String Structures, Reductions and T-duality

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16 February 2015

Outline

1. Topological T-duality and Courant algebroids

2. String structures and reduction of Courant algebroids

3. T-duality of string structures and heterotic Courant algebroids

Joint work with David Baraglia.

`arXiv:1308.5159, to appear in ATMP.`

Electromagnetic duality

Maxwell's equations in vacuum ($c = 1$):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

Duality of order 4: $(\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E})$

The duality still holds if <u>both</u> electric and magnetic charges are included.

Phase space duality

Harmonic oscillator:

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2$$

Duality of order 4:

$$(x, p) \mapsto (p, -x)$$
$$(m, k) \mapsto (\frac{1}{k}, \frac{1}{m})$$

These are examples of *S*-duality.

T-duality: a toy example

Topological T-duality arose in the study of string theory compactifications.

Let V be a real n-dimensional vector space with basis $\{v_k\}_{k=1,\ldots,n}$.

Let V^* be the dual space with basis $\{w_k\}_{k=1,\ldots,n}$.

Fix a volume form on V and V^*.

The Fourier-Mukai transform is an isomorphism

$$\mathcal{FM}\colon \wedge^\bullet V^* \to \wedge^\bullet V, \qquad \mathcal{FM}(\phi)(v) = \int \phi(w) e^{\sum_{k=1}^n w_k \wedge v_k}$$

Geometric formulation of $\mathcal{FM}$

Let $\Lambda \subset V$ be a lattice and $\Lambda^* \subset V^*$ the dual lattice.

Define the torus $T^n = V/\Lambda$ and the dual torus $\widehat{T}^n = V^*/\Lambda^*$.

Note that $\pi_1(T^n) = \Lambda = Irrep(\widehat{T^n})$ and $\pi_1(\widehat{T^n}) = \Lambda^* = Irrep(T^n)$.

In particular, $\widehat{T}^n = \mathrm{Hom}(\pi_1(T^n), S^1)$, so it parametrizes flat S^1-bundles on T^n.

$T^n \times \widehat{T}^n$ carries a universal S^1-bundle $\mathcal{P}$ called the Poincaré line bundle:

$$\mathcal{P}|_{T^n \times w} \cong \text{flat } S^1\text{-bundle on } T^n \text{ associated to } w$$

$$\mathcal{P}|_{v \times \widehat{T}^n} \cong \text{flat } S^1\text{-bundle on } \widehat{T^n} \text{ associated to } v$$

Geometric formulation of $\mathcal{FM}$

Now we have

$$H^\bullet(T^n, \mathbb{R}) = \wedge^\bullet V^*, \qquad H^\bullet(\widehat{T}^n, \mathbb{R}) = \wedge^\bullet V$$

and

$$ch(\mathcal{P}) = e^{\sum_{k=1}^n w_k \wedge v_k} \in H^\bullet(T^n \times \widehat{T}^n, \mathbb{R})$$

The Fourier-Mukai transform is an isomorphism

$$\mathcal{FM} \colon H^\bullet(T^n, \mathbb{R}) \to H^{\bullet - n}(\widehat{T}^n, \mathbb{R}), \qquad \mathcal{FM}(\phi) = \hat{p}_! \big(p^*(\phi) \wedge ch(\mathcal{P}) \big)$$

$$T^n \xleftarrow{\;p\;} T^n \times \widehat{T}^n \xrightarrow{\;\widehat{p}\;} \widehat{T}^n$$

Topological T-duality

Idea: Replace T^n by a family of tori.

Possibilities include:

- $X = M \times T^n$
- $X \to M$ a principal T^n-bundle
- $X \to M$ an affine T^n-bundle
- X a T^n-space (non-free action)
- singular fibrations (e.g. the Hitchin fibration, CY manifolds)

We shall consider the case when $X \to M$ is a principal torus bundle.

It turns out that an additional structure is needed on X, namely a bundle gerbe classified by its Dixmier-Douady class $[H] \in H^3(X, \mathbb{Z})$.

Topological T-duality

Theorem (Bouwknegt–Evslin–Mathai (2004), Bunke-Schick (2005))

There exists a commutative diagram

$$
\begin{array}{ccccc}
 & & (X \times_M \widehat{X}, p^*H - \hat{p}^*\widehat{H}) & & \\
 & \swarrow^{p} & & \searrow^{\hat{p}} & \\
(X, H) & & & & (\widehat{X}, \widehat{H}) \\
 & \searrow_{\pi} & & \swarrow_{\hat{\pi}} & \\
 & & M & &
\end{array}
$$

and

$$
\mathcal{FM}\colon\ (\Omega^\bullet(X)^{T^n}, d_H) \to (\Omega^{\bullet - n}(\widehat{X})^{\widehat{T}^n}, d_{\widehat{H}}), \qquad \mathcal{FM}(\omega) = \int_{T^n} e^{\mathcal{F}} \wedge \omega
$$

is an isomorphism of the differential complexes, where $p^*H - \hat{p}^*\widehat{H} = d\mathcal{F}$ *and* $\mathcal{F} = \langle p^*\theta \wedge \widehat{p}^*\widehat{\theta}\rangle$ *for connections* θ *and* $\widehat{\theta}$ *on* X *and* $\widehat{X}$ *respectively.*

Remarks

- As a corollary, we have an isomorphism in twisted cohomology

$$H^{\bullet}(X, H) \cong H^{\bullet - n}(\widehat{X}, \widehat{H})$$

- This can be refined to an isomorphism in twisted K-theory,

$$K^{\bullet}(X, H) \cong K^{\bullet - n}(\widehat{X}, \widehat{H})$$

- For circle bundles, the T-dual is unique up to isomorphism.

- For higher rank torus bundles, an additional condition on H is needed and the T-dual is not unique.

Example: Lens spaces L_p

Consider the action of $\mathbb{Z}_p$ on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ given by

$$e^{\frac{2\pi i}{p}}(z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{p}} z_2)$$

The quotient $L_p = S^3/\mathbb{Z}_p$ is an S^1-bundle over S^2 with the Chern class

$$c_1(L_p) = p \in H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$$

Let $H = q \in H^3(L_p, \mathbb{Z}) \cong \mathbb{Z}$, then the T-dual pair is (L_q, p).

In particular $L_0 = S^2 \times S^1$, so

$$(S^3, 0) \Longleftrightarrow (S^2 \times S^1, 1)$$

Note that

$$K^0(S^3) = K^1(S^3) = \mathbb{Z}$$

$$K^0(S^2 \times S^1, 1) = K^1(S^2 \times S^1, 1) = \mathbb{Z}$$

while

$$K^0(S^2 \times S^1) = K^1(S^2 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$$

Courant algebroids

A Courant algebroid on a smooth manifold X consists of a vector bundle $E \to X$ equipped with

- a bundle map $\rho\colon E \to TX$ called the anchor,
- a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon E \otimes E \to \mathbb{R}$,
- an $\mathbb{R}$-bilinear operation $[\ ,\ ]\colon \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$,

satisfying the following properties

- $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$
- $[a, b] + [b, a] = d\langle a, b\rangle$
- $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$

- $[a, fb] = f[a, b] + \rho(a)(f)b$
- $\rho[a, b] = [\rho(a), \rho(b)]$

Exact Courant algebroids

A Courant algebroid E is **transitive** if the anchor ρ is surjective.

E is **exact** if it fits into an exact sequence

$$0 \to T^*X \to E \to TX \to 0$$

Exact Courant algebroids are classified by their Ševera class $H \in H^3(X, \mathbb{R})$.

An isotropic splitting $s\colon\ TX \to E$ fixes an isomorphism

$$E \cong TX \oplus T^*X$$

where

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$$

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H$$

with

$$H(X, Y, Z) = \langle [s(X), s(Y)], s(Z) \rangle.$$

Symmetries and generalised metric

Spin module $\Omega^\bullet(X)$: $\quad (X+\xi)\cdot\omega = \iota_X\omega + \xi\wedge\omega.$

Abelian extension: $\quad \mathrm{Aut}(E) = \mathrm{Diff}(X) \ltimes \Omega^2_{cl}(X)$

$$e^B(X+\xi) = X + \xi + \iota_X B$$

Extension class: $\quad c(X, Y) = d\iota_X\iota_Y H$

A generalised Riemannian metric is a self-adjoint orthogonal bundle map $G \in \mathrm{End}(TX \oplus T^*X)$ for which $\langle Gv, v\rangle$ is positive definite.

$G^2 = Id$ determines an orthogonal decomposition

$$TX \oplus T^*X = G_+ \oplus G_-$$

where $G_\pm = \{X + B(X,\cdot) \pm g(X,\cdot) \mid X \in TX\}$.

Simple reduction

Consider a Lie group K acting freely on X.

Suppose the action lifts to a Courant algebroid E on X.

The simple reduction E/K is a vector bundle on X/K, which inherits the Courant algebroid structure on E.

E/K is not an exact Courant algebroid.

Buscher rules

Theorem (Cavalcanti-Gualtieri)

The map

$$\phi\colon (TX \oplus T^*X)/T^n \to (T\widehat{X} \oplus T^*\widehat{X})/\widehat{T}^n$$

$$X + \xi \quad \mapsto \quad \hat{p}_*(\hat{X}) + p^*(\xi) - \mathcal{F}(\hat{X})$$

is an isomorphism of Courant algebroids.

The Buscher rules for (g, B) are given by

$$\widehat{G} = \phi(G)$$

Heterotic string theory

Conceived by the *Princeton String Quartet* in 1985.

Combines 26-dimensional bosonic left-moving strings with 10-dimensional right-moving superstrings.

The theory includes a principal G-bundle $P \to X$ equipped with a connection.

The Green-Schwarz anomaly cancellation:

$$dH = \frac{1}{2}p_1(TX) - \frac{1}{2}p_1(P)$$

String structures

A spin structure on an oriented manifold X is a lift:

$$\begin{array}{ccc} & & BSpin(n) \\ & \overset{s}{\nearrow} & \downarrow \\ X & \longrightarrow & BSO(n) \end{array}$$

A string structure on a spin manifold X is a lift:

$$\begin{array}{ccc} & & BString(n) \\ & \overset{S}{\nearrow} & \downarrow \\ X & \longrightarrow & BSpin(n) \end{array}$$

A string structure exists if and only if $\left[\frac{1}{2}p_1(S)\right] = 0$.

Equivalently, a string structure is $[H] \in H^3(P, \mathbb{Z})$, where $P \to X$ is the spin structure, such that the restriction of $[H]$ to any fiber of P is the generator of $H^3(Spin(n), \mathbb{Z}) \cong \mathbb{Z}$.

String classes H are intimately related to extended actions and certain transitive Courant algebroids.

Heterotic Courant algebroids

Let G be a compact connected simple Lie group and $P \to X$ a principal G-bundle.

The Atiyah algebroid $\mathcal{A} := TP/G \to TX$ is a quadratic Lie algebroid,

$$\langle x, y \rangle = -k(x, y)$$

where k denotes the Killing form on $\mathfrak{g}$.

A transitive Courant algebroid $\mathcal{H}$ is a heterotic Courant algebroid if

$$\mathcal{H}/T^*X \cong \mathcal{A}$$

is an isomorphism of quadratic Lie algebroids, where $\mathcal{A}$ is the Atiyah algebroid of some principal G-bundle P.

Classification of heterotic Courant algebroids

The obstruction for the Atiyah algebroid of P to arise from a transitive Courant algebroid $\mathcal{H}$ is the first Pontryagin class $p_1(\mathcal{P}) \in H^4(X, \mathbb{R})$.

Theorem

Let $P \to X$ be a principal G-bundle and A a connection on P with curvature F. Let H^0 be a 3-form on X satisfying

$$dH^0 + k(F, F) = 0.$$

Any heterotic Courant algebroid is isomorphic to one of the form

$$\mathcal{H} = TX \oplus \mathfrak{g}_P \oplus T^*X,$$

where

$$\langle (X, s, \xi), (Y, t, \eta) \rangle = \frac{1}{2}(i_X\eta + i_Y\xi) + \langle s, t \rangle$$

$$\begin{aligned}[X + s + \xi, Y + t + \eta]_{\mathcal{H}} = [X, Y] + \nabla_X t - \nabla_Y s - [s, t] - F(X, Y) \\ + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H^0 \\ + 2\langle t, i_X F \rangle - 2\langle s, i_Y F \rangle + 2\langle \nabla s, t \rangle,\end{aligned}$$

Extended action on Courant algebroids

Let E be an exact Courant algebroid on a G-manifold X and assume that the action lifts $G \to Aut(E)$.

If the infinitesimal action $\mathfrak{g} \to Der(E)$ on E is by inner derivations, we could consider a lift $\mathfrak{g} \to \Gamma(E)$.

A trivially extended action is a map $\alpha\colon \mathfrak{g} \to \Gamma(E)$ such that

- α is a homomorphism of Courant algebras,
- $\rho \circ \alpha = \psi$, where $\psi\colon \mathfrak{g} \to \Gamma(TX)$ denotes the infinitesimal G-action on X,
- the induced adjoint action of $\mathfrak{g}$ on E integrates to a G-action on E.

Reduction by extended action

For an exact Courant algebroid $E \cong TX \oplus T^*X$ with a G-invariant Ševera class H, the extended action

$$\alpha\colon \mathfrak{g} \to \Gamma(E), \quad v \mapsto \psi(v) + \xi(v)$$

corresponds to solutions to $d_G(H+\xi) = c$, with the non-degenerate form $c(\cdot,\cdot) = -\langle \alpha(\cdot), \alpha(\cdot)\rangle \in \Omega^0(X, S^2\mathfrak{g}^*)^G$.

Two extended actions ξ, ξ' are equivalent if there exists an equivariant function $f\colon M \to \mathfrak{g}^*$ such that $\xi' = \xi + df$

Changing the invariant splitting of E corresponds to

$$H' + \xi' = H + \xi + d_G(B)$$

where $B \in \Omega^2(X)^G$ is the invariant 2-form relating the splittings.

The reduced Courant algebroid on X/G is defined by $E_{red} = \mathrm{Im}(\alpha)^\perp / G$.

Heterotic Courant algebroids by reduction

Let $\sigma \colon P \to X$ be a G-bundle equipped with a G-invariant closed 3-form H on P and $E = TP \oplus T^*P$ with the H-twisted Dorfman bracket.

Since $\mathfrak{g}$ comes with a natural pairing, it is natural to consider $c = -k$.

Proposition

Equivalence classes of solutions to $d_G(H + \xi) = -k$ *are represented by pairs* (H^0, A) *satisfying*

$$dH^0 + k(F, F) = 0.$$

The corresponding pair (H, ξ) *is given by*

$$H = \sigma^*(H^0) + CS_3(A), \qquad \xi = kA.$$

Hence, every heterotic Courant algebroid is obtained from an exact Courant algebroid via a trivially extended action.

Relation to string structures

The restriction of $H = \sigma^*(H^0) + CS_3(A)$ to any fibre of P is given by

$$\omega_3 = -\frac{1}{6}k(\omega, [\omega, \omega])$$

where $\omega \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan form.

A real string class is a class $H \in H^3(P, \mathbb{R})$ such that the restriction of H to any fibre of P coincides with ω_3. Imposing integrality, (P, H) defines a string structure on X.

Let $\mathcal{EA}(P)$ and $\mathcal{SC}(P)$ denote the sets of equivalence classes of trivially extended actions and string classes on P respectively. The map

$$(H, \xi) \to [H]$$

is an isomorphism of $H^3(X, \mathbb{R})$-torsors.

Heterotic T-duality

Consider a T^n-bundle $X \to M$ equipped with a string structure (P, H).

We assume that the T^n-action on X lifts to a T^n-action on P by principal bundle automorphisms, so we can view P as a principal $T^n \times G$-bundle over M. Then $P_0 = P/T^n$ is a principal G-bundle over M.

Choose H to be a $T^n \times G$-invariant representative for the string class.

Strategy

- Since $P \to P_0$ is a principal T^n-bundle, we can apply ordinary T-duality to the pair (P, H) to obtain a dual pair $(\widehat{P}, \widehat{H})$.
- The existence of a T-dual imposes the usual constraints on H.
- However, there is no guaranty that the G-action on P_0 lifts to an action on $\widehat{P}$ commuting with the $\widehat{T}^n$-action.
- The restriction of H to the $G \times T^n$-fibres of $P \to M$ defines a class in $H^2(G, H^1(T^n, \mathbb{Z}))$, which is the obstruction to $\widehat{P} \to P_0$ being a pullback under $\sigma_0 \colon P_0 \to M$ of a $\widehat{T}^n$-bundle $\widehat{X} \to M$.

T-duality commutes with reduction

Proposition

For commuting group actions, the simple reduction and reduction by extended action commute.

Theorem

The T-duality isomorphism

$$\phi\colon (TP \oplus T^*P)/T^n \to (T\widehat{P} \oplus T^*\widehat{P})/\widehat{T}^n$$

exchanges extended actions (H, ξ) and $(\widehat{H}, \widehat{\xi})$, and we have the desired isomorphism

$$\mathcal{H}/T^n \cong ((TP \oplus T^*P)/T^n)_{red} \cong ((T\widehat{P} \oplus T^*\widehat{P})/\widehat{T}^n)_{red} \cong \widehat{\mathcal{H}}/\widehat{T}^n$$

The proof hinges on establishing the following identity,

$$\widehat{H} + \widehat{\xi} = H + \xi + d_G\langle\theta, \hat{\theta}\rangle$$

where $\widehat{A} - A = -\iota\langle\theta, \hat{\theta}\rangle$.

Remarks

- T-duality can be adapted to incorporate:
  - String structures
  - Trivially extended actions
  - Heterotic Courant algebroids
- Heterotic Buscher rules are recovered via generalised metrics.
- The Pontryagin class $\frac{1}{2}p_1(TX)$ can be included.
- The heterotic Einstein equations are preserved under T-duality.
- String structures allow for more flexibility in the possible changes in topology under T-duality.

Examples

Proposition

Let $c \in H^2(M, H^1(\hat{T}^n, \mathbb{Z}))$ *and* $\hat{c} \in H^2(M, H^1(T^n, \mathbb{Z}))$ *be the Chern classes of* $X \to M$ *and* $\hat{X} \to M$*. Then the following holds in* $H^4(M, \mathbb{R})$*:*

$$\langle c, \hat{c} \rangle = p_1(P_0).$$

Ordinary T-duality corresponds to $\langle c, \hat{c} \rangle = 0$.

- Higher dimensional Lens spaces.
- Homogeneous spaces $G \to G/H$.