{\text{bas}}, d) = W_{\text{bas}} = S(\mathfrak{g}^*)^G$. □

1.3.2.1 The Chern-Weil Map

Given any G^* algebra A, it is useful to be able to make A into an algebra over W.

Theorem 1.3.8. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $W(\mathfrak{g})$ its Weil algebra. Let A be a type (C) G^* algebra. Then there exists a G^* algebra homomorphism $\rho : W \to A$. Any two such homomorphisms are chain homotopic.*

Proof. Choose connection elements θ^i of $W(g)$ and θ^i_A of A. Define ρ on connection elements by

$$\rho(\theta^i) = \theta^i_A \tag{1.58}$$

ρ has a unique extension to a G^* homomorphism on $W(\mathfrak{g})$, since the $\theta^i, d\theta^i$ generate $W(\mathfrak{g})$ and ρ commutes with δ. For uniqueness of ρ up to chain homotopy, see ([5], Theorem 3.3.1, p. 37-38). □

Let $\omega \in W(\mathfrak{g})_{\text{bas}}$. Since ρ is a G^*-morphism, for each $\xi \in \mathfrak{g}$, $g \in G$, we have

$$\begin{aligned}
&0 = [\iota_\xi, \rho](\omega) = \iota_\xi\rho(\omega) \\
&0 = [(L_g^{-1})^*, \rho](\omega) = (L_g^{-1})^* \circ \rho(\omega) - \rho \circ (L_g^{-1})^*\omega = (L_g^{-1})^* \circ \rho(\omega) - \rho(\omega) \\
\implies &(L_g^{-1})^* \circ \rho(\omega) = \rho(\omega)
\end{aligned}$$

So ρ maps $W(\mathfrak{g})_{\text{bas}} = S(\mathfrak{g}^*)^G$ into A_{bas}. Further, any other choice of G^* homomorphism ρ' is chain homotopic in ρ (i.e. identical in cohomology). This gives a canonical map, called the **Chern-Weil map** or **characteristic homomorphism**

$$\kappa_G : S(\mathfrak{g}^*)^G \to H_G(A) \tag{1.59}$$

By taking the image of elements in $S(\mathfrak{g}^*)^G$ in cohomology followed by their image under ρ_*. The elements in the image of κ_K are called **equivariant characteristic classes**. We will touch more on this later.

1.3.2.2 Commuting Actions

The Chern-Weil map has a useful application to commuting actions. Suppose G, K are compact Lie groups with smooth commuting actions on a manifold M. If the K action is free, M/K is a manifold, and the Chern-Weil map gives a map $\kappa_K : S(\mathfrak{k}^*) \to H_G(M/K)$. For more details see ([5], p.49-50,104).

1.3.2.3 W^* Modules

Given any type (C) G^* algebra A, the homomorphism $\rho : \to A$ given in Eq. (1.58) makes A into an algebra over W. Obviously this method fails if A is not of type C because we cannot construct the connection elements. We present a more general way of turning a G^* algebra (or module) into a module over W.

Definition 1.3.9. *A W^* **module (algebra)** is a G^* module (algebra) A which is also a module over W such that the map*

$$W \otimes A \to A,\ w \otimes a \mapsto wa \tag{1.60}$$

is a G^* module (algebra) morphism.

Again, the canonical example of a W^* module is the de Rham complex $\Omega(M)$. The following theorem gives a general visualization of W^* modules.

Theorem 1.3.10 ([5], Theorem 3.4.1, p.40)**.** *Let B be a W^* module and let $\theta^1, ..., \theta^n \in W_1$ be connection elements. Every element of B can be written uniquely as a sum of monomials $\theta^I \omega_I$, where I is a multi-index and $\omega_I \in B_{hor}$.*

1.3.3 The Cartan Model

We will now discuss the Cartan model. But first, a progress check. Suppose we have a G^* module B, and wish to find $H_G(B)$. We have seen so far that the Weil algebra W is acyclic, i.e. all of its cohomologies are trivial apart from H^0. Hence we can define $H_G(B) = H((B \otimes W)_{\text{bas}}, d)$ as the algebraic equivalent of $H^*_G(B) = H^*(B \times_G E)$. Unfortunately, $(B \otimes W)_{\text{bas}}$ is in general difficult to compute. The purpose of the Cartan model is to make this computation more tractable by introducing a more intuitive model called the *Cartan model*, which we can link to the Weil model via the *Mathai-Quillen isomorphism*. For ease of computation, throughout this section we will work with $(W \otimes B)_{\text{bas}}$ instead of $(B \otimes W)_{\text{bas}}$.

1.3.3.1 The Mathai-Quillen Isomorphism

Definition 1.3.11. *Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra with basis $\xi_1, ..., \xi_n$. Let W be its Weil algebra. Suppose A is a W^* module and B is a G^* module, and let*

$$\theta^1, ..., \theta^n \in W_1, \quad \mu^1, ..., \mu^n \in W_2$$

be connection and curvature generators of W. We define the endormorphism $\gamma \in End(A \otimes B)$ by

$$\gamma := \theta^a \otimes \iota_a \tag{1.61}$$

which has degree zero since θ^a raises bidegree by 1 and ι_a lowers bidegree by 1. Further, γ is nilpotent ($\gamma^{n+1} = 0$), for after $n+1$ iterations the ι_a part kills off any element. The ***Mathai-Quillen isomorphism*** *$\phi \in Aut(A \otimes B)$ is defined as*

$$\phi := exp\, \gamma = 1 + \gamma + \frac{1}{2}\gamma^2 + ... + \frac{1}{n!}\gamma^n \tag{1.62}$$

ϕ is an automorphism in the sense that is preserves W^* and G^* structure. Equivariance of ϕ follows from equivariance of γ. The following two properties of ϕ follow from a series of uninteresting calculations.

Theorem 1.3.12 ([5], Theorem 4.1.1, p.42-44)**.** *The Mathai-Quillen isomorphism satisfies*

$$\begin{aligned} \phi(1 \otimes \iota_\xi + \iota_\xi \otimes 1)\phi^{-1} &= \iota_\xi \otimes 1 \; \forall \xi \in \mathfrak{g} \phi d \phi^{-1} && = d - \mu^a \otimes \iota_a + \theta^a \otimes \mathcal{L}_a \\ &= d_A \otimes 1 + 1 \otimes d_B - \mu^a \otimes \iota_a + \theta^a \otimes \mathcal{L}_a \end{aligned} \tag{1.63}$$

We are now ready to define the Cartan complex.

1.3.3.2 The Cartan Complex

Theorem 1.3.13. *Let B be a G^* algebra. Then ϕ maps the basic elements of $W \otimes B$ onto the G-invariant elements of $S(\mathfrak{g}^*) \otimes B$:*

$$\phi : (W \otimes B)_{bas} \to (S(\mathfrak{g}^*) \otimes B)^G \tag{1.64}$$

Proof. Let A, B be arbitrary W^* and G^* algebras respectively. Let $\alpha \otimes \beta \in A \otimes B$. Eq. (**??**) gives

$$\begin{aligned} &\phi(1 \otimes \iota_a + \iota_a \otimes 1)(\alpha \otimes \beta) = \iota_a \otimes 1(\alpha \otimes \beta) \\ \implies &\phi(\alpha \otimes \iota_a \beta + \iota_a \alpha \otimes \beta) = \iota_a \phi(\alpha) \otimes \phi(\beta) \\ \implies &\phi(0) = 0 = \iota_a \phi(\alpha) \otimes \phi(\beta) \\ \implies &\iota_a \phi(\alpha) = 0 \end{aligned}$$

since ϕ is an isomorphism. So we have the mapping $\phi : (A \otimes B)_{\text{hor}} \to A_{\text{hor}} \otimes B$. Now let $A = W$. Then

$$\phi : (W \otimes B)_{\text{hor}} \to W_{\text{hor}} \otimes B = S(\mathfrak{g}^*) \otimes B$$

Note the restriction of d_W to W_{hor} is given by $d = \theta^a \mathcal{L}_a$ (Eq. (1.57)). Combining this with (1.63) gives

$$\phi d \phi^{-1} = \theta^a \mathcal{L}_a + 1 \otimes d_B + \theta^a \otimes \mathcal{L}_a - \mu^a \otimes \iota_a = (\theta^a \otimes 1)(\mathcal{L}_a \otimes 1 + 1 \otimes \mathcal{L}_a) + 1 \otimes d_B - \mu^a \otimes \iota_a \tag{1.65}$$

Since ϕ is G-equivariant, for each $g \in G, \omega \in W \otimes B$ we have

$$(L_g^{-1})^* \phi(\omega) = \phi((L_g^{-1})^* \omega) = \phi(\omega)$$

i.e. ϕ maps invariant elements to invariant elements. Therefore

$$\phi : (W \otimes B)_{\text{bas}} \to (W_{\text{hor}} \otimes B)^G = (S(\mathfrak{g}^*) \otimes B)^G \tag{1.66}$$

Observe that on $(S(\mathfrak{g}^*)\otimes B)^G$ is important, the $\mathcal{L}_a\otimes 1+1\otimes\mathcal{L}_a$ part of (1.65) vanishes on this space. Hence

$$\phi d\phi^{-1}=1\otimes d_B-\mu^a\otimes\iota_a$$

on $(S(\mathfrak{g}^*)\otimes B)^G$. Further, given $\alpha\otimes\beta\in(S(\mathfrak{g}^*)\otimes B)^G$,

$$\phi d\phi^{-1}(\alpha\otimes\beta)$$

is certainly G-invariant, since ϕ preserves invariance and d commutes with pullbacks. Further

$$(1\otimes d_B-\mu^a\otimes\iota_a)(\alpha\otimes\beta)=\alpha\otimes d_B\beta-\mu^a\alpha\otimes\iota_a\beta$$

is clearly in $S(\mathfrak{g}^*)\otimes B$ since the μ^a generate $S(\mathfrak{g}^*)$. □

This motivates the definition of the *Cartan complex*.

Definition 1.3.14. *Let B be a G^* module. The space*

$$C_G(B):=(S(\mathfrak{g}^*)\otimes B)^G \tag{1.67}$$

together with the differential

$$d_G:C_G(B)\to C_G(B),\quad d_G=1\otimes d_B-\mu^a\otimes\iota_a \tag{1.68}$$

is called the ***Cartan complex*** *or the* ***Cartan model*** *for the equivariant cohomology of B. We have shown that the Mathai-Quillen isomorphism ϕ restricts to an isomorphism $(W\otimes B)^G_{hor}=(W\otimes B)_{bas}$ onto $(S(g^*)\otimes B)^G=C_G(B)$ and conjugates $d|_{W_{hor}\otimes B}$ into d_G. This induces an isomorphism of cohomology groups*

$$H^*((W\otimes B)_{bas},d)\cong H^*(C_G(B),d_G) \tag{1.69}$$

One can view elements $\omega\in C_G(B)$ as polynomials $\omega(\xi)$ in $\mathfrak{g}$ with coefficients in B. In this sense we can write d_G as

$$d_G(\omega)(\xi)=d_B(\omega(\xi))-\iota_\xi(\omega(\xi)) \tag{1.70}$$

Note that d_G is a derivation of degree one, since

$$1\otimes d_B:S^k(g^*)\otimes B_\ell\to S^k(\mathfrak{g}^*)\otimes B_{\ell+1} \tag{1.71}$$

$$\mu^a\otimes\iota_a:S^k(g^*)\otimes B_\ell\to S^{k+1}(\mathfrak{g}^*)\otimes B_{\ell-1} \tag{1.72}$$

So $1\otimes d_B$ and $\mu^a\otimes\iota_a$ change the total degree $2k+\ell$ of a form to $2k+\ell+1$ and $2(k+1)+(\ell-1)=2k+\ell+1$. Either way d_G raises the degree of forms by 1. Note that the $\mu^a\otimes\iota_a$ is a sum over the basis of $\mathfrak{g}$. In the one dimensional case, for example $G=\mathbb{S}^1$, $\mathfrak{g}=i\mathbb{R}$, we could simply write $d_G=1\otimes d_B+\mu\otimes\iota_\xi$ where ξ,μ generate $i\mathbb{R}$, $S(\mathfrak{g}^*)$ respectively.

Remark 1.3.15. *G acts on $C_G(A)$ via a diagonal action, acting on the $S(\mathfrak{g}^*)$ part via the adjoint action and on A via the representation ρ on A as a G^* module. Let $g\in G$ and $\omega=\xi\otimes\omega_i$, $\xi\in\mathfrak{g}$, $\omega_i\in A$. Then*

$$g\cdot\omega=Ad_g\xi_i\otimes\rho(g)\omega_i \tag{1.73}$$

In the case of $A=\Omega(M)$, we have $g\cdot\omega=Ad_g\xi_i\otimes(L_{g^{-1}})^*\omega_i$.

1.3.3.3 Equivariant Cohomology of W^* Modules

A core result in the Borel construction is that, given a smooth free action of a compact Lie group G on a topological space M, we have

$$H^*_G(M) = H^*(M/G)$$

The following is an equivalent statement about equivariant cohomology in the context of W^* modules, which essentially says that the definition of $H_G(A)$ in Eq. (1.45) is independent of the choice of acyclic G^* algebra E.

Theorem ([5], Theorem 4.3.1, p.46). Let A be a W^* module and E an acyclic W^* algebra. Then

$$H^*((A \otimes E)_{\text{bas}}, d) = H^*(A_{\text{bas}}) \tag{1.74}$$

Therefore it is conventional to define equivariant cohomology groups in terms of the Weil algebra W.

Definition 1.3.16. *Let A be a W^* module. We define the equivariant cohomology of A by*

$$H_G(A) = H^*_{bas}(A \otimes W) = H^*(A_{bas}) \tag{1.75}$$

1.3.3.4 Equivariant Forms and Equivariantly Closed Extensions

In the case of the G^* algebra $\Omega(M)$ we have the following crucial definition.

Definition 1.3.17. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M a smooth G-manifold. The set of **equivariant differential forms** is $\Omega_G(M)$ the subspace of elements of $S(\mathfrak{g}^*) \otimes \Omega(M)$ that are equivariant with respect to the coadjoint representation of G and the representation $\rho(g) = (L_{g^{-1}})^*$ in the sense that, given a form $\omega = \sum_i \mu^i \otimes \omega_i \in S(\mathfrak{g}^*) \otimes \Omega(M)$ with $\omega_k \in \Omega^k M)$, we have $(L_{g^{-1}})^*\omega = Ad^*_g \omega$, or*

$$\sum_{k=0}^{m} \mu^{m-k} \otimes \omega_k \circ L_{g^{-1}} = \sum_{k=0}^{m} Ad^*_g \mu^{m-k} \otimes \omega_k \tag{1.76}$$

Note that if G is abelian, e.g. a torus, then the coadjoint action is trivial and $\Omega_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G = C_G(\Omega(M))$. We will show in the next section that WLOG this can be assumed to be the case.

It is possible to extend any ordinary closed differential form $\omega \in \Omega(M)$ to an equivariantly closed form $\widetilde{\omega}$ by adding on forms so that $d_G\widetilde{\omega} = (1 \otimes d + \mu^a \otimes \iota_a)\widetilde{\omega} = 0$. This will be useful at the end once we have proved the localization formula. For example, suppose M is a smooth $2m$-dimensional manifold and $\omega \in \Omega_G(M)$, where G, and hence $\mathfrak{g}$, is one-dimensional. Suppose we write the extension more generally as $\widetilde{\omega} = \sum_{k=0}^{m} \omega_k$, where each $\omega_k \in S(\mathfrak{g}^*) \otimes \Omega(M)$. By comparing same-degree terms and referring to Eqs. (1.71)-(1.72), we see that the requirement that $d_G\widetilde{\omega} = 0$ is equivalent to

$$d\omega_m = 0, \quad \iota_\xi \omega_m = d\omega_{m-2}, \quad \iota_\xi \omega_{m-2} = d\omega_{m-4}, \quad ..., \quad \iota_\xi \omega_2 = d\omega_0 \tag{1.77}$$

In the case that G is abelian and $\Omega_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G$, we have

$$H_G(\Omega(M)) = H^*((\Omega(M) \otimes W)_{\text{bas}}, d) = H^*(\Omega(M)_{\text{bas}}, d)$$

But also

$$H^*((\Omega(M) \otimes W)_{\text{bas}}, d) = H^*(C_G(\Omega(M)), d_G) = H^*(\Omega_G(M), d_G)$$

So

$$H^*(\Omega(M)_{\text{bas}}, d) = H^*(\Omega_G(M), d_G) \tag{1.78}$$

In other words, the de Rham cohomology of $\Omega_G(M)$ with respect to d_G is the same as the de Rham cohomology of $\Omega(M)_{\text{bas}}$ with respect to d.

1.3.3.5 The Cartan Map

We take a brief digression into defining the *Cartan map*, which will be useful later in Section 2.3.3 in constructing an equivariant Thom form.

Proposition/Definition 1.3.18. *Let A be a type (C) W^* module with connection elements $\theta^1, ..., \theta^n \in W_1$ and curvature elements $\mu^1, ..., \mu^n \in W_2$ that act on A in the standard way described in Eq.* (1.60)*. Define the horizontal projection operator*

$$H : C_G(A) \to (S(\mathfrak{g}^* \otimes A_{hor})^G, \quad x^I \otimes \omega \mapsto x^I \otimes (\prod_a \iota_a \theta^a)\omega \tag{1.79}$$

where I is a multi-index, and the basic projection operator

$$B : (Sg^* \otimes A)^G \to A_{bas}, \quad x^I \otimes \omega \mapsto \mu^I \omega \tag{1.80}$$

We define the Cartan map

$$\pi_C : C_G(A) \to A_{bas}, \quad \pi_C = H \circ B \tag{1.81}$$

Proof. Let $\omega \in A$. We show that $(\prod_a \iota_a \theta^a)\omega$ is indeed horizontal. From Eq. (1.8) we have $\iota_b \iota_a = -\iota_b \iota_a$, and in particular $\iota_a^2 = 0$. So

$$\iota_b \prod_a \iota_a \theta^a w = 0$$

Since if $a \neq b$ then ι_a, ι_b commute, and $\iota_a^2 = 0$. Further, G-invariance follows immediately from G-invariance of ω. Finally, the fact that the elements $\mu^I a$ in the image of B follows from the fact that the μ^i are basic. □

Example 1.3.19. *Let G be a compact Lie group with a smooth free action on a smooth manifold M. Consider the manifold $X = M/G$ and the projection $\pi : M \to X$. By definition, the set of basic forms in $\Omega(M)$ is $\Omega(M)_{bas} = \pi^*(\Omega(X))$. Now consider the map*

$$\kappa_X = \pi_C \circ (\pi^*)^{-1} \tag{1.82}$$

induced by $(\pi^*)^{-1}$. Certainly π_C is G-equivariant in the sense that, for all $g \in G$, $f \otimes \omega \in C_G(A)$, $(L_{g^{-1}})^* \pi_C = \pi_C (L_{g^{-1}})^*$. Further, by definition, both $\Omega(X)$ and $\Omega(M)_{bas}$ are G-invariant. Hence we have a G-equivariant map $\kappa_X : \Omega_G(M) \to \Omega(X)$.

1.4 Abelianization

The key result of this section is an isomorphism between the equivariant cohomology groups $H_G(A)$ for a compact connected Lie group G, and the Weyl group invariant elements of $H_T(A)$ for the maximal tori $T \subseteq G$. In other words, the equivariant cohomology of a smooth manifold M with respect to a compact connected Lie group G is identical to the equivariant cohomology with respect to the Weyl-invariant subset of any maximal abelian subgroup of G.

We first need an important definition.

Definition 1.4.1 (Weyl Group)**.** *Let T be a Cartan subgroup of G, and $K = N_G(T)$. Then T is abelian, hence a normal subgroup. The quotient group*

$$W := K \backslash T \tag{1.83}$$

*is called the **Weyl group** of T.*

Theorem 1.4.2 (Abelianization of compact connected Lie groups)**.** *Let G be a compact connected Lie group, T a maximal torus in G and W its Weyl group. Then for any G^*-module A we have*

$$H_G(A) \cong H_T(A)^W \tag{1.84}$$

Proof. It is well known that the Weyl group is finite. For a proof see ([6], Theorem IV.1.5, p.158). Let $W = \{w_1T, ..., w_rT\}$ for some $w_i \in \mathbb{K}$. Then $K = \bigcup_{i=1}^r w_iT$. K contains e since K is a group, but the cosets w_iT are disjoint, so exactly one of the w_iT contains e. Thus

$$\text{Lie}(K) = T_eK = T_ew_iT \cong T_eT = \text{Lie}(T)$$

So K and T have isomorphic Lie algebras $\mathfrak{k}$,$\mathfrak{t}$. But $K \leq T \implies \mathfrak{k} \leq \mathfrak{t}$, which have have the same dimension, so $\mathfrak{k} = \mathfrak{t}$. Since T is abelian, its coadjoint action on $\mathfrak{t}^*$, hence on $S(\mathfrak{t}^*)$, is trivial, since $tXt^{-1} = Xtt^{-1} = X \; \forall t \in T, X \in \mathfrak{t}$. So we have

$$S(\mathfrak{k}^*)^K = S(\mathfrak{t}^*)^K = S(\mathfrak{t}^*)^W \tag{1.85}$$

By the Chevalley restirction theorem (Thm. C.1.15), $S(\mathfrak{g}^*)^G \cong S(\mathfrak{t}^*)^W$, so by Thm. B.2.1, $H_G(A) \cong H_K(A)$. Now, the inclusion $T \hookrightarrow K$ induces a morphism of double complexes $C_K(A) \to C_T(A)^W$, which induces a morphism $H_K(A) \to H_T(A)^W$, and hence a morphism at each stage of the spectral sequences. By (1.85), the E_1 morphism is just the identity morphism

$$\rho_1 : S(\mathfrak{t}^*)^W \otimes H(A) \to S(\mathfrak{t}^*)^W \otimes H(A)$$

since K, and hence W, acts trivially on $H(A)$. Provided the spectral sequence converges (which it does in the case of compact connected G by Thm. B.1.20, the gives an isomorphism at every stage of the spectral sequence. Hence, by Thm. B.1.14,

$$H_K(A) = H_T(A)^W$$

combining this with Eq. (B.2.1) gives the desired result. $\square$

In light of this result we can assume that our compact connected Lie groups G are also abelian, i.e. G is a torus.

Chapter 2

Localization

2.1 Equivariant Tubular Neighbourhoods

Definition 2.1.1. *Let S be a submanifold of a smooth manifold M, and let ι denote the embedding $S \hookrightarrow M$. A* ***tubular neighbourhood*** *of S in M is a vector bundle $\pi : E \to S$, together with a smooth map $J : E \to M$, such that*

1. *$J \circ i_0 = \iota$, where $i_0 : S \to E$ is the zero section mapping every element $s \in S$ to the zero element of the fiber $\pi^{-1}(s)$.*
2. *There exists $U \subseteq E$, $V \subseteq M$ satisfying $i_0(S) \subseteq U$ and $S \subseteq V$, such that $J|_U : U \to V$ is a diffeomorphism.*

Theorem 2.1.2 (Equivariant Tubular Neighbourhood Theorem, [7], Theorem 2.2, p. 306)**.** *Let G be a compact Lie group acting smoothly on a smooth manifold M. Let A be an embedded G-invariant submanifold of M. Then A has a G-invariant tubular neighbourhood U.*

By G-invariant we mean for all $p \in U$, $g \in G$, $g \cdot p \in U$. Tubular neighbourhoods are advantageous because sometimes we wish to define differential forms on subspaces of manifolds that cannot be endowed with the structure of a smooth submanifold (e.g. on the connected components of the fixed point set M^G, which we will see later in the proof of the localization formula). But tubular neighbourhoods, being open, *can* be endowed with the structure of a smooth submanifold. Further, since U is G-invariant, the action $L^*_{g^{-1}}\omega$ of G on forms $\Omega_G(U)$ does not change the domain of ω. The most important example of an equivariant tubular neighbourhood is the normal bundle of an embedded submanifold, discussed below.

Corollary 2.1.3 ([10], Corollary 1.28, p.12)**.** *The normal bundle $\mathcal{N}$ of A is a tubular neighbourhood of A. In particular, there exists a tubular neighbourhood $A \subseteq U \subseteq M$ such that we have a K-invariant diffeomorphism $J : \mathcal{N} \to U$. In this way we can think of $\mathcal{N}$ as a subspace of M.*

2.2 Characteristic Classes

In this section we give a brief overview of the theory of characteristic classes. This is necessary in order to define the *equivariant Euler class*, which is crucial in the statement of the localization formula.

We know from the theory of prinipal G-bundles that, given a principal G-bundle $P \to X$, the following diagram commutes

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

where $f : X \to BG$ is a classifying map. Consider a ring R. In singular cohomology R could be $\mathbb{Z}, \mathbb{R}, \mathbb{C}$, etc. From f we get an induced map in cohomology $H^*(\cdot, R)$ (the cohomology classes of $\cdot$ with coefficients in R) via pullback:

$$f^* : H^*(BG, R) \to H^*(X, R) \tag{2.1}$$

A **characteristic class** of $P \to X$ is an element $f^*[c] \in H^*(X, R)$, where $[c] \in H^*(BG, R)$. The elements of $H^*(BG, R)$ are aptly named **universal characteristic classes**. These characteristic classes can in fact be constructed in terms of arbitrary vector bundles, and go by different names depending on the ring R, the type of principal G-bundle, and the order of the cohomology class they belong to. Examples of characteristic classes (some of which we will see below) include:

- Stiefel-Whitney classes $w_n(P) \in H^n(X, \mathbb{Z}_2)$ for real vector bundles
- Chern classes $c_n(P) \in H^{2n}(X, \mathbb{Z})$ for complex vector bundles
- Pontryagin classes $p_n(P) \in H^{2n}(X, \mathbb{Z})$ for complex vector bundles
- Euler classes $e_n(P) \in H^n(X, \mathbb{Z})$ for an oriented n-dimensional real vector bundle

Of course, if we are only considering even-dimensional vector bundles then real and complex vector bundles are interchangeable.

We have already seen that an important group in the theory of equivariant cohomology is $U(n)$, since any compact finite-dimensional Lie group can be embedded in $U(n)$ for some n. Two other important groups are the group $O(n)$ and the subgroup of determinant 1 matrices $SO(n)$. Note that $U(2n) \leq SO(2n) \leq O(2n)$. The injective homomorphisms $U(2n) \to SO(2n) \to O(2n)$ induce Lie algebra homomorphisms $\mathfrak{u}(2n) \to \mathfrak{o}(2n) \to \mathfrak{so}(2n)$. These induce homomorphisms in the dual Lie algebras in the opposite direction which can be extended linearly to homomorphisms $S(\mathfrak{o}^*(2n))^{O(2n)} \to S(\mathfrak{so}^*(2n))^{SO(2n)} \to S(\mathfrak{u}^*(2n))^{2n}$. Here we can identify $\mathfrak{u}(n)$ with the space of complex self-adjoint matrices, and $\mathfrak{o}(2n), \mathfrak{so}(2n)$ with the space of complex skew-Hermitian (skew-symmetric) $n \times n$ complex ($2n \times 2n$ real) matrices.

We have seen an abstract topological definition of characterisitic classes. In the next section we describe how certain characteristic classes can be constructed algebraically on principal G-bundles for certain compact matrix Lie groups G.

Definition 2.2.1. *To any vector bundle $E \to X$ there is an associated principal fiber bundle called a **frame bundle**, given by the set of fiberwise frames of E. The fiber of $F(E)$ over a point $x \in X$ is the set of bases of E_x. In other words we have a fiber structure $\pi : X(E) \to X$ given by $\pi((x, \boldsymbol{e} = (e_1, ..., e_n))) = x$.*

Definition 2.2.2. *Let $E \to X$ be a complex vector bundle. Given a hermitian structure on E (that is, a smooth, positive-definite complex inner product on each fiber E_x), let $M = \mathcal{F}(E)$ denote the unitary frame bundle of E, i.e. the sub-bundle of $F(E)$ consisting of orthonormal frames of E. As before, the points of M are tuples $(x, \boldsymbol{e})$ where $x \in E$ and $\boldsymbol{e} = (e_1, ..., e_n)$ is an orthonormal basis for E_x.*

Consider the Lie compact group $G = U(n)$. We define a right action of G on M given, for all $A \in G$, by

$$(x, e_1, ..., e_n) \mapsto (x, e_1 A, ..., e_n A) \tag{2.2}$$

Clearly this gives M the structure of a principal G-bundle over X, and hence the characteristic homomorphism gives a map

$$\kappa : S(\mathfrak{g}^*)^G \to H^*(X) = H^*_G(M) \tag{2.3}$$

whose image im κ is a subring of $H^*(X)$ which we call the **ring of characteristic classes** of the vector bundle E.

2.2.1 Chern Class

Let G be a matrix Lie group and $A \in \mathfrak{g}$. It is well known that the characteristic polynomial takes the form

$$\det(\lambda I - A) = \lambda^n - c_1(A)\lambda^{n-1} + ... + c_n(A) \tag{2.4}$$

where the c_i are the coefficients of $(-1)^k \lambda^{n-k}$ and are polynomials in the eigenvalues of A, for example $c_1(A) = \text{trace}(A)$ and $c_n(A) = \det(A)$. It is known that for $G = U(n), O(2n), SO(2n)$, the c_i generate the ring $S(\mathfrak{g}^*)$.

2.2.2 Pontryagin Class

Let $G = SO(n)$. $\mathfrak{g} := \mathfrak{so}(n)$ and hence $\mathfrak{g}^*$ can be regarded as the space of skew-symmetric real $n \times n$ matrices. We first note that *any* matrix A satisfies $\det(A) = \det(A^T)$, so

$$\det(\lambda I - A^T) = \det(\lambda I^T - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A)$$

So, for a real skew-symmetric matrix $A \in \mathfrak{g}*$, we have

$$\det(\lambda I - A) = \det(\lambda I - A^T) = \det(\lambda I + A)$$

and hence

$$\begin{aligned} &\lambda^n - c_1(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} + ... + (-1)^n c_n(A) \\ =&\lambda^n - c_1(-A)\lambda^{n-1} + c_2(-A)\lambda^{n-2} + ... + (-1)^n c_n(-A) \end{aligned}$$

It is known that, again for any matrix A over a field $\mathbb{K}$, the polynomials c_i satisfy $c_i(\alpha A) = (\alpha)^i c_i(A)$ for all $\alpha \in \mathbb{K}$, so we have

$$\begin{aligned} &\lambda^n - c_1(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} + ... + (-1)^n c_n(A) \\ =&\lambda^n + c_1(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} + ... + (-1)^{2n} c_n(A) \\ =&\lambda^n + c_1(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} + ... + c_n(A) \end{aligned}$$

The even degree terms cancel, and so the odd degree terms must vanish. Thus we can write

$$\det(\lambda I - A) = \lambda^n + p_1(A)\lambda^{n-2} + p_2(A)\lambda^{n-4} + ... \tag{2.5}$$

where $p_i(A) = c_{2i}(A)$. The characteristic classes corresponding to the p_i for a real vector bundle are called its **Pontryagin classes**. It is known (see [5], p.97) that the p_i are invariant under the adjoint action of G.

2.2.3 Pfaffian

Consider the Lie group $\mathrm{SO}(2n)$. Let $V = \mathbb{R}^{2n}$. Note that $SO(2n)$ acts on $\mathfrak{so}(2n)$ via the adjoint representation, given for fixed $U \in \mathrm{SO}(2n)$ and any $A \in \mathfrak{so}(2n)$ by $A \mapsto UAU^{-1}$. For each $A \in \mathfrak{so}(2n)$ we define a map ω_A given, for each $v, w \in V$, by

$$\omega_A(v, w) := \langle Av, w \rangle \tag{2.6}$$

By the standard relationship of inner products with skew-symmetric matrices, we have

$$\omega_A(w, v) = \langle Aw, v \rangle = \langle v, Aw \rangle = -\langle Av, w \rangle = -\omega_A(w, v) \tag{2.7}$$

So ω_A defines a skew-symmetric bilinear form on V, i.e. $\omega_A \in \bigwedge^2(V^*)$. Now consider the form

$$\frac{1}{n!}\omega_A^n \tag{2.8}$$

of $\bigwedge^{2n}(V^*)$. Note that ω_A only depends on the inner product on V and is hence invariant under the action of $SO(2n)$. We next define the degree $2n$ **volume form** $\mathrm{vol} \in \bigwedge^{2n} V^*$

$$\mathrm{vol} := e_1^* \wedge e_2^* \wedge ... \wedge e_{2n}^* \tag{2.9}$$

where the $e_1^*, ..., e_{2n}^*$ are a dual basis for V^*. We define the **Pfaffian** of A, denoted by $\mathrm{Pfaff}(A)$, to satisfy

$$\frac{1}{n!}\omega_A^n = \mathrm{Pfaff}(A)\mathrm{vol} \tag{2.10}$$

Since $\frac{1}{n!}\omega_A^n$ and vol are $\mathrm{SO}(2n)$-invariant ($\mathrm{SO}(2n)$ preserves the volume form), clearly $\mathrm{Pfaff}(A)$ is $\mathrm{SO}(2n)$-invariant.

It is a well-known fact that, given a $2n \times 2n$ skew-symmetric matrix A over $\mathbb{R}$, there exists a basis of V such that, in terms of this basis, A can be transformed into a block diagonal matrix via a linear transformation that leaves $\det A$ unchanged ([5], p.98). That is, for each $A \in \mathfrak{so}(2n)$, we can find an orthonormal oriented basis of V, and hence an orthonormal dual basis $e_1^*, ..., e_{2n}^*$ of V^* such that A can be represented as a block-diagonal matrix in terms of this basis by (see [5], p.98 or [11], p.16)

$$\begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix} \tag{2.11}$$

for some eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$. ω_A can be expressed in terms of this basis as

$$\omega_A = \lambda_1 e_1^* \wedge e_2^* + \lambda_2 e_3^* \wedge e_4^* + ... + e_{2n-1}^* \wedge e_{2n}^* \tag{2.12}$$

ω_A^n is a sum of products of $2n$ elements of the dual basis, all except for one of which contain repeated occurrences of some e_j^*. Therefore the only surviving term in ω_A^n is

$$n!(\lambda_1 e_1^* \wedge e_2^*) \wedge (\lambda_2 e_3^* \wedge e_4^*) \wedge ... \wedge (\lambda_n e_{2n-1}^* \wedge e_{2n}^*) = n!\lambda_1\lambda_2...\lambda_n e_1^* \wedge e_2^* \wedge \lambda_2 e_3^* \wedge e_4^* \wedge ... \wedge e_{2n-1}^* \wedge e_{2n}^*$$

Thus

$$\frac{1}{n!}\omega_A^n = \lambda_1 ... \lambda_n \text{vol} \tag{2.13}$$

Hence we equate

$$\text{Pfaff}(A) = \lambda_1 ... \lambda_2 \tag{2.14}$$

Note that $\det(A) = \lambda_1^2 ... \lambda_n^2$. So it is conventional to write

$$\text{Pfaff}(A) = \det^{\frac{1}{2}}(A) \tag{2.15}$$

The Pfaffian is a polynomial in the entries of A (in particular the eigenvalues of A) and has an associated characteristic class. More specifically, for a $2n$-dimensional vector bundle, we define the characteristic classes $c_{2n}(E) = \text{Pfaff}^2(E)$. Another important characteristic class corresponding to the Pfaffian is the following.

Definition 2.2.3. *Let $G = U(n)$ and let $E \to X$ be a real, oriented $2n$-dimensional principal G-bundle. The **Euler class** of E, denoted $e(E) \in H^{2n}(X)$, is the characteristic class given by*

$$e(E) = \frac{1}{(2\pi)^n} c_{2n}(E) \tag{2.16}$$

If $G = O(2n)$ or $SO(2n)$, we define $e(E) = \frac{1}{(2\pi)^n} p_n(E)$ or $e(E) = \frac{1}{(2\pi)^n}$Pfaff respectively.

For an n-dimensional real principal G-bundle $E \to X$ with n odd, $c_n(E)$ is defined to be 0.

Up until this point we've defined characteristic classes algebraically in terms of matrices. Just now we've defined the Euler and Chern classes in terms of a vector bundle - the traditional topological definition. We will now describe how these two definitions can be reconciled.

Theorem 2.2.4 ([10], Theorem 5.1, p.44)**.** *Let $P \to X$ be a principal G-bundle and let θ be a connection on P. Let $\mu^\theta = d\theta + \frac{1}{2}[\theta, \theta]$ be the corresponding curvature. Given any $p \in S^k(\mathfrak{g}^*)^G$, define the $\mathfrak{g}$-matrix-valued $2k$-form*

$$c^\theta(p) = p(\mu^\theta) \tag{2.17}$$

The cohomology class $[c^\theta(p)]$ is independent of the choice of connection θ. Further, the class $[c^\theta(p)]$ given by $p \in S^k(\mathfrak{g}^*)^G$ is exactly the kth characteristic class of P.

It is useful at this point to define the concept of *equivariant characteristic classes*, which we will need later. Previously we defined characteristic classes as the pullback via the classifying map of elements of $H^*(BG, R)$ for some principal G-bundle $P \to X$. Equivariant characteristic classes are simply characteristic classes a K-equivariant vector bundle $\widetilde{P} \to \widetilde{X}$, where $\widetilde{P} = P \times_K EK$ and $\widetilde{X} = X \times_K EK$ for some subgroup $K \leq G$. The name "equivariant" comes from the fact that the equivariant characteristic classes belong to $H^*(X \times_K EK) = H_K^*(X)$.

Alternatively, let G be a compact Lie group acting on a manifold X and let $E \to X$ be a vector bundle on which G acts as vector bundle automorphisms. That is, given $g \in G$, $x \in X$, G sends the fiber E_x to the fiber E_{gx}. Let $K = U(n)$. We can define a K-action on the unitary frame bundle as described above in Eq. (2.2). Given a G-invariant hermitian inner product on E, we get an associated left G-action on $\mathcal{F}(E)$, given by $(x, \mathbf{e}) \mapsto (ax, a\mathbf{e})$, which commutes with the right K action (that is, $\mathcal{F}(E) \to X$ is a K-equivariant vector bundle). The Chern-Weil map then gives a map $\kappa_K : S(\mathfrak{k}^*)^K \to H_G(X/K) = H_G(X)$. The images of the Chern classes under κ_K are the *equivariant Chern classes*. Alternatively, if $K = O(2n)$ or $SO(2n)$, the image of p_i under κ_K gives the *equivariant pontryagin classes*.

These characteristic classes are uniquely defined, though describing what they look like in more detail takes more work. For more explanation on how the Chern classes (from which the Euler class follows) are constructed for an arbitrary complex vector bundle $E \to X$, see ([9], Theorem 3.2, p.78-81).

Note that, given a G^*-module A, if K is a closed subgroup of G, the restriction mapping $H_G(A) \to H_K(A)$ sends G-characteristic classes to K-characteristic classes. We will use this fact later.

Example 2.2.5 (Vector bundle of a point)**.** *If the base space X of the vector bundle $E \to X$ has trivial cohomology, then certainly its characterisitic classes vanish. The same need not be true for the equivariant characteristic classes. For example, consider the principal G-bundle $E \to pt$. If G is compact it is isomorphic to a subgroup K of $U(n)$ for some n. The Chern-Weil map gives a homomorphism $\kappa_K : S(\mathfrak{k}^*)^K \to H_G(pt/G) = H_G(pt) = S(\mathfrak{g}^*)^G$. The K-equivariant chern classes are the images of the $c_i \in S(\mathfrak{k}^*)^K$ under this map. Alternatively we can see the equivariant chern classes as the image of the transpose map $h : S(\mathfrak{k}^*)^K \to S(\mathfrak{g}^*)^G$.*

2.2.4 Euler Class

Let G be a compact connected Lie group and M be G-manifold. Let $X \subseteq M$ be a submanifold and $\mathcal{N} \to X$ be its normal bundle. Suppose we have a torus $T \leq G$ and a T-equivariant n-dimensional complex principal bundle $P \to X$. Then the equivariant Chern classes are denoted by $c_n^T(P)$, and the equivariant Euler classes by

$$e^T(P) = (2\pi)^{-n} c_n^T(P) \tag{2.18}$$

2.3 Fiber Integration and the Thom Isomorphism

The equivariant localization formula is a statement about integration, hence we need to describe what we mean by integration. In this chapter we define *fiber integration*, along with an important equivariant differential form called the *Thom form*. In addition, give an isomorphism theorem called the *Thom isomorphism*, which we will use in the proof of the localization formula.

2.3.1 Fiber Integration

Definition 2.3.1. *Let K be a compact Lie group acting smoothly and properly on oriented manifolds X, Y with $m = \dim Y$, $n = \dim X$ such that $k = m - n \geq 0$. Note that orientation is needed in order to define a volume form with which to integrate with respect to. Suppose that $\pi : Y \to X$ is a K-equivariant fibration (meaning its fiber preserving map is K-equivariant). Let $\Omega^*(Y)_0, \Omega^*(X)_0$ denote the spaces of compactly supported degree-$*$ forms on Y, X respectively. For all $\ell \geq k \in \mathbb{N}$ there is a map*

$$\pi_* : \Omega^\ell(Y)_0 \to \Omega^{\ell-k}(X)_0 \tag{2.19}$$

*called **fiber integration**, where, for each $\mu \in \Omega^\ell(Y)_0$, the form $\pi_*\mu$ is the unique form which satisfies the property that, for all $\beta \in \Omega^{m-\ell}(Y)_0$*

$$\int_Y \pi^*\beta \wedge \mu = \int_X \beta \wedge \pi_*\mu \tag{2.20}$$

Clearly $\pi_*\mu$ is well defined since, if $\alpha \in \Omega^{\ell-k}(X)_0$ is any other form such that $\int_Y \pi^*\beta \wedge \mu = \int_X \beta \wedge \alpha$, then

$$0 = \int_X \beta \wedge \alpha - \int_X \beta \wedge \pi_*\mu = \int_X \beta \wedge (\alpha - \pi_*\mu)$$

So it must be the case that $\alpha - \pi_*\mu = 0$, i.e. $\alpha = \pi_*\mu$. It remains to show existence of a $\pi_*\mu$ that satisfies Eq. (2.20). This can be done locally. Let $(U, \varphi = (x^1, ..., x^n, t^1, ..., t^k)$ be a coordinate patch of Y, where $x = (x^1, ..., x^n)$ are the coordinates in X and $t^1, ..., t^k$ are the remaining $m - n$ coordinates in Y. Let $\pi : Y \to K$ be the fibration with $\pi(x, t) = x$ (obviously π is K-equivariant). Then for any form $\mu \in \Omega^\ell(Y)_0$ given by

$$\mu = a_I(x, t)x^1 \wedge ... \wedge x^n \wedge t^1 \wedge ... \wedge t^k + ... \tag{2.21}$$

where the remaining terms involve wedge products of less than k of the t coordinates, we see that $\pi_*\mu := (\int_Y a_I(x, t)t^1 \wedge ... \wedge t^k) \wedge a^I$ satisfies Eq. (2.20).

2.3.2 The Thom Class and Thom Form

Let Y be an oriented d-dimensional manifold, and X be a compact oriented manifold of codimension k. The Poincaré duality gives a non-degenerate map

$$\langle \cdot, \cdot \rangle : H^{d-k}(Y) \times H_0^k(Y) \to \mathbb{C}, \quad \langle a, b \rangle = \int_Y a \wedge b \tag{2.22}$$

i.e. $a = 0$ is the only element in $H^{d-k}(Y)$ such that $\langle a, b\rangle = 0\ \forall b \in H^k(Y)_0$. Furthermore, there is a unique cohomology class $\tau(X)$, called the **Thom class** associated with X, such that, $\forall a \in H^{d-k}(Y)$,

$$\int_Y a = \langle a, \tau(X)\rangle \tag{2.23}$$

Any closed form $\tau_X \in \tau(X)$ representing this class is called a **Thom form**.

2.3.3 Construction of an Equivariant Thom Form

Let Y be an oriented d-dimensional manifold on which K acts, and let X be a compact submanifold of even codimension k (i.e. $n := \dim X = d - k$). Let $i : X \to Y$ denote the inclusion map. Let $\mathcal{N}$ be the normal bundle of X as a submanifold of Y (note $\mathcal{N}$ has $\dim k$). As before, we can treat $\mathcal{N}$ as a subspace of Y.

Let $\langle\cdot,\cdot\rangle$ be an inner product on X. Since X is compact, WLOG we assume $\langle\cdot,\cdot\rangle$ is K-equivariant. Let P be the bundle of oriented orthonormal frames of $\mathcal{N}$, where

$$P_x = \{\mathbf{e} = (e_1, ..., e_k) : e_i \in \mathcal{N},\ \langle e_i, e_j\rangle = \delta^i_j\}$$

Note that P can be viewed a smooth manifold. Let M denote the compact Lie group $SO(k)$. As with $U(n)$ we have a right action of M on P given, for each $A \in M$, by $(x, \mathbf{e}) \mapsto (x, \mathbf{e}A)$. Now, consider $V = \mathbb{R}^k$ as a module over M. We have that $P \to X$ is a principal M-bundle. Because the inner product on $\mathcal{N}$ is K-invariant, we have an action of K on P commuting with M given by $(x, e_1, ..., e_k) \mapsto (ax, ae_1, ..., ae_k)$, which can be extended to an action of K on V by letting K act trivially on V. We then define a principal M-bundle

$$\psi : P \times V \to \mathcal{N},\ ((x, \mathbf{e}), \mathbf{a} = (a_1, ..., a_n)) \mapsto (x, a_1e_1 + ... + a_ne_n) \tag{2.24}$$

To explain the intuition behind this map, given a point $x \in X$ and a basis $\mathbf{e} = \{e_1, ..., e_n\}$ of $\mathcal{N}_x$, $\mathcal{N}_x$ is just the set of linear combinations $a_1e_1 + ... + a_ne_n$, $(a_1, ..., a_n) \in V$. Thus each $\mathcal{N}_x \in \mathbb{N}$ can be equated (under ψ) with the set of points $((x, \mathbf{e}), \mathbf{a}) \in P \times V$, i.e. the set $\{(x, \mathbf{e})\} \times V$. Next, let $A \in M$, $g \in K$. Then

$$\begin{aligned} gA((x, \mathbf{e}), \mathbf{a}) &= g((x, eA), \mathbf{a}A) \\ &= ((gx, g\mathbf{e}A), g\mathbf{a}A) \\ &= ((gx, g\mathbf{e}A), \mathbf{a}A) \quad (K \text{ acts trivially on } V) \\ &= Ag((x, \mathbf{e}), \mathbf{a}) \end{aligned}$$

i.e. the actions of M and K commute on $P \times V$. Hence the map (2.24) gives rise to a K-equivariant diffeomorphism $\Psi : (P \times V)/M \to \mathcal{N}$. Similarly, the map $P \to X$ descends to a K-equivariant diffeomorphism $P \to X/M$. Following the logic of 1.3.19, these two maps induce maps between the compactly supported equivariant de Rham complexes

$$\begin{aligned} \kappa_N &: \Omega_{M\times K}(P \times V)_0 \to \Omega_K(N)_0 \\ \kappa_X &: \Omega_{M\times K}(P)_0 \to \Omega_K(X)_0 \end{aligned}$$

Lastly, since K acts trivially on V by assumption, we have

$$\Omega_M(V)_0 = \Omega_{\mathrm{SO}(k)}(V)_0 \to \Omega_{M\times K}(V)_0 \otimes S(\mathfrak{k}^*)^K,$$

which gives rise to an embedding

$$\iota : \Omega_M(V)_0 = \Omega_{\mathrm{SO}(k)}(V)_0 \hookrightarrow \Omega_{M\times K}(V)_0, \quad \alpha \mapsto \alpha \otimes 1 \tag{2.25}$$

To proceed we need the universal Thom-Mathai-Quillen form $\nu \in \Omega_M(V)$, defined in ([5], p.85-89). Note that ν has the property of being d_M closed and has the form

$$\nu = e^{-\frac{1}{2}\|u\|^2} p_I(x) du^I \tag{2.26}$$

where the u is a variable in V, x is a variable in $m = \mathfrak{o}(k)$ (the Lie algebra of the orthogonal group $\mathrm{O}(k)$), and the p_I are polynomials in the x.

It is clear that the the $e^{-\frac{1}{2}\|u\|^2}$ part of ν, and hence ν vanishes at infinity in V. We will use a modified version of ν that has compact support (to make it integrable), specifically whose support is the open unit ball B in V. Consider the diffeomorphism

$$\rho : B \to V, \quad u \mapsto \frac{1}{1-\|u\|^2}u$$

and the form

$$\rho^*\nu = \rho^*(e^{-\frac{1}{2}\|u\|^2})p_I(x)\rho^*(du^I) = \rho^*(e^{-\frac{1}{2}\|u\|^2})p_I(x)d\rho^*(u^I)$$

Note that the

$$\rho^* e^{-\frac{1}{2}\|u\|^2} = e^{-\frac{1}{2}\|\frac{1}{1-\|u\|^2}u\|} = e^{-\frac{\|u\|^2}{2(1-\|u\|^2)^2}}$$

part of $\rho^*\nu$ vanishes faster than the $\rho^* du^I$ part of $\rho^*\nu$ can grow, so $\rho^*\nu$ vanishes as $\|u\| \to 1$. Extending $\rho^*\nu$ to zero outside of B gives a compactly supported version of ν (also d_M-closed) which we denote by ν_0. We have $\iota(\nu_0) = \nu_0 \otimes 1 \in \Omega_{M\times K}(V)_0$. Let pr_2 denote the projection $\mathrm{pr}_2 : P \times V \to V$, the pullback of which induces a map $\mathrm{pr}_2^* : \Omega_{M\times K}(V) \to \Omega_{M\times K}(P \times V)$.

Finally we define our candidate for the equivariant Thom form

$$\tau := \kappa_N(\mathrm{pr}_2^*(\nu_0 \otimes 1)) \in \Omega_K(\mathcal{N}) \tag{2.27}$$

It can be shown that τ is indeed an equivariant Thom form. See ([5], p.156-158) for details.

2.3.4 Further Properties of the Thom Class and Thom Form

Theorem 2.3.2. *Let Y be an oriented d-dimensional manifold on which K acts, and let X be a compact submanifold of even codimension k (i.e. $n := \dim X = d - k$). Let $i : X \to Z$ be the inclusion map, $\tau(X)$ be the Thom class of X and $e(\mathcal{N})$ be the equivariant Euler class of the normal bundle $\mathcal{N}$ of X. Then*

$$i^*\tau(X) = e(\mathcal{N})$$

Proof. Recalling from the last section that there is a diffeomorphism $(P \times V)/M \to \mathcal{N}$, where $M = SO(k)$, we represent $\mathcal{N}$ by $(P \times V)/M$. We have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{j} & P \times V \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{\sigma} & \mathcal{N} \end{array}$$

where ψ is the diffeomorphism in the previous section, j denotes the inclusion $(x, \mathbf{e}) \mapsto (x, \mathbf{e}) \times \{0\}$, ρ is the projection $(x, \mathbf{e}) \mapsto x$, and σ is the zero-section $x \mapsto \{x\} \times \{0\}$. To see that this diagram commutes, observe that for any $(x, \mathbf{e}) \in P$, we have

$$\begin{aligned} &\sigma \circ \rho((x, e)) = \sigma(x) = \{x\} \times \{0\} \\ &\psi \circ j((x, \mathbf{e})) = \psi((x, \mathbf{e}) \times \{0\}) = \{x\} \times \{0\} \quad \text{(by Eq. (2.24))} \end{aligned}$$

It is clear that $\rho : P \to X$ is an isomorphism, so, since we have define the action of M to be trivial on x, we have an isomorphism $H^*_{M \times K}(P) \cong H^*_{M \times K}(X) = H^*_K(X)$ which we denote by ρ_*. Further, recall that the actions of M and K commute on $P \times V$. Let EG be a contractible space on which $M \times K$ acts freely. Then, referring to ([5], Section 4.6, p.49-50), the diffeomorphism $\Psi : (P \times V)/M \to \mathcal{N}$ induces an isomorphism

$$\begin{aligned} H^*_{M \times K}(P \times V) &= H^*(((P \times V) \times EG)/(M \times K)) \\ &= H^*((((P \times V) \times EG)/M)/K) \cong H^*((\mathcal{N} \times EG)/K) = H^*_K(\mathcal{N}) \end{aligned}$$

which we denote by Ψ_*. Lastly, the pullback maps j^*, σ^* induce maps $(i^*)_* : H^*_{M \times K}(P \times V) \to H^*_{M \times K}(P)$, $(\sigma^*)_* : H^*_K(\mathcal{N}) \to H^*_K(X)$. We then get the following induced commutative diagram in cohomology

$$\begin{array}{ccc} H^*_{M \times K}(P \times V) & \xrightarrow{\Psi_*} & H^*_K(\mathcal{N}) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle\sigma^*} \\ H^*_{M \times K}(P) & \xrightarrow{\rho_*} & H^*_K(X) \end{array}$$

By definition, we must have $\tau(X) = \Psi_*(\mathrm{pr}_2^*(\nu_0 \otimes 1))$, and so $i^*\tau(X) = \rho_*(\mathrm{pr}_2^*(i^*\nu_0 \otimes 1))$. Recalling that $V = \mathbb{R}^k$ where k is even, we look to ([5], Section 7.2), in particular (Eq. 7.20, p.88), to recover the desired result.

$\square$

Theorem 2.3.3 ([5], Theorem 10.6.1, p.159)**.** *Let K be a compact Lie group and let $\pi : \ \to X$ be a K-equivariant vector bundle of rank k over a compact oriented manifold X, with N also oriented. Then we have an induced map in cohomology*

$$\pi_* : H^\ell_K(N)_0 \to H^{\ell-k}_K(X) \tag{2.28}$$

satisfying $\pi_*[\tau] = 1$.

2.4 Localization

We are almost ready to tackle the equivariant localization formula. Before we do so we one small result.

Proposition 2.4.1. *Let X be a submanifold of a smooth manifold M, and let $i : X \to Z$ denote inclusion. Let $\pi : U \to X$ be a tubular neighbourhood of X. Let $\omega \in \Omega_G(Z)$ be closed. Then*

$$[\omega|_U] = [\pi^* i^* \omega] \tag{2.29}$$

Proof. This follows from the fact that $i \circ \pi$ is a deformation retract, hence we get an isomorphism in cohomology $H_G(\Omega(Z)) \xrightarrow{(i\circ\pi)^* = \pi^* i^*} H_G(\Omega(U))$ □

We now move onto proving the localization formula, starting with a local statement.

Theorem 2.4.2. *Let G be a compact Lie group with an action on a compact, oriented, d-dimensional manifold M. It is important also that the action of G preserves the orientation on M for ease of integration. Let M^G denote the set of points in M fixed by the G-action. Let $i : X \to M$ denote the inclusion of some connected component X of M^G into M, and let $\pi : U \to X$ be a G-invariant tubular neighbourhood of X. Then,*

$$\int_U \omega = \int_X \frac{i^* \omega}{e(\mathcal{N})} \tag{2.30}$$

for any closed $\omega \in \Omega_G(M)$, where $\mathcal{N}$ denotes the normal bundle of X in M.

Proof. Let $\tau \in \Omega_G(U)_0$ be an equviariant Thom form. We recall $\pi_*[\tau] = 1$. Further, $[\omega_U] = [\pi^* i^* \omega]$ since ω is closed.

$$\int_U \omega \wedge \tau = \int_U \pi^* i^* \omega \wedge \tau = \int_X i^* \omega \wedge \pi_* \tau = \int_X i^* \omega$$

Similarly, since $i^*[\tau] = e(\mathcal{N})$, we have

$$\int_X i^* \omega = \int_U \omega \wedge \tau = \int_U \omega \wedge \pi^* i^* \tau = \int_U \omega \wedge \pi^* e(\mathcal{N}) \tag{2.31}$$

Note that if N has odd dimension then $e(\mathcal{N}) = 0$ and so $\int_X i^* \omega = 0$ if and only if X has even codimension. Further, since G acts trivially on X, the equivariant cohomology groups decompose nicely as $H_G(X) = (S(\mathfrak{g}^*) \otimes H(X))^G = S(\mathfrak{g}^*)^G \otimes H(X)^G = S(\mathfrak{g}^*)^G \otimes H(X)$. So $e(\mathcal{N})$ can be expressed as

$$e(\mathcal{N}) = f_m + f_{m-1}\alpha_1 + ... + \alpha_m \tag{2.32}$$

where codim $X = 2m$, $f_i \in S^i(\mathfrak{g}^*)^G$, $\alpha_i \in \Omega^{2i}(X)$, with each α_i d_G-closed. Suppose the leading term f_m of $e(\mathcal{N})$ is non-zero, and let

$$\alpha = -(e(\mathcal{N}) - f_m) = -(f_{m-1}\alpha_1 + ... + \alpha_m)$$

Then we can rewrite $e(\mathcal{N})$ as $e(\mathcal{N}) = f_m(1 - \frac{\alpha}{f_m})$. Given an element x in a commutative ring R (in our case the ring $H_G(\Omega(X))$, the inverse of the element $1 - x$ is given by $\sum_{n=0}^{\infty} x^n$, provided the sum converges.

In the case that the element x is nilpotent we have guaranteed convergence. In our case the element $\frac{\alpha}{f_m}$ is certainly nilpotent, as we cannot have higher than top-degree forms on X. Since each α_i has even degree, the smallest $q \in \mathbb{N}$ such that $(\frac{\alpha}{f_m})^q = 0$ is at most $\frac{1}{2}\dim X$. On that note, let $q = \lfloor \frac{1}{2}\dim X \rfloor + 1$ and consider

$$\beta(\mathcal{N}) = f_m^q * e(\mathcal{N})^{-1} = f_m^q * \frac{1}{f_m}(1 + \frac{\alpha}{f_m} + \frac{\alpha^2}{f_m^2} + ... + \frac{\alpha^{q-1}}{f_m^{q-1}}) = f_m^{q-1} + f_m^{q-2}\alpha + ... + \alpha^{q-1}$$

Note that α is d_G-closed, since $d_G(\alpha(\xi)) = \sum_{i=1}^m f_{i-1}(\xi) d_G(\alpha) = 0$. So

$$\begin{aligned} d_G(\beta(\mathcal{N}))(\xi) &= d_G \sum_{k=0}^{q-1} f_m^{q-1-k}(\xi)\alpha^k \\ &= \sum_{k=0}^{q-1} f^{q-1-k}(\xi) d_G(\alpha^k) \end{aligned}$$

For each k, every term in the expansion of $d_G(\alpha^k)$ contains a $d_G(\alpha)$ term by the Leibniz rule. Hence every term in the series vanishes. So $\beta(\mathcal{N})$ is d_G-closed. Further,

$$\begin{aligned} e(\mathcal{N}) * \beta(\mathcal{N}) &= f_m(1 - \frac{\alpha}{f_m}) * (f_m^{q-1} + f_m^{q-2}\alpha + ... + \alpha^{q-1}) \\ &= (f_m^q + f_m^{q-1}\alpha + ... + f_m\alpha^{q-1}) - (f_m^{q-1}\alpha + f_m^{q-2}\alpha^2 + ... + \alpha^{q-1}) \\ &= f_m^q \end{aligned}$$

We proceed by substituting ω for $\pi^*\beta(\mathcal{N}) \wedge \omega$ in Eq. (2.31) and evaluating:

$$\int_X i^*(\pi^*\beta(\mathcal{N}) \wedge \omega = \int_U \beta(\mathcal{N}) \wedge \omega \wedge \pi^* e(\mathcal{N}) \tag{2.33}$$

$\beta(\mathcal{N}) \in \Omega_G(X)$, So $\pi^*\beta(\mathcal{N}) \in \Omega_G(U)$. Sinde we can map M onto U via a deformation retract, M and U have the same cohomology. Thus $\pi^*\beta(\mathcal{N})$ is cohomologous to a form in $\Omega_G(M)$, and so $i^*\pi^*\beta(\mathcal{N}) \in \Omega_G(X)$ is cohomologous to $\beta(\mathcal{N})$. Since $e(\mathcal{N})$ is a degree $2m$-form, (2.33) becomes

$$\begin{aligned} \int_X i^*(\pi^*\beta(\mathcal{N}) \wedge \omega) &= \int_U \pi^*\beta(\mathcal{N}) \wedge \omega \wedge \pi^* e(\mathcal{N}) \\ &= \int_U (-1)^{2m|\omega|} \pi^*\beta(\mathcal{N}) \wedge \pi^* e(\mathcal{N}) \wedge \omega \\ &= \int_U \pi^*(\beta(\mathcal{N}) \wedge e(\mathcal{N})) \wedge \omega \\ &= \int_U \pi^*(f_m^q e(\mathcal{N})^{-1} \wedge e(\mathcal{N})) \wedge \omega \\ &= \int_U \pi^*(f_m^q) \wedge \omega = f_m^q \int_U \omega \end{aligned}$$

But $[i^*\pi^*\beta(\mathcal{N})] = [\beta(\mathcal{N})]$, so multiplying by $(f_m^q)^{-1}$ gives

$$\int_U \omega = (f_m^q)^{-1} \int_X \beta(\mathcal{N}) \wedge i^*\omega = \int_X e(\mathcal{N})^{-1} \wedge i^*\omega \tag{2.34}$$

□

Eq. (2.34) is a ”local” form of the equivariant localization formula that relates the integral of an equivariant differential form over a connected component X of the fixed point set M^G to an integral over an invariant tubular neighbourhood of X. We want to extend this to a localization formula over the whole manifold, which will be the goal of the remainder of this section.

Eq. (2.34) assumes that the polynomial f_m is invertible, i.e. $f_m \neq 0$. We state a useful equivalent condition for this to be true below, but first state a useful theorem.

Theorem 2.4.3. *Let X be a connected component of M^G and $x_0 \in X$ be arbitrary. Let $\mathcal{N}_0$ be the fiber of the normal bundle $\mathcal{N} \to X$ above x_0. The restriction of f_m to $\mathfrak{t}$ is*

$$f_m|_{\mathfrak{t}} = (2\pi)^{-m}\ell_1 ... \ell_m \tag{2.35}$$

where the ℓ_i are the exponents of the isotropy representation of T on N_0. Hence $f_m \neq 0$ if and only if the ℓ_i are all non-zero.

Proof. Let $\iota : \{x_0\} \to \mathcal{N}_0$. The tangent space of a point manifold is zero-dimensional, so the normal bundle $\mathcal{N}_0$ of x_0 is isomorphic to $T_{x_0}M$, and hence the isotropy representation of G on $T_{x_0}M$ gives a representation $\rho : G \to \mathrm{GL}(\mathcal{N}_0)$. It is known that, for any $a \in G$, left multiplication L_a is orientation-preserving, and that the differential of an orientation-preserving map is orientation-preserving, hence the isotropy action of G on $\mathcal{N}_0$ is orientation-preserving.

Since G is compact, we can equip N_0 with a G-invariant inner product. In this way we get a new representation of G on $\mathcal{N}_0$ by $G \to \mathrm{SO}(\mathcal{N}_0) = \mathrm{SO}(2m)$. This Lie group homomorphism induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{so}(2m)$, and hence a homomorphism between the invariant polynomial rings $h : S(\mathfrak{so}(2m)^*)^{\mathrm{SO}(2m)} \to S(\mathfrak{g}^*)^G$.

Let $j : \mathcal{N}_0 \to \mathcal{N}$ denote inclusion. Then

$$f_m|_{\mathcal{N}_0} = j_0^* f_m = j_0^*(f_m(1 + \frac{\alpha}{f_m})) = j_0^* e(\mathcal{N}) = e(\mathcal{N}_0)$$

since the ”form part” $\frac{\alpha}{f_m}$ of $j_0^* e(\mathcal{N})$ is an element of $H_G(\mathrm{pt}) = S(\mathfrak{g}^*)^G$, and hence must vanish. Now, let T be a maximal torus of G and $\mathfrak{t}$ its Lie algebra. In the case of $G = SO(2m)$ and the vector bundle $\mathcal{N}_0 \to x_0$ over a point, we have identified the equivariant Euler class as $e(\mathcal{N}_0) = f_m = (2\pi)^{-m}h(\mathrm{Pfaff})$ (see Eg.2.2.5). The invariant f_m is a functional on $\mathfrak{g}$, hence an element of $\mathfrak{g}^*$, which is determined uniquely by its restriction $\lambda \circ f_m$ to $\mathfrak{t}^*$ under $\lambda : S(\mathfrak{g}^*) \to S(\mathfrak{t}^*)$.

$\lambda \circ f_m = \lambda \circ (2\pi)^{-m}h(\mathrm{Pfaff})\ \mathfrak{t}$ of the mth equivariant Chern class of the bundle $\mathcal{N}_0 \to \{x_0\}$ which is equal to (see [5], Theorem 10.8.1, p.162-163)

$$f_m = (2\pi)^{-m} \prod_{i=1}^{m} \ell_{i,p} \tag{2.36}$$

where the $\ell_{i,p}$ are the roots of the isotropy representation of T on T_pM for a given point $p \in M^T$. Note that the product $\prod_{i=1}^m \ell_{i,p}$ is well-defined only up to sign, but if M is oriented then the product is well-defined by the orientation of M. The desired result follows immediately. $\square$

Corollary 2.4.4 ([5], p.163). *Let T be a maximal torus of G. Then f_m is non-zero if and only if $M^G = M^T$.*

Therefore, in order for the Euler class to be invertible and for Eq. (2.34) to be applicable, we will require, and from here on assume that $G = T$.

Before we prove the general localization theorem we need one last result.

Theorem 2.4.5 ([5], Proposition 10.9.1, p.165). *The connected components of M^G are orientable and of even codimension.*

The significance of the connected components being orientable is that we can integrate over them without trouble. We will see the significance of dimensionality later in Section 2.4.2.2.

2.4.1 The Equivariant Localization Theorem

Theorem 2.4.6 (Localization formula for a torus action, Berline-Vergne, Atiyah-Bott). *Let G be a compact connected Lie group acting smoothly on a compact oriented d-dimensional manifold M. Let $T \subseteq G$ be a maximal torus. Suppose $\omega \in \Omega_T^k(M)$ ($k \geq d$) is d_T closed. Then*

$$\int_M \omega(\xi) = \sum_{X \subseteq M^T} \int_X \frac{i_X^* \omega(\xi)}{e^T(\mathcal{N}_X)} \tag{2.37}$$

where the sum ranges over the connected components $X \subseteq M^T$, $i_X : X \to M$ denotes inclusion, and $e^T(\mathcal{N}_X)$ is the equivariant Euler class of the normal bundle of X.

Proof. We will first prove the formula for the case of a circle action $T = S^1$ and refer the reader to ([5], Section 10.9, p.164-166) for the extension to a general torus. Suppose ω is d_{S^1} closed. A subgroup of S^1 is either one-dimensional, which case it is exactly S^1, or a zero-dimensional proper subgroup, in which case it must be discrete. The former case cannot happen if the assumption in 2.4.3 holds true. If the stabilizer groups are discrete then the action is locally free. Let $U = M \backslash M^{S^1}$. Then we have $H_{S^1}^k(U) = H^k(U/S^1)$. But $\dim U \backslash S^1 = d - 1$, so for $k \geq d$ we have $H_{S^1}^k(U) = 0$. Hence ω is exact in U, i.e. $\omega|_U = d_{S^1}\nu$ for some $\nu \in \Omega_{S^1}^{k-1}(U)$.

Note that the above definition $H_{S^1}^k(U) = H^k(U/S^1)$ is the cohomology of the orbifold U/S^1. In the same way the quotient M/G of a manifold by a smooth, proper free G-action is a manifold, the quotient U/S^1 of the manifold U by a smooth, proper (since S^1 is compact) *locally free* S^1-action is an *orbifold*. There is of course a cohomology theory for orbifolds, but we omit the details here.

We would like to consider differential forms on X, however it is not immediately clear how to define $\Omega(X)$ as X may not have the structure of a smooth manifold. To this end we choose open neighbourhoods W_X of each X, which assume the structure of submanifolds of M. By the equivariant tubular neighbourhood theorem, there exists an S^1-invariant tubular neighbourhood U_X of W_X. Note that the connected components X are closed. Since the G-action is smooth it must send points sufficiently close to X to points that are also sufficiently close to X. Because of this we can WLOG assume the U_X are separated by shrinking the W_X sufficiently close to X such that we can shrink each U_X to a tubular neighbourhood of W_X that is still S^1-invariant.

For each X let $\rho_X \in C^\infty(U_X)_0$ be an S^1-invariant function such that $\rho_X|_{W_X} = 1$. Note that for each $X, U \cap W_X \neq \emptyset$, so we can extend $d_{S^1}\nu$ so that $\omega = d_{S^1}\nu$ on all of M. With this in mind we now define

$$\nu' = \nu - \sum_X \rho_X \nu \tag{2.38}$$

We claim that for each X there exists $\omega_X \in \Omega^{k-1}_{S^1}(U_X)_0$ satisfying $d_{S^1}\rho_X\nu = \omega_X$ (and hence $\omega_X|_{W_X} = \omega$) such that

$$\omega = d_{S^1}\nu' + \sum_X \omega_X \tag{2.39}$$

We check this case by case.

Case 1 $\mu|_U$. We have $U = (U \setminus \bigcup_X U_X) \cup (U \cap \bigcup_X U_X)$. On $U \setminus \bigcup_X U_X$, each ρ_X and ω_X vanishes, so $\omega = d_{S^1}\nu' = d_{S^1}\nu$ as stated before.

Case 2. Now consider $\omega|_{W_X}$ for some X. In this region $\mu_{X'}, \rho_{X'}$ vanish for all other connected components X'. So we have

$$d_{S^1}\nu - d_{S^1}\sum_X \rho_X\nu + \sum_X \omega_X = d_{S^1}\nu - d_{S^1}(\rho_X\nu) + \omega_X = d_{S^1}\nu - d_{S^1}\nu + \omega = \omega$$

In particular $\nu|_{W_X} = 0$.

Case 3. Finally, consider $\omega_{U_X \cap W_X}$ for some X. In this region $\omega_{X'}, \rho_{X'}$ vanish for all other connected components X'. So we have

$$d_{S^1}\nu - d_{S^1}\rho_X\nu + \omega_X = d_{S^1}\nu = \omega$$

So our chosen reformulation of ω is consistent. Writing ω in this way we can now evaluate

$$\int_M \omega = \sum_X \int_M \omega = \int_M d_{S^1}\nu' + \sum_X \int_M \omega_X$$

The first integral is zero by equivariant Stokes' theorem, since

$$\int_M d_{S^1}\nu' = \int_{\partial M} \nu' \tag{2.40}$$

and $\partial M = \emptyset$. We are then left with

$$\sum_X \int_M \omega_X = \sum_X \int_{U_X} \omega_X = \sum_X \int_X \frac{i_X^*\omega_X}{e^T(\mathcal{N}_X)} = \sum_X \int_X \frac{i_X^*\omega}{e^T(\mathcal{N}_X)} \quad \text{by Eq. (2.34)}$$

and so, for all $\xi \in \mathfrak{g}$, we have

$$\int_M \omega(\xi) = \sum_X \int_X \frac{i_X^*\omega(\xi)}{e^T(\mathcal{N}_X)(\xi)} \tag{2.41}$$

$\square$

2.4.2 Examples

2.4.2.1 Isolated Fixed Points

We discuss an important special case of the localization theorem. Let T be a torus acting on a compact $2m$-dimensional, oriented, compact smooth manifold M. Suppose the set of fixed points F of the group action is discrete (hence finite, since M is compact). For each $p \in M$, Then the LHS of the localization formula reduces to a finite summation

$$\int_M \omega(\xi) = \sum_{p \in F} \frac{i_p^* \omega(\xi)}{e^T(\mathcal{N}_p)(\xi)} = (2\pi)^m \sum_{p \in F} \frac{i_p^* \omega(\xi)}{\prod_{i=1}^m \alpha_{i,p}(\xi)} \tag{2.42}$$

Note that this case can occur only when M is even-dimensional ([20], Remark 9.4, p.7).

2.4.2.2 Equivariantly Closed Extensions

Consider $G = \mathbb{S}^1$ acting on M. Let $\omega_{2m} \in \Omega^{2m}(M)$ be an ordinary closed top-degree differential form, and let $\widetilde{\omega} = \sum_{k=0}^m \mu^{(m-k)} \otimes \omega_k$ be an equivariantly closed extension, where $\omega_k \in \Omega^k(M)$ and μ is a curvature generator of $S(\mathfrak{g}^*)$. We see that integrating $\widetilde{\omega}$ "picks out" only the top degree form ω_{2n}, i.e.

$$\int_M \widetilde{\omega}(\xi) = \int_M \omega_{2m}(\xi) \tag{2.43}$$

So the integral of the equivariantly closed extension is equal to the integral of the original form. Suppose the fixed point set F is finite. Then the localization formula gives, for all $\xi \in \text{Lie}(S^1) = \mathbb{R}$,

$$\int_M \omega_{2m}(\xi) = \sum_{p \in F} \frac{i_p^* \widetilde{\omega}(\xi)}{e^{S^1}(\mathcal{N}_p)(\xi)} \tag{2.44}$$

.

The restriction of any k-form to a point p vanishes unless $k = 0$. Furhter, it is known that the equivariant euler class $e^{S^1}(\mathcal{N}_p)$ turns out to be exactly $(\prod_{i=1}^m \ell_{i,p})\mu^m$ (see [19], p.3), where the ℓ_i are the exponents of the isotropy representation of S^1 on T_pM. So we get

$$\int_M \omega_{2m}(\xi) = (2\pi)^m \sum_{p \in F} \frac{\mu^m(\xi)\omega_0}{\prod_{i=1}^m \ell_{i,p}\mu^m(\xi)} = (2\pi)^m \sum_{p \in F} \frac{\omega_0(p)}{\prod_{i=1}^m \ell_{i,p}} \tag{2.45}$$

That is, the integral of $\widetilde{\omega}$ depends only on its ω_0 part.

Thm. 2.4.5 says that the connected components of M^G are always of even codimension, and so, for an even dimensional manifold, the integrals on the RHS of the localization formula will always pick out *some* contribution from the equivariantly closed extension, thought it might not be from the ω_0 term. For example, if the connected components have dimension 2, then for an ordinary closed top-degree form $\omega \in \Omega(M)$ and its equivariantly closed extension $\widetilde{\omega} = \sum_{k=0}^m \mu^{(m-k)} \otimes \omega_k$, the localization formula gives

$$\int_M \widetilde{\omega}(\xi) = \int_M \omega(\xi) = \sum_{X \subset M^T} \int_X \frac{\omega_2 \mu^{m-1}(\xi)}{e^T(\mathcal{N}_X)} \tag{2.46}$$

2.4.2.3 Surface Area of a Sphere

Let us look at a specific case of the previous example. We return to the example at the beginning of Section 1.1.1 of $G = \mathbb{S}^1$ acting on $\mathbb{S}^2 \subset \mathbb{R}^3$ by rotation about the z-axis ($m = 1$). Note that the volume form on $\mathbb{S}^2$ is given by

$$\text{vol} = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

Choose $\xi = -2\pi i \in \mathfrak{s} = \text{Lie}(S^1)$. For any $(x, y, z) \in \mathbb{S}^2$, we have

$$\begin{aligned} \xi^{\#}_{(x,y,z)} = \frac{d}{dt}|_{t=0} \exp(2\pi i t) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) & 0 \\ \sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2\pi & 0 \\ 2\pi & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= 2\pi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\pi(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) \end{aligned}$$

We want to apply the localization formula to calculate the surface area of the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ as $\int_{\mathbb{S}^2} \text{vol}$. It is a straightforward computation to verify that vol is closed i.e. $d\text{vol} = 0$, but to apply the localization formula we need to find an *equivariantly closed* extension of vol.

Let μ be a curvature generator of $S(\mathfrak{s}^*)$. The Cartan derivative is given by $d_G = 1 \otimes d - \mu \otimes \iota_\xi$. We have

$$\begin{aligned} \iota_\xi \text{vol} &= \iota_{2\pi(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})}(xdy \wedge -ydx \wedge dz + zdx \wedge dy) \\ &= 2\pi(y^2dz - yzdy + x^2dz - xzdx) \\ &= 2\pi(-z(xdx + ydy) + (x^2 + y^2)dz) \\ &= 2\pi(-z(x+y)dz + (1 - z^2)dz) \quad \text{(since } x^2 + y^2 + z^2 = 1\text{) on } \mathbb{S}^2 \\ &= 2\pi(dz - z(xdz + ydy + zdz) \\ &= 2\pi dz \end{aligned}$$

The second to last line follows from the fact that $x^2 + y^2 + z^2 = 1$ implies $d(x^2 + y^2 + z^2) = xdx + ydy + zdz = 0$. Now let $\widetilde{\text{vol}} = 1 \otimes \text{vol} + \mu \otimes 2\pi z$. We have

$$\begin{aligned} d_G(\widetilde{\text{vol}}) &= 1 \otimes d(1 \otimes \text{vol} + \mu \otimes 2\pi z) - \mu \otimes \iota_\xi(1 \otimes \text{vol} + \mu \otimes 2\pi z) \\ &= 1 \otimes d\text{vol} + \mu \otimes 2\pi dz - \mu \otimes \iota_\xi \text{vol} - \mu^2 \otimes \iota_\xi 2\pi z \\ &= 1 \otimes d\text{vol} + \mu \otimes 2\pi dz - \mu \otimes \iota_\xi \text{vol} \\ &= \mu \otimes 1(1 \otimes 2\pi dz - 1 \otimes \iota_\xi \text{vol}) = 0 \end{aligned}$$

So $\widetilde{\mathrm{vol}}$ is the required equivariantly closed extension, with the vol_0 in the previous section being 2, so Let $N = (0,0,1)$, $S = (0,0,-1)$ denote the north and south poles of $\mathbb{S}^2$. The isotropy representation of S^1 on T_pS^2 is an irreducible represetnation of the form $\rho : S^1 \to \mathbb{C}^\times$, i.e. has a single exponent ℓ_p. It is known (see [2], p.125), that $\ell_N = \frac{1}{2\pi}$, $\ell_S = -\frac{1}{2\pi}$. By Eq. (2.45), we have, as expected

$$\int_{S^2} \mathrm{vol} = \int_{S^2} \widetilde{\mathrm{vol}} = 2\pi\left(\frac{2\pi * 1}{2\pi} + \frac{2\pi * -1}{-2\pi}\right) = 4\pi \tag{2.47}$$

Conclusion

To summarize, we have outlined the theory of equivariant cohomology in the Borel construction and the Weil and Cartan models for a compact connected Lie group G. We discussed the connections between these models and defined the equivariant cohomology and $W*$ algebra structure of the de Rham complex $\Omega(M)$. For a more tractible model of equivariant cohomology, we defined the space of equivariant differential forms $\Omega_G(M)$ and the Cartan complex $C_G(\Omega(M))$, which are equivalent due to the abelianization theorem.

We applied the theory of equivariant cohomology to prove the equivariant localization formula which says that, given a torus acting on a compact oriented smooth manifold, the integral of an equivariantly closed form can be expressed as a sum of potentially simpler integrals over the connected components of the fixed point set. The equivariant localization formula is applicable to a wide range of problems involving actions of compact Lie groups, and in the case of non-compact Lie groups one can restrict to a maximal torus. Further, in many cases one can extend a closed form in the orindary de Rham complex to an equivariantly closed form.

The applications of the equivariant localization formula extend far beyond those discussed here. For instance, while the results given in this thesis apply to general torus actions, we focused centrally on circle actions. We remind the reader that there is an alternative approach to localization, called *abstract localization*, which we have not discussed in this thesis. We refer the reader to ([5], Chapter 11). Furthermore, the reader can seek more applications of localization (for example to symplectic geometry, the Duistermaat-Heckman formula and stationary phase approximation and path integrals), in various sources such as [5], [17], [20], [21],[24].

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Appendix A

Differential Geometry

We recall some useful equations from differential geometry.

A.1 G^* Modules

A.1.1 G^* Modules as Categories

To be able to think of the set of G^* algebras and G^* modules as categories, we need to establish what is meant by morphisms between them.

Definition A.1.1. *Let A, B be G^* modules and $f : A \to B$ be a continuous linear map. We say that f is a* ***morphism*** *of G^* modules if, for all $g \in G$, $\xi \in \mathfrak{g}$, we have*

$$[\rho(g), f] = 0 \tag{A.1}$$
$$[\mathcal{L}_\xi, f] = 0 \tag{A.2}$$
$$[\iota_\xi, f] = 0 \tag{A.3}$$
$$[d, f] = 0 \tag{A.4}$$

This is roughly equivalent to saying that f "preserves the G^* action". As expected, if, $\forall i \in \mathbb{Z}$, we have $f : A_i \to B_{i+k}$, we say that f is a **degree** k morphism. If f is even then the commutator is independent of the degree of $\rho(g)$ and we can rewrite Eqs. (A.1) $-$ (A.4) as saying that, $\forall a \in A$,

$$\rho(g)f(a) = f(\rho(g)a) \tag{A.5}$$
$$\mathcal{L}_\xi f(a) = f(\mathcal{L}_\xi) \tag{A.6}$$
$$\iota_\xi f(a) = f(\iota_\xi a) \tag{A.7}$$
$$df(a) = f(da) \tag{A.8}$$

A.1.2 Chain Homotopies

Definition A.1.2. *Let A, B be G^* modules. A linear map*

$$Q : A \to B$$

*is called a **chain homotopy** if it is of odd degeee, G-equivariant, and satisfies*

$$\iota_\xi Q + Q\iota_\xi = 0, \ \forall \xi \in \mathfrak{g} \tag{A.9}$$

Note that for each $\xi \in \mathfrak{g}$, G-equivariance of Q implies $\mathcal{L}_\xi Q - Q\mathcal{L}_\xi = 0$.

Proposition A.1.3. *Let $Q : A \to B$ is a chain homotopy, then the map*

$$\tau := dQ + Qd \tag{A.10}$$

is a morphism of G^* modules.

Proof. Note that τ is even by construction. For any $g \in G$, we have, using the properties of G^*-modules and that Q is a morphism,

$$\tau\rho(g) = dQ\rho(g) + Qd\rho(g) = d\rho(g)Q + Q\rho(g)d = \rho(g)dQ + \rho(g)Qd = \rho(g)\tau$$

So, since each $\rho(g)$ has degree 0, we have

$$[\rho(g), \tau] = \rho(g)\tau - (-1)^0\tau\rho(g) = \rho(g)\tau - \tau\rho(g) = 0$$

We then rewrite τ as

$$\tau := [d, Q]$$

For each $\xi \in \mathfrak{g}$, Eq. (A.9) is equivalent to $[\iota_\xi, Q] = 0$, since ι_ξ, Q are both odd. Similarly, Eq. (A.1.2) is equivalent to $[\mathcal{L}_\xi, Q] = 0$. We now evaluate the three remaining commutators. Using the Jacobi identity and the fact that τ is even, we have, $\forall \xi \in \mathfrak{g}$,

$$\begin{aligned}
[\mathcal{L}_\xi, \tau] &= [L_\xi, [d, Q]] \\
&= [[L_\xi, d], Q] + [d, [L_\xi, Q]] = [0, Q] + [d, 0] = 0 \\
[\iota_\xi, \tau] &= [\iota_\xi, [d, Q]] \\
&= [[\iota_\xi, d], Q] - [d, [\iota_\xi, Q]] = [L_\xi, Q] - 0 = 0 \\
[d, \tau] &= [d, [d, Q]] \\
&= [[d, d], Q] + [d, [d, Q]] = [0, Q] - [d, [d, Q]] = -[d, \tau] \\
\implies [d, \tau] &= 0
\end{aligned}$$

$\square$

Definition A.1.4. *Let A, B be G^* modules. We say two morphisms τ_0, τ_1 are **chain homotopic** and write*

$$\tau_0 \cong \tau_1$$

if there exists a chain homotopy $Q : A \to B$ such that

$$Qd + dQ = \tau_1 - \tau_0 \tag{A.11}$$

This implies that $\tau_1 - \tau_0 = 0$ in cohomology, i.e.

$$\tau_{0*} = \tau_{1*} \tag{A.12}$$

A.2 Connections and Curvatures

In this section we present a review of the theory of connections and curvatures needed for equivariant cohomology. For a more detailed review see ([2], Chapters 15-17).

Definition A.2.1. . *Let G be a Lie group and $\pi : P \to M$ be a principal G-bundle. The **vertical tangent space** $\mathcal{V}_p$ at a point $p \in P$ is the kernel of the differential $D_p\pi : T_pP_{\pi(p)}M$. Elements of $\mathcal{V}_p$ are called **vertical vectors** at p.*

Definition A.2.2. . *A **horizontal distribution** on a principal G-bundle $\pi : P \to M$ is a vector subbundle $\mathcal{H}$ of the tangent bundle TP such that, $\forall p \in P$,*

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \tag{A.13}$$

Proposition A.2.3 ([5], p.24)**.** *Alternatively we can define the vertical and horizontal bundles $\mathcal{V}, \mathcal{H}$ by an inner product $\langle \cdot, \cdot \rangle$ on TP. In general $\mathcal{V}$ will not be G-invariant. However, if $\langle \cdot, \cdot, \rangle$ is G-invariant, so will $\mathcal{V}$ be. If G is compact, we can always choose $\langle \cdot, \cdot \rangle$ to be so.*

Definition A.2.4. *Let $\pi : P \to M$ be a principal G-bundle and $\mathcal{H}$ be a horizontal distribution on P. For each $p \in P$, define a map $j_p : G \to P$ by $j_p(g) = pg$. Then we have a pushforward $j_{p*} : T_eG = \mathfrak{g} \to T_pP$.*

Proposition A.2.5.

$$\mathcal{V} \cong P \times \mathfrak{g}, \quad \mathcal{H} \cong P \times \mathfrak{g}^* \tag{A.14}$$

Or, pointwise,

$$\mathcal{V}_p \cong \mathfrak{g} \,(\textit{via } j_{p*}) \quad \mathcal{H}_p \cong \mathfrak{g}^* \tag{A.15}$$

The composition θ_p of the projection $\mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p$ with j_{p*}

$$\theta_p : T_pP = \mathcal{V}_p \otimes \mathcal{H}_p \to \mathcal{V}_p$$

is a linear map that takes vectors in T_pP to $\mathfrak{g}$. Hence θ is a $\mathfrak{g}$-valued 1-form on P. Further, any vector $X_p \in T_pP$ decomposes uniquely into a sum of horizontal and vertical components $X_p = v(X_p) + h(X_p)$, so we can write $\theta_p(X_p) = j_{p*}^{-1}(v(X_p))$.

Theorem A.2.6. *([2], Theorem 16.3, p.65). Let $P \to M$ be a principal G-bundle and let $\mathcal{H}$ be a smooth horizontal distribution on $P \to M$ that is* ***right-invariant*** *in the sense that it is preserved under R_{g*} for any $g \in G$. Then the $\mathfrak{g}$-valued 1-form θ defined as above satisfies the following:*

1. $\theta(\xi^{\#}) = \xi \; \forall \xi \in \mathfrak{g}$
2. $R_g^* \theta = (Adg^{-1})\theta \; \forall g \in G$ *(equivariance)*
3. θ *is* C^∞

Definition A.2.7. . *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to M$ be a principal G-bundle with a smooth right-invariant horizontal distribution $\mathcal{H}$. A* ***connection*** *on P is a C^∞ $\mathfrak{g}$-valued 1-form on P satisfying properties (1) and (2) above. Note that left- and right-invariant are the same in the context of abelian groups, which we will see in the next chapter are the only groups we need to consider here.*

Definition A.2.8. . *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to M$ be a principal G-bundle. Let ω be a connection on P. We denote the* ***curvature*** *of the connection ω by the $\mathfrak{g}$-valued 2-form on P given by*

$$\mu = d\omega + \frac{1}{2}[\omega, \omega] = 0 \tag{A.16}$$

Below are two important identities relating connections and curvatures.

Theorem A.2.9 (Second Structural Equation, [2], p.68)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to M$ be a principal G-bundle. Let θ be a connection on P and μ its curvature. With respect to a basis $\xi_1, ..., \xi_n$ of $\mathfrak{g}$, we write*

$$\theta = \sum_k \theta^k \xi_k \quad \mu = \sum_k \mu^k \xi_k$$

where the θ^k, μ^k are $\mathbb{R}$-valued 1-forms and 2-forms respectively. Then

$$d\theta^k = \mu^k - \frac{1}{2} \sum_{i,k} c_{ij}^k \theta^i \wedge \theta^j \tag{A.17}$$

Theorem A.2.10 (Second Bianchi Identity, [2], Theorem 17.2, p.69)**.** *Let ω, Ω be connection and curvature forms on a principal G-bundle. Then*

$$d\mu = [\mu, \theta] \tag{A.18}$$

or, in basis-dependent form,

$$d\mu^k = \sum_{i,j} c_{i,j}^k \theta^i \wedge \mu^j$$

A.3 Pushforward in Equivariant Cohomology

This section is taken from ([23], 4.2, p.4).

Suppose we have a map $f : X \to Y$ between finite-dimensional oriented compact manifolds X and Y, with $\dim X = m$, $\dim Y = n$, $k = m - n$. The Poincaré duality gives a pushforward map $f_* : H^*(X) \to H^{*+(m-n)}(Y)$. However, suppose we have a Lie group G acting smoothly on X and Y. We want a pushforward map between the cohomology groups of $X_G = (X \times EG)/G$ and $Y_G = (Y \times EG)/G$, but Poincaré duality fails because EG is infinite dimensional, so this requires further work.

Let $f : X \to Y$ be a fibre integral and G be a Lie group acting on X and Y. Let $\omega = \sum_i \mu \otimes \omega_i \in C_G(\Omega(X))$. In this case we define the pushforward of ω by f, also called the of ω, by

$$f_*\omega = \sum_i \mu_i \otimes (f_*\omega_i) \tag{A.19}$$

where $f_*\omega_i$ is just the fibre integral in ordinary cohomology. This gives a map fm $H^*_G(X) \to H^{*-k}_G(Y)$. Now consider the case where X is a subspace of Y and not just compact, but closed. Let $f : X \hookrightarrow Y$ denote inclusion. Let $\mathcal{N}$ be the normal bundle of X. We have a fiber bundle $\mathcal{N} \to X$. The Thom isomorphism gives, by the inverse π^{-1} of $\pi : \mathcal{N} \to X$, a map $\pi^{-1} : H^*_G(X) \to H^{*+k}_G(\mathcal{N})_0$. We then have a map

$$H^*(X) \to H^{*+k}(Y) \tag{A.20}$$

as the composition $H^*(X) \to H^{*+k}(\mathcal{N})_0 \to H^{*+k}(Y)$. Finally, we define the equivariant pushforward for an arbitrary map $f : X \to Y$. Let

$$\begin{aligned} \Gamma &: X \to X \times Y, \quad x \mapsto (x, f(x)) \\ \pi &: X \times Y \to Y, \quad (x, y) \mapsto y \end{aligned}$$

be the graph map and second projection. Then $f = \pi \circ \Gamma$. X is a subspace of $X \times Y$ and Γ is the inclusion of X into $X \times Y$, so we have the pushforward $\Gamma_* : H^*_G(X) \to H^{*+((n+m)-m)}(X \times Y)$. Further, since $X \times Y \to Y$ is a fiber bundle, we have the pushforward $\pi_* : H^*_G(X \times Y) \to H^{*-((n+m)-n)}_G(Y) = H^{*-m}_G(Y)$. We can then define the pushforward f_* as the composition

$$f_* = \pi_* \circ \Gamma_* : H^*_G(X) \to H^{*+n-m}_G(Y) = H^{*+k}_G(Y) \tag{A.21}$$

Appendix B

Spectral Sequences and Abelianization

B.1 Spectral Sequences

B.1.1 Constructing a Spectral Sequence

A **double complex** is a bigraded vector space

$$C = \bigoplus_{p,q \in \mathbb{Z}} C^{p,q} \tag{B.1}$$

with coboundary operators $d : C^{p,q} \to C^{p+1,q}$, $\delta : C^{p,q} \to C^{p,q+1}$ satsifying

$$d^2 = 0, \qquad d\delta + \delta d = 0, \qquad \delta^2 = 0 \tag{B.2}$$

From this we define the **total complex** C^n for a given $n \in \mathbb{N}$ by

$$C^n := \bigoplus_{p+q=n} C^{p,q} \tag{B.3}$$

with coboundary operator

$$d + \delta : C^n \to C^{n+1}$$

Note that $(d+\delta)^2 = 0$ by the above relations, so $(C, d+\delta)$ is a cochain complex. In the usual way, define

$$\begin{aligned} Z^n &= \ker(d + \delta : C^n \to C^{n+1}) = \{z \in C^n : (d+\delta)z = 0\} \\ B^n &= \operatorname{im}(d + \delta : C^{n-1} \to C^n) = (d+\delta)C^{n-1} \end{aligned}$$

Then we can define a cohomology on C by

$$H^n(C, d+\delta) = Z^n \backslash B^n \tag{B.4}$$

Definition B.1.1. *Let*

$$C_k^n := \bigoplus_{p+q=n,\, p\geq k} C^{p,q} \tag{B.5}$$

and

$$\begin{aligned} Z_k^n &= Z^n \cup C_k^n = \{z \in C_k^n : (d+\delta)z = 0\} \\ B_k^n &= B^n \cup C_k^n \subseteq Z_k^n \end{aligned}$$

with the analogous cohomology groups given by $H_k^n = Z_k^n \backslash B_k^n$. This gives a filtration of H^n ... $\supseteq H_{k-1}^n \supseteq H_k^n \supset H_{k+1}^n \supseteq$ ... from which we can define

$$H^{k,n-k} := H_k^n \backslash H_{k+1}^n \qquad gr\ H^n := \bigoplus_{k\in\mathbb{Z}} H^{k,n-k}$$

Each element of $\mathbf{a} = C^n$ has some leading term at position (p,q), where p is the largest k such that $a \in C_k^n$ (equivalently the smallest k such that the $C^{k,n-k}$ component of a doesn't vanish). For example if the leading term a_0 lies in $C^{p,q}$, then **a** will have no component in $C^{p-1,q+1}$ (i.e. applying d to a gives 0). But if the $C^{p+1,q-1}$ component is non-zero, then there must be an $a_1 \in C^{p+1,q-1}$ such that $(d+\delta)a = a_1$ Let $Z^{p,q}$ denote the set of components of cocycles in C^n with at position p, q, i.e. the set of $a \in C^{p,q}$ such that the system of equations

$$\begin{aligned} da &= 0 \\ \delta a &= -da_1 \\ \delta a_1 &= -da_2 \\ \delta a_2 &= -da_3 \\ &\dots \end{aligned} \tag{B.6}$$

has a solution

$$\mathbf{a} = (a_0, a_1, a_2 \dots), \qquad a_i \in C^{p+i,q-i} \tag{B.7}$$

as a consequence the element $z = a \oplus a_1 \oplus a_2 \oplus \dots$ clearly lies Z_p^n since $(d+\delta)z = 0$. Suppose the following "boundedness condition" holds true:

$$\text{There exists } m_\ell \in \mathbb{N} \text{ such that } C^{i,j} = 0 \text{ when } |i-j| > m_\ell. \tag{B.8}$$

Then it is sufficient to solve the system of equations for a bounded range of i.

Next, let $B^{p,q} \subset C^{p,q}$ consist of all b with the property that the system of equations

$$\begin{aligned} dc_0 + \delta c_{-1} &= b \\ dc_{-1} + \delta c_{-2} &= 0 \\ dc_{-2} + \delta c_{-3} &= 0 \\ &\dots \end{aligned} \tag{B.9}$$

has a solution

$$(c_0, c_{-1}, c_{-2}, ...), \qquad c_{-i} \in C^{p-i,q+i-1} \tag{B.10}$$

Again, if the boundedness condition holds, it is sufficient to solve the system of equations for a bounded range of i.

Example B.1.2. *The Cartan complex $C_G(\Omega(M)) = ((\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G, d_C)$ can be thought of as a double complex with bigrading $C^{p,q} = (\mathbb{C}[\mathfrak{g}]^p \otimes \Omega^{q-p}(M))^G$, i.e.*

$$C_G(\Omega(M)) = \bigoplus_{p,q\in\mathbb{Z}} (\mathbb{C}[\mathfrak{g}]^p \otimes \Omega^{q-p}(M))^G \tag{B.11}$$

Note $C^{p,q} = 0$ if $p < 0$ or $q < 0$ by definition. Also note that elements of the subspace $\mathbb{C}[\mathfrak{g}]^p \otimes \Omega^m(M)$ have bidegree (p,p+m), i.e. each of its elements has total degree 2p + m. This agrees with how degrees of elements in the Cartan complex were previously defined. The d and δ operators can be thought of the generalizations of the $1 \otimes d$ and $-\mu \otimes \iota$ parts of the Cartan derivative d_G. Notice also the similarities between Eqs. (B.6) *and Eqs.* (1.77) *for the equivariantly closed extension of an ordinary closed form.*

In the rest of this chapter we will describe and apply a cohomology theory for the Cartan complex based on spectral sequences.

Proposition B.1.3. *It is a matter of computation (see [11], p.48) to show that $H^{p,q} = H_p^{p+q} \backslash H_{p+1}^{p+q}$, where $H_p^{p+q} = Z_p^{p+q} \backslash B_p^{p+q}$ and $H_{p+1}^{p+q} = Z_{p+1}^{p+q} \backslash B_{p+1}^{p+q}$. We can then define a component-wise cohomology on C by*

$$H^{p,q} = Z^{p,q} \backslash B^{p,q} \tag{B.12}$$

Definition B.1.4. *Let $Z_r^{p,q} \subseteq C^{p,q}$ be the set of all $a \in C^{p,q}$ for which the first $r-1$ of the equations* (B.6) *can be solved. That is, those elements that can be joined by a sequence of zigzags to some $a_{r-1} \in C^{p_r,q_r}$, where $(p_r, q_r) = (p+r-1, q-r+1)$. Similarly, let $B_r^{p,q} \subseteq B^{p,q}$ be the set of $b \in C^{p,q}$ such that the equations* (B.9) *have a solution with $c_{-i} = 0$ for $i \geq r$.*

Remark B.1.5. *$a \in Z^{p,q}$ can be thought of as the (p,q) component of an element of Z^{p+q}, and $b \in B^{p,q}$ can be thought of as the (p,q) component of $B^{p+q} \cap C_p^{p+q}$.*

The next theorem tells us when we can also extend this by a sequence of zigzags to an element of C^{p_r+1,q_r-1}.

Theorem B.1.6. *Let $a \in Z_r^{p,q}$. Then*

$$a \in Z_{r+1}^{p,q} \iff \delta a_{r-1} \in B_r^{p_r+1,q_r} \tag{B.13}$$

for any solutions $(a, a_1, ..., a_{r-1})$.

Proof. Given a partial solution $\mathbf{a}_r = (a_0, a_1, ..., a_{r-1})$, we may need to replace $\mathbf{a}_r$ with a different partial solution $(a_0, a_1', ...)$ such that the differences $a_i'' := a_i' - a_i$ satisfy

$$
\begin{aligned}
da''_1 &= da'_1 - da_1 = 0 \\
\delta a''_1 + da''_2 = (\delta a'_1 - da'_2) + (\delta a'_1 - da''_2) &= 0 + 0 = 0 \\
&\dots \\
\delta a''_{r-2} + da''_{r-1} &= 0
\end{aligned}
$$

For the inductive step we want to zigzag down one more step. In this case we need an $a''_r \in C^{p_r+1, q_r-1}$ such that

$$\delta a_{r-1} = -\delta a''_{r-1} - da''_r \tag{B.14}$$

so that we can solve $\delta a_{r-1} = -da_r$ for a_r. Now we set

$$b = \delta a_{r-1},\ c_0 = -a''_r,\ c_{-i} = -a''_{r-i} \text{ for } i = 1, ..., r-1 \text{ and } c_{-i} = 0 \text{ for } i \geq r \tag{B.15}$$

We therefore get a system of equations

$$
\begin{aligned}
dc_0 + \delta c_{-1} &= b \\
dc_{-1} + \delta c_{-2} &= 0 \\
dc_{-2} + \delta c_{-3} &= 0 \\
&\dots \\
dc_{-r+2} + \delta c_{-r+1} &= 0 \\
dc_{-r+1} + 0 &= 0 \\
&\dots
\end{aligned}
\tag{B.16}
$$

Note $b = \delta a_{r-1} \in C^{p_r, q_r}$, so if this system of equations admits a solution, then $b \in B^{p_r+1, q_r}$.

If there exists such a $b \in B_r^{p_r+1, q_r}$, then we have our δa_{r-1} and can solve for a a_r. Thus $a \in Z_r^{p,q}$. On the other hand, if $a \in Z_r^{p,q}$, then choose another partial solution $(a'_1, ..., a'_r)$ and define a''_i as before for $1 \leq i \leq r$. Then the set of equations (B.16) must have a solution. □

Proposition B.1.7. *Every element $b \in B_{r+1}^{p_r+1, q_r}$ can be written in the form*

$$b = \delta a_{r-1} + dc \tag{B.17}$$

for some $c \in C^{p_r+1, q_r-1} = C^{p+r, q-r}$ and some solution $(a_0, a_1, ..., a_{r-1})$ to the first $r-1$ equations of (B.6).

Proof. Given an arbitrary $b \in B_r^{p_r+1, q_r}$ satisfying (B.16), we can reverse the definitions in (B.15) to get a sequence

$$(a_1, ..., a_r), \qquad a_i = -c_{r-i} \tag{B.18}$$

satisfying the first r equations of (B.6) with $b = \delta a_{r-1} + da_r$. Since $a_r \in C^{p+r, q-r}$, we are done. □

Definition B.1.8. *Let*

$$E_r^{p,q} := H_r^{p,q} := Z_r^{p,q} \backslash B_r^{p,q} \tag{B.19}$$

Proposition/Definition B.1.9. *Let $a \in Z_r^{p,q}$. By Theorem B.1.6 we have*

$$\delta a_{r-1} \in B_{r+1}^{p_r+1,q_r} \subseteq Z^{p_r+1,q_r} \subseteq Z_{r+1}^{p_r+1,q_r} \subseteq Z_r^{p_r+1,q_r}$$

We define a projection $\delta_r : Z_r^{p,q} \to E_r^{p_r+1,q_r}$ by

$$\delta_r a = \delta a_{r-1} B_r^{p_r+1,q_r} \tag{B.20}$$

With this definition the above theorem can be restated as saying $a \in Z_{r+1}^{p,q} \iff \delta a_{r-1} \in B_r^{p_r+1,q_r} \iff \delta_r a = 0$.

Proposition B.1.10. *. $\delta_r(a)$ only depends on the equivalence class of a modulo $B^{p,q}$.*

Proof. Observe that if an element of $B^{p,q}$ is added to a, then the solution $(a_0, a_1, ..., a_{r-1})$ to the first $(r-1)$ equations of (B.6) changes by an element of $B^{p,q}$. To see this, suppose we add to a an element $b_0 = dc_0 + \delta c_{-1}$. Then the second equation of (B.6) becomes

$$\delta(a + b_0) = \delta(a + dc_0 + \delta c_{-1}) = \delta a + \delta dc_0 = -da_1$$

Note $c_0 \in C^{p,q-1}$, so $-da_1$ changes by an element δdc_0 of $C^{p+1,q}$, and hence a_1 must change by an element of $C^{p+1,q-1}$ to keep the equations consistent. Call this element b_1 and let $a_1' = a_1 - b_1$.

Adding b_1 to a_1 means that in the third equation of (B.6), $-da_2$ changes by an element $\delta b_1 \in C^{p+2,q-1}$, so a_2 must change by some element $b_2 \in C^{p+2,q-2}$ satisfying. Let $a_2' = a_2 - b_2$. In general, on the $(r+1)$th line of Eq. (B.6) we add an element $b_r \in C^{p_r,q_r}$ to the RHS and define $a_r' = a_r - b_{r-1}$. Now define $c_{-i} = a_{r-i}'$ in the analogous way to in the proof of Theorem B.1.6. The first $r+1$ equations can then be written in the form of Eqs. (B.16). Hence for each r we have $b_r \in B_r^{p,q} \subset B^{p,q}$. □

Since $B^{p,q} \supset B_r^{p,q}$ for any r. we can consider δ_r as a map $\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with bidegree $(r, -(r-1))$.

Proposition/Definition B.1.11. *Let*

$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q} \tag{B.21}$$

$$H(E_r, \delta_r) = \mathit{ker}\ \delta_r \backslash \mathit{im}\ \delta_r \tag{B.22}$$

Then $\forall r \in \mathbb{N}$, the sequence of complexes (E_r, δ_r) is a spectral sequence, i.e. it satisfies

$$H(E_r, \delta_r) = E_{r+1} \tag{B.23}$$

Proof. By (B.17), im δ_r is the projection of $B_{r+1}^{p+r,q-r+1}$. By theorem (B.13), ker δ_r is the projection of $Z_{r+1}^{p,q}$ onto $E_r^{p,q}$ Therefore, the sequence

$$\dots \xrightarrow{\delta_r} E_r^{p,q} \xrightarrow{\delta_r} \dots$$

satisfies ker $\delta_r \supset$ im δ_r and

$$(\text{ker } \delta_r \backslash \text{im } \delta_r)^{p,q} = E_{r+1}^{p,q}$$

It follows that $H(E_r, \delta_r) = E_{r+1}$.

Theorem B.1.12. *If the boundedness condition* (B.8) *holds, then the spectral sequence* (E_r) *eventually stabilizes (i.e. has a tail in which the terms are all equal), and*

$$E_\infty^{p,q} = \lim_{r\to\infty} E_r^{p,q} = H^{p,q} \tag{B.24}$$

Proof. If (B.8) holds, there exists $r \in \mathbb{N}$ such that $C^{p,q} = 0\ \forall p, q \in \mathbb{Z}$ such that $|p-q| > \frac{r}{2}$. Thus, if the first r equations of (B.6) are solvable, then the first $r+1$ equations of (B.6) are solvable, and so on. The same logic applies to the set of equations (B.9). In this case, $\exists r \in \mathbb{Z}$ such that

$$Z_r^{p,q} = Z_{r+1}^{p,q} = \dots \text{ and } B_r^{p,q} = B_{r+1}^{p,q} = \dots$$

and so

$$E_r^{p,q} = E_{r+1}^{p,q} = \dots \tag{B.25}$$

Moreover, by definition the limit of this complex is just $H^{p,q}$:

$$E_\infty^{p,q} = \lim_{r\to\infty} E_r^{p,q} = \lim_{r\to\infty} Z_r^{p,q} / \lim_{r\to\infty} B_r^{p,q} = Z^{p,q}/B^{p,q} = H^{p,q}$$

B.1.2 Additional properties

B.1.2.1 Convergence and Double Complexes as Categories

Let (C, d, δ) and $(C'.d', \delta')$ be double complexes, and $\rho : C \to C'$ a morphism of double complexes of bidegree (m, n). This gives a cochain map

$$\rho : (C, d+\delta) \to (C', d'+\delta') \tag{B.26}$$

of degree $m+n$, which induces a map in the cohomology of the total complex

$$\rho_\# : H(C, d+\delta) \to H(C', d'+\delta') \tag{B.27}$$

of degree $m+n$, sending $H(C^i, d+\delta) \mapsto H((C')^{i+m+n}$. Similarly ρ maps the cochain complex $(C^{p,*}, d)$ into the cochain complex $((C')^{p+m,*}, d')$, and so induces a map $\rho_1 : E_1 \to E_1'$ in cohomology of bidegree (m, n) mapping $E_1^{p,q} \to E_1'^{p+m,q+n}$, and intertwining δ_1 and δ_1'. Similarly ρ_1 induces a map in $\rho_2 : H(E_1, \delta_1) \to H(E_1', \delta_1')$ in cohomology. Following this we can inductively construct the maps

$$\rho_r : (E_r, \delta_r) \to (E'_r, \delta'_r) \tag{B.28}$$

From here the we have the following two important theorems.

Theorem B.1.13. . *If the spectral sequences $(E_r, \delta_r), (E'_r, \delta'_r)$ converge, then*

$$\lim_{r\to\infty} \rho_r = gr\ \rho_\# \tag{B.29}$$

Theorem B.1.14. . *If ρ_r is an isomorphism for some $r = r_0 \in \mathbb{N}$, then it is an isomorphism $\forall r \geq r_0$ and so, if both spectral sequences converge, $\rho_\#$ is an isomorphism.*

B.1.2.2 Gaps

A recurring pattern of zeroes in the spectral sequence can simplify calculations. Here is one example.

Proposition B.1.15. *Suppose $E_r^{p,q} = 0$ when $p + q$ is odd (the even case holds aswell). Then the spectral sequence "collapses" at the E_r stage, i.e. $E_r = E_{r+1} = \ldots.$*

Proof. $\delta_r : E_r^{p,q} \to E^{p_r+1,q_r}$, so δ_r reverses the parity of $p+q$. Thus either the range or domain of δ_r is 0 and so δ_r. Hence $E_{r+1} = H(E_r, \delta_r) = H(E_r, 0) = E_r$, and similarly $E_k = E_r$ for all $k \geq r$. □

B.1.3 The Cartan Complex as a Double Complex

Let G be a compact Lie group and $A = \bigoplus_k A^k$ be a G^* module. Its Cartan complex

$$C_G(A) = (S(\mathfrak{g}^*) \otimes A)^G, d_C)$$

can be thought of as a double complex with bigrading

$$C^{p,q} := (S^p(\mathfrak{g}^*) \otimes A^{q-p})^G$$

with vertical and horizontal operators $d = 1 \otimes d_A$ and $\delta = -\mu^a \otimes \iota_a$ respectively. Note that the subspace $(S^p(\mathfrak{g}^*) \otimes A^m)^G$ has bidegree $(p, p+m)$ and so total degree $2p+m$, which is consistent with the grading of $(S(\mathfrak{g}^*) \otimes A)^G$ as a commutative superalgebra.

Remark B.1.16. *If $A^k = 0$ for $k < 0$, we see that $C_G(A)$ satisfies the boundedness condition* (B.8) *and hence has convergent spectral sequences. We explore the possibilities for the limit of the spectral sequence by first finding the E_1 term.*

Theorem B.1.17. *The E_1 term in the spectral sequence of $C_G(A)$ is*

$$E_1 = (S(\mathfrak{g}^*) \otimes H(A))^G \tag{B.30}$$

So by the grading of $C_G(A)$,

$$E_1^{p,q} = (S(\mathfrak{g}^*)^p \otimes H^{q-p}(A))^G$$

Proof. Note that $(C_G(A), 1 \otimes d_A)$ is contained in $(S(\mathfrak{g})^* \otimes A, 1 \otimes d_A)$, and so the cohomology groups of $C_G(A)$ are just the G-invariant components of the cohomology of $S(\mathfrak{g})^* \otimes A$, which have the grading stated above.

Proposition B.1.18. *The connected component of the identity in G acts trivially on $H(A)$.*

Proof. Note that, $\forall a \in \mathfrak{g}$,

$$L_a - 0 = L_a = \iota_a d + d\iota_a = 0 \text{ on the connected component of the identity}$$

So L_a is chain homotopic to 0, and is hence trivial in cohomology, on the connected component of the identity of G. $\square$

Theorem B.1.19. *Let G be a compact, connected Lie group. Then*

$$E_1^{p,q} = S^p(\mathfrak{g}^*)^G \otimes H^{q-p}(A) \tag{B.31}$$

Proof. The connected component of the identity of G is G, so every element of $H(A)$ is G-invariant, and so because of the way G acts on $C_G(A)$ $(S(\mathfrak{g}^*)^G \otimes H(A))^G = S(\mathfrak{g}^*)^G \otimes H(A)$ and the theorem holds. $\square$

Theorem B.1.20. *Let G be a compact, connected Lie group. Suppose $H^p(A) = 0$ for p odd. Then the spectral sequence of $C_G(A)$ collapses at the E_1 stage.*

Proof. If $p - q$ (and hence $p + q$) is odd, then $E_1^{p,q} = S^p(\mathfrak{g}^*)^G \otimes H^{q-p}(A) = 0$. The theorem holds by the gaps argument discussed previously. $\square$

For the rest of this section we assume that the Lie group G is connected and compact so that (B.31) holds.

Example B.1.21. *Suppose G acts freely and smoothly on a smooth manifold M. We have the G^* algebra $\Omega(M)$. The E_1 term of the spectral sequence of its Cartan complex $C_G(\Omega(M)$ is, by de Rham's theorem,*

$$(S(\mathfrak{g})^* \otimes H(\Omega(M)))^G = S(\mathfrak{g}^*)^G \otimes H(\Omega(M))) = S(g^*) \otimes H^*(M)$$

The spectral sequences converges to a graded version of $H^*(M/G)$. Describing what the higher pages of a spectral sequence look like is technical (there is no general intuitive interpretation) and so we have skipped some details here. For more details see ([5], Chapter 6), in particular p.66-67,69,71.

For more on the link between the Cartan complex and double complexes outside of spectral sequences, we refer the reader to ([5], Chapter 5).

B.1.4 Morphisms of G^* Modules

Definition B.1.22. *Let $\rho : A \to B$ be degree morphism between G^* modules A, B. We get an induced map*

$$\rho' : (S(\mathfrak{g}^*) \otimes A)^G \to (S(\mathfrak{g}^*) \otimes B)^G$$

between the Cartan double complexes. This gives an induced map

$$\rho_* : H(A, d) \to H(B, d)$$

in ordinary cohomology and, by analogy with (B.27)*, another induced map*

$$\rho_\# = (\rho')_* : H_G(A) \to H_G(B)$$

Theorem B.1.23. *If the induced map ρ_* on ordinary cohomology is bijective, then so is the induced map $\rho_\#$ on equivariant cohomology.*

Proof. Note that $(S(\mathfrak{g}^*) \otimes H(A))^G$ and $(S(\mathfrak{g}^*) \otimes H(B))^G$ sit inside $S(\mathfrak{g}^*) \otimes H(A)$ and $S(\mathfrak{g}^*) \otimes H(B)$ respectively. So clearly if ρ_* is bijective, we have an induced bijection

$$\rho_1 : (S(\mathfrak{g}^*) \otimes H(A))^G \to (S(\mathfrak{g}^*) \otimes H(B))^G$$

or in other words, a bijection $\rho_1 : E_1^A \to E_1^B$ where E_1^A, E_1^B are the E_1 terms of the spectral sequences of $C_G(A), C_G(B)$ respectively. Thus $\rho_r : E_r^A \to E_r^B$ is a bijection for all $r \in \mathbb{N}$ and so $\rho_\#$ is a bijection. □

B.2 Abelianization

Let G be a compact connected Lie group and $K \leq G$ be a closed (hence compact, but not necessarily connected) subgroup of G. We get an injection $\mathfrak{k} \to \mathfrak{g}$ of their Lie algebras by inclusion, and so an $\tilde{\mathfrak{k}} \to \tilde{\mathfrak{g}}$ injection of their superalgebras. So by restricting the $\mathfrak{g}$ to $\mathfrak{k}$ we can turn any G^* module into a K^* module. In the usual way, the injection $\mathfrak{g} \to \mathfrak{k}$ gives a projection $\mathfrak{g}^* \to \mathfrak{k}^*$ which can be extended linearly to a map

$$S(\mathfrak{g}^*) \otimes A \to S(\mathfrak{k}^*) \otimes A$$

Certainly if an element of $S(\mathfrak{g}^*) \otimes A$ is invariant with respect to the G action it is invariant with respect to the K action. So we get a map

$$(S(\mathfrak{g}^*) \otimes A)^G \to (S(\mathfrak{k}^*) \otimes A)^K$$

or in other words a map $C_G(A) \to C_K(A)$, which can be checked to be a map of double complexes. This restricts to a mapping on horizontal elements

$$H_G(A) \to H_K(A)$$

which, following B.1.4 and using the identity morphism $\rho : A \to A$, restricts to a morphism at each stage of the corresponding spectral sequences. Further, since G is connected, it acts trivially on $H(A)$, and so K as a subgroup of G acts trivially on $H(A)$, despite K not necessarily being connected. Therefore

$$E_1^G = S(\mathfrak{g}^*)^G \otimes H(A), \quad E_1^K = S(\mathfrak{k}^*)^K \otimes H(A)$$

and we have a restriction morphism $E_1^G \to E_1^K$ or

$$S(\mathfrak{g}^*)^G \otimes H(A) \to S(\mathfrak{k}^*)^G \otimes H(A)$$

The next theorem follows by similar logic as before.

Theorem B.2.1. *Suppose that the restriction map*

$$S(\mathfrak{g}^*)^G \to S(k^*)^K$$

is bijective. Then the restriction map in equivariant cohomology gives an isomorphism

$$H_G(A) \cong H_K(A)$$

Appendix C

Lie Theory and Representation Theory

C.1 Lie Theory

C.1.1 Standard Results

Proposition C.1.1 ([4], Proposition 7.14, p.156). *Suppose G is a Lie group, and $W \subseteq G$ is any neighbourhood of the identity. Then*

- *W generates an open subgroup of G*
- *If W is connected, it generates an open connected subgroup of G*
- *If G is connected, then W generates G.*

Theorem C.1.2 (First Isomorphism Theorem for Lie Groups, [4], Theorem 21.27, p.556). *If $F : G \to H$ is a Lie group homomorphismm, then the kernel of F is a closed normal Lie subgroup of G, the image of F has a unique smooth manifold structure making it into a Lie subgroup of H, and F descends to a Lie group isomorphism $\tilde{F} : G \backslash KerF \to ImF$. If F is surjective, then $G \backslash KerF$ is smoothly isomorphic to F.*

Theorem C.1.3 (Lie's third theorem, [4], Theorem 19.26, p.506). *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then there is a bijection between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

Proposition C.1.4 ([4], Exercise 20-22, p.539). *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Let K be a Lie subgroup and $\mathfrak{k}$ its Lie algebra. Let $Z_G(K)$ and $Z_{\mathfrak{g}}(\mathfrak{k})$ denote the centers of K,$\mathfrak{k}$ in G,$\mathfrak{g}$ respectively. Then $Lie(Z_G(K)) = Z_{\mathfrak{g}}(\mathfrak{k})$.*

C.1.2 Maximal Tori

In this section we give two definitions of a torus and show the relationship between them.

Definition C.1.5. *An n-dimensional real (or complex) torus is denoted by $\mathbb{T}^n = \mathbb{S}^1 \times ... \times \mathbb{S}^1$. Note $\mathbb{T}^n \cong \mathbb{R}^n \backslash \mathbb{Z}^n$. Further, each $\mathbb{S}^1$ can be considered a Lie group, hence so can $\mathbb{T}^n$.*

Definition C.1.6. *A **torus** is a compact connected abelian Lie group.*

Definition C.1.7. *Let G be a compact Lie group. A **Cartan subgroup** of G of a Lie group G is a maximal torus in G (with respecft to inclusion). A **Cartan subalgebra** is a maximal abelian Lie subalgebra of g. Clearly Cartan subgroups correspond to Cartan subalgebras by Lie's third theorem.*

Theorem C.1.8. *Let G be a compact Lie group. Then there is a bijection between the maximal tori of G and the Cartan subalgebras of $\mathfrak{g}$.*

Proof. Recall that there is a bijection between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$ (Theorem 19.26 of Lee). So any connected Lie subgroup T of G corresponds to a unique Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Further, recall that $\text{Lie}(Z_G(T)) = Z_{\mathfrak{g}}(\mathfrak{t})$, so that T is abelian if and only if $\mathfrak{t}$ is abelian. So we have established a bijection between maximal connected abelian Lie subgroups of G and Cartan subalgebras of $\mathfrak{g}$. Finally, let $T \geq G$ be any connected abelian Lie subgroup. Then its closure $\bar{T}$ is connetced. It is also abelian since, for any $g, h \in \partial T$, we can pick sequences $(g_j), (h_j) \subset T$ converging to g, h respectively. Since multiplication in T is continuous, we have

$$gh = \lim_{j\to\infty} g_j h_j = \lim_{j\to\infty} h_j g_j = hg$$

Since $\bar{T}$ is closed and G is compact, $\bar{T}$ is compact, connected and abelian (i.e. a torus). So the maximal connected abelian subgroups of G correspond to maximal tori, which correspond to abelian Lie subalgebras in $\mathfrak{g}$. Finally, these corresponding abelian Lie subalgebras must themselves be maximal, or there would be other connected Lie subalgebras containing them, corresponding to connected abelian Lie subalgebras containing T, violating maximality of T. □

Clearly Cartan subalgebras exist, since any one-dimensional space of $\mathfrak{g}$ is abelian. So certainly maximal tori exist for any compact Lie group.

We now aim to show that an n-dimensional torus T (in definition 2) is isomorphic to $\mathbb{T}^n$.

Definition C.1.9. *Let V be a finite dimensional vector space over $\mathbb{R}$ and let $e_1, ..., e_n$ be a set of linearly independent vectors in V. The **lattice** of V corresponding to the $e_1, ..., e_n$ is the discrete additive subgroup L of V generated by the e_i. In other words,*

$$L = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$$

Note that V can be considered as an additive abelian Lie group, and thus L a normal Lie subgroup. For more details see ([16], Section 6.1, p.129-131).

Theorem C.1.10. *Let $e_1, ..., e_n$ be a basis for V and $L \subset V$ the corresponding lattice, then there exists a Lie isomorphism $V \backslash L \cong \bigoplus_n \mathbb{S}^1$.*

Proof. By the previous lemma, $V \backslash L \cong \mathbb{R}^n \backslash \mathbb{Z}^n \cong \bigoplus_n \mathbb{S}^1$. □

Theorem C.1.11. *Let G be an n-dimensional connected, abelian Lie group. Let $L = exp^{-1}(e) \subset \mathfrak{g}$. Then*

- *$G \cong \mathfrak{g} \backslash L \cong \mathbb{T}^d \times \mathbb{R}^{n-d}$, where $d = rank(L)$.*

- *If G is compact, then $G \cong \mathbb{T}^n$.*

Proof. We recall that exp : $\mathfrak{g} \to G$ is a Lie group homomorphism between the additive Lie group $\mathfrak{g}$ and G, and note two observations:

- $L = \exp^{-1}(e) = \ker(\exp)$ is discrete. To see this, it is known that there exists an open $U \subseteq \mathfrak{g}$ such that $\exp|_U$ is a diffeomorphism (isomorphism). Now, the exponential map is equivariant (with respect to the transitive adjoint action of G) and smooth, so by the equivariant rank theorem it has constant rank (which must be n since it has rank n in a neighbourhood of 0). Therefore, it is a smooth submersion by the global rank theorem. Hence $\ker(\exp) = \exp^{-1}(e)$ has dimension 0 by the implicit function theorem, and is therefore discrete.

- Since exp is a local diffeomorphism, it is a local homoemorphism, and so an open map. Therefore $\exp(\mathfrak{g})$ is an open subgroup of the topological group G, hence closed. But G is connected, so $\exp(\mathfrak{g}) = G$

Now since $L = \exp^{-1}(e) = \ker(\exp)$, it follows by the first isomorphism theorem for Lie groups (Lee Theorem 21.27) that $\mathrm{im}(\exp) = G$ is smoothly isomorphic to $\mathfrak{g}\backslash\ker(\exp) = \mathfrak{g}\backslash L$. Now, let $W = \mathrm{span}_{\mathbb{R}} L$ (that is, L is a lattice in W). Let $d = \dim_{\mathbb{R}}(W)$ $(0 \leq d \leq n = \mathrm{rank}(L))$. Then $G \cong \mathfrak{g}\backslash L = (W \oplus W^{\perp})\backslash L = (W\backslash L) \times W^{\perp} \cong \mathbb{T}^d \times \mathbb{R}^{n-d}$.

Finally, if G is compact, it cannot be isomorphic to the direct product of any space with $\mathbb{R}^k$ for any $k > 0$. So this forces $d = n$ and $G \cong \mathbb{T}^n$. □

Remark C.1.12 ([14], Example 3.2, p.13)**.** *There is a well-known isomorphism $\mathbb{S}^1 \cong U(1)$. Hence the group structure of a torus is $T \cong U(1)^n$. In fact,*

$$T = \{diag(t_1, ..., t_n) \in U(1)^n : t_1...t_n = 1\} \tag{C.1}$$

$$\mathfrak{t} = \{(a_1, ..., a_n) \in \mathbb{R}^n : \sum_i a_i = 0\} \tag{C.2}$$

C.1.3 The Chevalley Restriction Theorem

Definition C.1.13. *A Lie group is **simple** if it is connected, non-abelian and has only trivial connected normal subgroups. A Lie algebra is **simple** if it is non-abelian and its only ideals are 0 and itself.*

Definition C.1.14. *A Lie algebra is **semisimple** if it is a direct sum of simple Lie algebras. A Lie group is **semisimple** if its Lie algebra is semisimple.*

Theorem C.1.15 (Chevalley Restriction Theorem)**.** *Let G be a complex connected semi-simple Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\mathbb{C}[\mathfrak{g}]^G$ be the ring of invariant (under the adjoint G-action) polynomials on $\mathfrak{g}$. Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and let $\mathbb{C}[\mathfrak{h}]^W$ be the ring of polynomials that are invariant under the canonical action of the Weyl group $W = N(G)/G$ of G. Then we have the following isomorphism:*

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W \tag{C.3}$$

Proof. See ([15], Section 23.1, p.126-128) or ([22], Theorem 4.9.2, p.335) □

C.2 Representations

Definition C.2.1. *Given a representation $\rho : G \to GL(V)$ of G, a linear subspace W is called G-**invariant** if $\rho(g)(w) \in W\ \forall g \in G, w \in W$. The restriction $\rho|_W : G \to GL(W)$ is called a **sub-representation**. A representation (ρ, V) is **irreducible** if it has only non-trivial sub-representations (i.e. $\{0\}$ or V). Otherwise it is said to be **reducible**.*

Definition C.2.2. *A representation is called **completely reducible** if it is a direct sum of irreducible representations.*

Theorem C.2.3 ([2], Theorem 26.3, 114)**.** *Every finite-dimensional representation of a compact Lie group is completely reducible.*

Definition C.2.4. *Let G be a compact connected Lie group and T be a maximal torus. A **weight** of T is an irreducible representation of T.*

Proposition C.2.5. *Let T be an abelian matrix Lie group and ρ be an irreducible complex representation of T (that is, an irreducible representation $\rho : T \to GL(\mathbb{C})$. Then ρ is a one-dimensional representation of the form $\rho(z) = \lambda(z)I$ for each $z \in T$ and for some functional $\lambda : T \to \mathbb{C}$, where I is the identity map. In other words, any irreducible representation of T is of the form*

$$\rho : T \to GL_1(\mathbb{C}) \cong \mathbb{C}^\times$$

Proof. See ([13], Corollaries 4.30,4.31, p.94). □

Note that if T is compact, the image of ρ must lie in $U(1)$ since $U(1)$ is the maximal compact subgroup of $\mathrm{GL}_1(\mathbb{C})$. So in this case every irreducible representation is of the form

$$\rho : T \to U(1) \tag{C.4}$$

We apply the above results to a torus T, which is compact and abelian, and recall the following well known result:

Proposition/Definition C.2.6. *Let T be an n-dimensional torus and $\rho : T = \bigoplus^n \mathbb{S}^1 \to U(1)^n$ be a complex representation. Then ρ decomposes into n irreducible representations $\rho_1 : \mathbb{S}^1 \to U(1)$ given by $\rho_i(z) = \lambda_i(z)$. The $\lambda_i : T \to \mathbb{C}$ are called the **weights** of the representation ρ of T. A weight that is nowhere vanishing is called a **root**. Alternatively we can write ρ in terms of the Lie algebra so that, for any $\xi \in \mathfrak{t}$, we have $\rho(\exp \xi) = e^{2\pi i \ell_i(\xi)} I$. The functionals $\ell_i : \mathfrak{t} \to \mathbb{R}$ are called **exponents**.*

C.2.1 Adjoint and Coadjoint Representations

Consider the mapping $\Psi : G \to \mathrm{Aut}(G)$, given by $g \mapsto \Psi_g$ where Ψ_g is the inner automorphism given by

$$\Psi_g(h) = g^{-1}hg \tag{C.5}$$

The map Ψ is a Lie group homomorphism. For each $g \in G$, $\Psi_g : G \to G$ is a Lie group automorphism, and so its differential at the identity, which we denote by

$$\mathrm{Ad}_g = D_e\Psi_g : \mathfrak{g} \to \mathfrak{g} \tag{C.6}$$

is a Lie algebra automorphism. Further, since $g \mapsto \Psi_g$ is a group homomorphism, so too is $g \mapsto \mathrm{Ad}_g$ by composition with the differential. Hence the map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \tag{C.7}$$

given by $g \mapsto \mathrm{Ad}_g$ is a group representation, called the **adjoint representation**. For any $g \in G, \xi \in \mathfrak{g}$, we have $Ad_g(X) = gXg^{-1}$

Similarly we define the **coadjoint representation**

$$Ad^* : G \to \mathrm{Aut}(\mathfrak{g}^*) \tag{C.8}$$

by $\mathrm{Ad}^*(g) = (\mathrm{Ad}(g^{-1}))^*$

C.2.2 Isotropy Representation

Definition C.2.7. *Let G be a Lie group and M be a G-manifold. Let $p \in M^G$ (the fixed point set of G). For each $a \in G$, $L_a(p) = p$, so $D_pL_a : T_pM \to T_pM$ is a linear action of G on T_pM, called the **isotropy action** of G. In other words, at each fixed point $p \in M$ of G, we have a representation $\rho : G \to GL(T_pM)$ mapping $g \mapsto L_{g*}$. This representation is called the **isotropy representation** or **tangent space representation**.*

Example C.2.8. *Suppose G acts on itself by conjugation, i.e. $g \mapsto aga^{-1}$. Let $p = e$. The isotropy action of G on $T_eG = \mathfrak{g}$ is simply the adjoint representation.*