# Group Cocycles and Higher Representation Theory

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# Group cohomology

- Group $G$ and a $G$-module $A$ $\rightsquigarrow$ $H^n(G, A)$
- $H^0(G, A) = A^G = \{a \in A \mid g.a = a \,\forall g \in G\}$
- $H^1(G, A) = \{\alpha\colon G \to A \mid \alpha(gh) = \alpha(g) + g.\alpha(h)\}/\sim$
- Example of 1-cocycle:

$$\frac{d}{dx}\colon \mathit{Diff}(\mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R}^*)$$

# 2-cocycles

- $H^2(G, A) = \{\alpha\colon G \times G \to A \,|\delta\alpha = 0\}/\sim$

$$\delta\alpha(g, h, k) = g.\alpha(h, k) - \alpha(gh, k) + \alpha(g, hk) - \alpha(g, h) = 0$$

- Abelian extensions

$$1 \to A \to \widehat{G} \to G \to 1$$

$$(g, a) \cdot (h, b) = \big(gh, a(g.b)\alpha(g, h)\big)$$

# Projective representations

- $V$ a $\mathbb{C}$-vector space and $\alpha \in H^2(G, \mathbb{C}^*)$
- $\Phi\colon G \to PGL(V)$ a group homomorphism
- $\Phi\colon G \to GL(V),\ \Phi(gh) = \Phi(g)\Phi(h)\alpha(g,h)$
- $\Phi\colon \widehat{G} \to GL(V)$ a group homomorphism

# 3-cocycles

- $H^3(G, A) = \{\alpha \colon G \times G \times G \to A \,|\, \delta\alpha = 0\} / \sim$

$$\delta\alpha(g_1, g_2, g_3, g_4) = g_1.\alpha(g_2, g_3, g_4) - \alpha(g_1 g_2, g_3, g_4) +$$

$$+\alpha(g_1, g_2 g_3, g_4) - \alpha(g_1, g_2, g_3 g_4) + \alpha(g_1, g_2, g_3) = 0$$

- Question: is there an extension and representation theoretic interpretation?

# Octonions

- Group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted by

$$\beta(\mathbf{u}, \mathbf{v}) = (-1)^{\sum_{i<j} u_i v_j + u_1 u_2 v_3 + u_3 u_1 v_2 + u_2 u_3 v_1}$$

- Generators $\{e(\mathbf{u}) \mid \mathbf{u} \in \mathbb{Z}_2^3\}$ of the twisted group algebra $\mathbb{R}[\mathbb{Z}_2^3]_\beta$ satisfy

$$e(\mathbf{u})e(\mathbf{v}) = \beta(\mathbf{u}, \mathbf{v})e(\mathbf{u} + \mathbf{v})$$

- Associator

$$\alpha(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \delta\beta(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (-1)^{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}$$

# Categories

- A category consists of “objects" that are linked by “arrows"
- There exists a binary operation $\circ$ to compose the arrows associatively and an identity arrow **1** for each object
- A functor is a homomorphism between categories

# Abelian categories

- An abelian category $\mathcal{C}$ has the property that objects and morphisms can be added

  More precisely, there is a zero object that is both initial and terminal, $\mathcal{C}$ contains all pullbacks and pushouts, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel

- The auto-equivalences of an abelian category $\mathbb{GL}(\mathcal{C})$ is a 2-group

# 2-groups

- A groupoid is a (small) category where every morphism is an isomorphism
- A monoidal category $\mathcal{C}$ is a category with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, unit object $I$ and three natural isomorphisms

  $$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad I \otimes X \cong X, \quad X \otimes I \cong X$$

  subject to two coherence conditions
- The 2-group $\mathbb{G}$ is a monoidal groupoid such that $\pi_0(\mathbb{G})$ is a group. Here $\pi_0(\mathbb{G})$ is the set of isomorphism classes of objects, and it acts naturally on $\pi_1(\mathbb{G}) = End_{\mathbb{G}}(I)$

# Classification of 2-groups

- Theorem: A 2-group $\mathbb{G}$ is classified by its Postnikov invariant in $H^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))$

- $\mathbb{GL}(\mathcal{C})$ is classified by a 3-cocycle in $H^3(\pi_0(\mathbb{GL}(\mathcal{C})), \mathcal{Z}(\mathcal{C})^*)$, where $\mathcal{Z}(\mathcal{C}) = End_{\mathcal{C}}(\mathbf{1})$ is the center

- Note that 2-groups are equivalent to crossed modules

# Gerbal representations (Frenkel-Zhu 2011)

- $\mathcal{C}$ an abelian category and $\alpha \in H^3(G, \mathcal{Z}(\mathcal{C})^*)$
- $\Phi\colon G \to \pi_0(\mathbb{GL}(\mathcal{C}))$ a group homomorphism
- $\Phi\colon G \to \mathbb{GL}(\mathcal{C})$ where $\Phi(gh) \cong \Phi(g) \circ \Phi(h)$.

  Two ways to identify $\Phi(g) \circ \Phi(h) \circ \Phi(k)$ with $\Phi(ghk)$:

$$\Phi(gh) \circ \Phi(k) = \Phi(g) \circ \Phi(hk)\alpha(g, h, k)$$

- $\Phi\colon \widehat{G} \to \mathbb{GL}(\mathcal{C})$ a 2-group homomorphism, where

$$1 \to B\mathcal{Z}(\mathcal{C})^* \to \widehat{G} \to G \to 1$$

# Basic properties

- $[\alpha] = 0 \Rightarrow$ the gerbal representation lifts to a genuine 2-group homomorphism $\Phi : G \to \mathbb{GL}(\mathcal{C})$
- A homomorphism of gerbal $G$-modules is a functor $F : \mathcal{C} \to \mathcal{D}$ satisfying $F \circ \Phi_g \cong \Phi_g \circ F$ for all $g \in G$
- The gerbal representations are equivalent if $F$ is an equivalence of categories
- A gerbal submodule $\mathcal{C} \subset \mathcal{D}$ is a $G$-invariant subcategory

# Basic example of gerbal representations

- Let $R$ be a $\mathbb{C}$-algebra and $R$-mod denote the abelian category of left $R$-modules. Note: $\mathcal{Z}(R\text{-mod}) = \mathcal{Z}(R)$

- Consider a $G$-action by outer automorphisms $\phi : G \to Out(R)$, where

$$1 \to Inn(R) \to Aut(R) \to Out(R) \to 1$$

- Fix a central extension

$$1 \to \mathcal{Z}(R)^* \to \widehat{Inn(R)} \to Inn(R) \to 1$$

# Basic example of gerbal representations

- The obstruction to the prolongation $\hat{\phi} : G \to \widehat{Aut(R)}$,

$$1 \to \widehat{Inn(R)} \to \widehat{Aut(R)} \to Out(R) \to 1$$

is a 3-cocycle $\alpha \in H^3(G, \mathcal{Z}(R)^*)$

- The gerbal representation $\Phi : G \to \pi_0(\mathbb{GL}(R\text{-mod}))$ is

$$(\Phi_g m)(r, x) = m(\tilde{s}(g)^{-1} r, x), \quad \Phi_g(f) = f$$

where $m : R \otimes M \to M$ denotes an $R$-module (object), $\tilde{s} : G \to Aut(R)$ is a lifting and $f$ is any morphism of $R$-modules.

# Example of 3-cocycles for finite groups

- $H^n(G, A)$ is always torsion for finite groups

- Dihedral group

$$H^3(D_n, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

- Every 3-cocycle in $H^3(\mathbb{Z}_3, \mathbb{C}^*)$ is of the form

$$\alpha(x, y, z) = \left(a^{(-1)^z + x - xz} b^{x-z}\right)^{(-1)^y} \begin{cases} 1 & \text{if } x = y = 1 \\ \omega^z & \text{if otherwise} \end{cases}$$

where $a, b \in \mathbb{C}^*$ and $\omega$ is a cubic root of the unity.

# Motivation

- $LG = C^\infty(S^1, G)$ has a well-studied class of projective highest weight representations and central extensions,

$$1 \to S^1 \to \widehat{LG} \to LG \to 1$$

- No such theory for $C^\infty(M, G)$, where $M$ is a compact manifold of dimension larger than 2

- $H^3(C^\infty(M, G), A)$ is non-trivial for certain modules $A$

# Open problems

- Develop a gerbal 2-character theory

- Appropriate notion of 2-group algebra and their modules

- Study “tensorial" gerbal representations

- For discrete groups, investigate the link to geometry:

$$H^n(G,\mathbb{Z}) \cong H^n(BG,\mathbb{Z})$$

  For instance, $H^3(\mathbb{Z}^3,\mathbb{Z}) = H^3(\mathbb{T}^3,\mathbb{Z}) \cong \mathbb{Z}$

  Gerbal representations $\Leftrightarrow$ bundle gerbe modules?

$$R_\alpha(G) \cong K^0(BG,\alpha)?$$