Generalised Stone Dualities

David Farrell
Department of Mathematics
The University of Auckland

Supervisor: Pedram Hekmati

A dissertation submitted in partial fulfillment of the requirements for the degree of BSc(Hons) in Mathematics, The University of Auckland, 2018.

Abstract

In 1938, Marshall H. Stone, motivated by his work in functional analysis, proved his *representation theorem for Boolean algebras*, which establishes a correspondence between *Boolean algebras*, and a certain class of topological spaces, called *Stone spaces*. This correspondence, after the inception of category theory in the 40s, was recognised to be a *duality of categories*.

More recently, it has been found that there exists a similar correspondence between the category of those topological spaces satisfying a weak separation condition called *sobriety* and the category of lattices, called *spatial locales*, which satisfy a set of axioms analogous to those for a topology. A slight weakening of the conditions on spatial locales yields *locales*, which can be shown to satisfy many nice properties. For example, the analogue for Tychonoff's theorem for locales has a constructive proof. The theory of locales is considered to be a model of *point-free topology*, since the spaces considered do not have a primitive notion of *point*.

The duality between sober topological spaces and spatial locales is one of a number of theorems given the name *Stone duality*. In this dissertation I give a further generalisation of the methods of Stone duality which provides a duality theorem for any category of lattices which satisfies a handful of weak conditions. Special cases include Stone's representation theorem for Boolean algebras, the duality between sober spaces and spatial locales, and a correspondence between a category of Boolean algebras and a subcategory of measurable spaces.

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Introduction

Consider the following observation. Let (X, T) and (Y, S) be topological spaces and $f : X \to Y$ a homeomorphism. Then f is open and hence induces a map $f^* : T \to S$ by mapping an open set $U \in T$ to its image $f(U)$ under f. Since f is a bijection, so is f^*. A homeomorphism thus induces a bijection of the corresponding topologies which respects intersections, unions and inclusions in T. So structure preserving maps of the underlying spaces give rise to structure preserving maps between the structures placed upon them. Is there a way to tell which such maps arise from maps of spaces? Perhaps not, but there are special cases where all such maps do indeed arise in this way.

A topology is a poset, when ordered by inclusion, and has the property that, for any finite set of pairs, there is a unique minimal set following them in the order: their union; and there is a unique set preceding them in the order: their intersection. This is a special case of a lattice, which is a poset containing suprema and infima for all of its finite subsets. A topology is then just a lattice of sets which, in addition, contains arbitrary unions of its elements. A structure preserving map between topologies should then be one which respects all unions and intersections. In general, for a continuous function $f : X \to Y$ of topological spaces, the image $f(U)$ of an open set is not open, and f^* does not respect unions and intersections. However, by definition of continuity, of $U \subset Y$ is open, then $f^{-1}(U)$ is open in X, and inverse images respect all unions and intersections.

Thus, for every continuous map $f : X \to Y$ there is a corresponding "homomorphism" of topologies $f^{-1} : S \to T$ going the other way. In the case considered above where f is a homeomorphism, f^* is the inverse image map of the inverse of f. Now, if the condition that a topology consists of *sets* is relaxed, and we consider just lattices with arbitrary supremums, plus a distributivity condition which will be given later, we arrive at *frames*. Since the maps between frames go in the opposite direction to those between topological spaces, we often consider instead the opposite category to the category of frames, called the category of locales[1]. To what extent do locales correspond to spaces, and to what extent do maps of locales correspond to maps of spaces? There

[1]Short for *local lattices*.

exist locales which do not arise from topological spaces, and some which do have maps which do not arise from continuous functions of spaces. However there exist conditions called *spatiality* and *sobriety* for locales and topological spaces respectively such that one has an equivalence between the full subcategories **SpLoc** and **SobTop** of *spatial locales* and *sober topological spaces.*

This equivalence is an example of a so-called *Stone Duality theorem*, which is a loose name for a collection of theorems which give dualities between categories thought to consist of *algebraic* objects, such as lattices, and categories thought to contain *geometric* objects, such as topological spaces.

The main difference between topological spaces and locales is the topological spaces come with a "primitive" notion of points; the elements of the point set. Locales do not. Hence the study of locales is often referred to as *point-free topology* or *pointless topology.* Locales have many analogies with topological spaces, and many definitions and theorems can be translated from topological spaces to locales either directly or through Stone duality. There are, however, some distinctions. For instance, for topological spaces, the statement *a product of spaces is compact if and only if each of its factors is*, known as Tychonoff's theorem, is equivalent to the axiom of choice. The notion of compactness carries over for lattice almost verbatim, but the corresponding analogue of Tychonoff's theorem has a constructive proof. For this and other reasons, a large source of the motivation for pointless topology comes from constructive mathematics, hence results relying on the axiom of choice are marked with a *.

In measure theory one of the main objects of study are so-called *measurable spaces.* Measurable spaces are, again, just sets with a particular type of lattice of subsets associated. Can Stone duality apply to measurable spaces? In fact, is the choice of *type* of lattice important at all? Is there a suitable notion of *type* of lattice such that each type comes with a Stone-like duality theorem? Are there conditions which ensure that maps of lattices correspond to maps of some type of space? In this dissertation I will answer these questions.

The structure is as follows. In chapter 1, I give a brief treatment of the fundamentals of lattice theory, including topics such as lattices, homomorphisms, distributivity, complements, completeness and Boolean algebras. In chapter 2, I cover some introductory Locale theory, including their associated point spaces and some of the nicer results. In chapter 3, I give generalisations for the concepts from locale theory, including the aforementioned notion of *type* of lattice, which takes the form of a choice of suitable category $\mathscr{K}$ of lattices. Chapter 3 culminates in the main result of this dissertation, Theorem 3.5.2, the extension of the Stone duality theorem to any choice of $\mathscr{K}$. Chapter 4 covers the Stone representation theorem for Boolean algebras, which, with a little extra work, can be seen as a special case of Theorem 3.5.2. Chapter 5 covers the details of the application of Theorem 3.5.2 to measurable spaces.

Chapter 1

Lattices

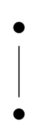
Definition 1.0.1. Let L be a poset. Then L is said to be a *lattice* if, every finite set $S \subset L$ has both an infimum $\wedge S$, called the *meet* of S, and a supremum $\vee S$, called the *join* of S.

If $x, y \in L$, we write $x \wedge y := \wedge\{x, y\}$ and $x \vee y := \vee\{x, y\}$. Hence one has associative commutative binary operations $\wedge, \vee : L \times L \to L$ for any lattice L. A subset $S \subset L$ of a lattice L is said to *join in* L if it has a supremum $\vee S$ in L, and *meet in* L if it has an infimum $\wedge S$ in L. One writes

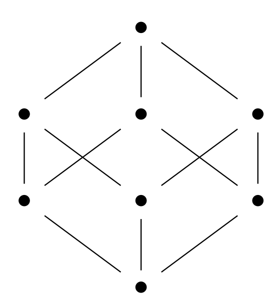
$$\bigwedge \varnothing = \top \quad \text{and} \quad \bigvee \varnothing = \bot.$$

and observes that every lattice is bounded.

Finite lattices can be given by Hasse diagrams, in which the elements are depicted as vertices with an edge running upwards from a node x to a node y if and only if y *covers* x, i.e., $x \leq y$ and $x \leq z \leq y$ implies $z = x$ or $z = y$. For instance, one has the lattices



$\mathbf{2} = \{0, 1\}$

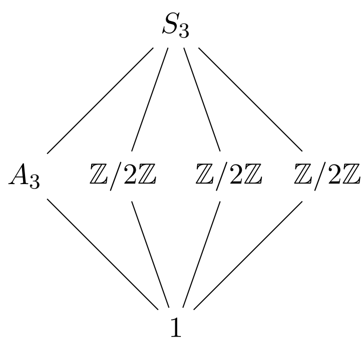


$\mathbf{2}^3$

where $\mathbf{2}$ is equipped with the unique order with $0 \leq 1$ and $\mathbf{2}^3$ has the order given by $(a, b, c) \leq (d, e, f)$ iff $a \leq d$ and $b \leq e$ and $c \leq f$.

Now linearly ordered bounded sets such as $\mathbf{2}$ and $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ are trivially lattices, with their meet and join being simply min and max. More interestingly, if X is a set then the power set $\mathcal{P}(X)$ is a lattice under inclusion with meet and join being to intersection and union. Many of the structures that are important in modern mathematics arise as sublattices of power sets. For one such example, let G be a group. Then the set $\mathcal{S}(G)$ of subgroups of G is a sublattice of $\mathcal{P}(G)$ and is subject to much of the attention paid by group theorists. Further, and more relevant, examples come from the aforementioned observation that a topology on a set X is nothing more than a sublattice of $\mathcal{P}(X)$ with a few extra properties, and similarly for a σ-algebra on X.

Example 1.0.1. *A Hasse diagram of the subgroup lattice of S_3, labelled by isomorphism type.*



Definition 1.0.2. Let L and J be lattices. Then a function $f : L \to J$ is said to be a lattice homomorphism if it preserves meet and join, i.e., whenever $x, y \in L$, one has

$$f(x \wedge y) = f(x) \wedge f(y) \quad \text{and} \quad f(x \vee y) = f(x) \vee f(y)$$

If f is a bijective lattice homomorphism then f^{-1} is a lattice homomorphism and f (and f^{-1}) is said to be an *isomorphism*.

Note that $x \wedge y = x$ if and only if $x \leq y$ (Similarly, $x \vee y = x$ iff $y \leq x$) hence if f is a lattice homomorphism and $x \leq y$ then

$$f(x) = f(x \wedge y) = f(x) \wedge f(y)$$

hence $f(x) \leq f(y)$, i.e., lattice homomorphisms are *isotone*;

Definition 1.0.3. Let P and L be posets and $f : P \to L$ a function. Then f is said to be *isotone*, *order preserving* or an *order homomorphism* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in P$. If $x \leq y$ implies $f(x) \geq f(y)$ then f is said to be *antitone* or *order reversing*.

Note that the inverse of an isotone map is not, in general, isotone. An *order isomorphism* is a map $f : P \to L$ where $x \leq y$ is *equivalent* to $f(x) \leq f(y)$.

Note also that a homomorphism f of lattices satisfies $f(\bot) = \bot$ and $f(\top) = \top$.

Definition 1.0.4. Let L be a lattice. Then the *opposite lattice*, L^{op}, is the poset whose underlying set is the same as that of L, but one has $x \leq y$ in L^{op} if and only if $y \leq x$ in L.

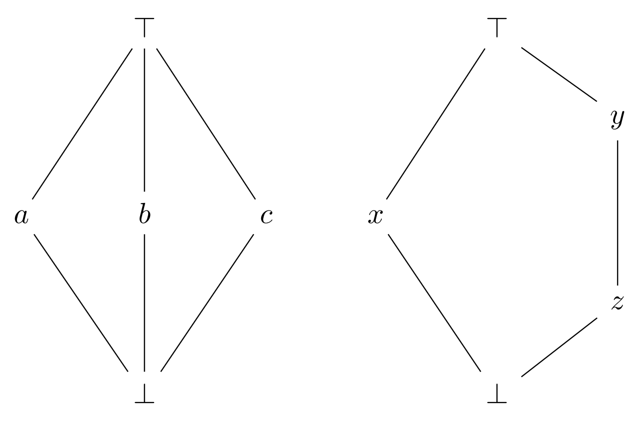
Due to the symmetry in the definition of a lattice, it is clear to see that the opposite lattice to any lattice is still a lattice. Hence one has a *duality principle* for lattices: any theorem for lattices has a *dual*. That is, if one proves a theorem for all lattices, then it holds in any lattice L and its opposite L^{op}. Thus if a theorem is true, so is the same theorem with all the inequalities reversed and meets and joins swapped.

Definition 1.0.5. Let L be a lattice. Then L is said to be *distributive* if any of the two equivalent statements

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{or} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

hold for all $x, y, z \in L$.

It is a theorem [2] that a lattice L is distributive if and only if it does not contain either of the nondistributive lattices



M_3 $\qquad\qquad$ N_5

as a sublattice. For instance, the subgroup lattice of S_3 contains a copy of M_3, hence it is not distributive (moreover, while it is contained in the set $\mathcal{P}(S_3)$, it is not a *sublattice* of $\mathcal{P}(S_3)$; the join in $\mathcal{P}(S_3)$ is union, but the join of two subgroup G and H is the subgroup $\langle G, H\rangle$). Examples of distributive lattices, crucially, include the power sets and hence their sublattices.

Definition 1.0.6. Let L be a lattice. Two elements $x, y \in L$ are *disjoint* if $x \wedge y = \bot$ and *complementary* if $x \wedge y = \bot$ and $x \vee y = \top$, in which case we also say that y is a complement to x.

Proposition 1.0.1. *Let L be a distributive lattice. Then any element $x \in L$ has at most one complement.*

Proof. Let y and z be complements for x. Then

$$\begin{aligned} y &= \top \wedge y \\ &= (z \vee x) \wedge y \\ &= (z \wedge y) \vee (x \wedge y) \\ &= (z \wedge y) \vee \bot \\ &= z \wedge y \end{aligned}$$

hence $y \leq z$. Interchanging y and z in the above argument yields $z \leq y$ thus $y = z$. □

This result shows immediately why the lattices M_3 and N_5 are nondistributive: in M_3, the element a has distinct complements b and c, and in N_5, the element x has distinct complements y and z.

A lattice is said to be *complemented* if each element has a complement. A *uniquely complemented lattice* is a complemented lattice in which every element x has a unique complement x^c.

Proposition 1.0.2. *Let L and J be uniquely complemented lattices and $f : L \to J$ a homomorphism. Then, for all $x \in L$, $f(x^c) = f(x)^c$.*

Proof. Let $x \in L$. Then

$$f(x) \wedge f(x^c) = f(x \wedge x^c) = f(\bot) = \bot$$

and

$$f(x) \vee f(x^c) = f(x \vee x^c) = f(\top) = \top$$

hence $f(x^c)$ is complementary to $f(x)$. □

1.1 Ideals and Filters

Definition 1.1.1. Let L be a lattice. An *ideal* is a subset $I \subset L$ such that

1. $\bot \in I$,
2. $x \vee y \in I$ for all $x, y \in I$, and

3. I is *lower closed*, i.e., whenever $x \in I$, $\{y \in L : y \leq x\} \subset I$.

and a *filter* in L is an ideal in L^{op}. That is, a subset $F \subset L$ such that

1. $\top \in F$,
2. $x \wedge y \in F$ for all $x, y \in F$, and
3. F is *upper closed*, i.e., whenever $x \in F$, $\{y \in L : x \leq y\} \subset F$.

An ideal (or filter) is said to be *proper* if it is not the whole lattice L.

Example 1.1.1. *(Examples of filters and ideals)*

1. *Let L be a lattice and $a \in L$. Then the sets*
$$a^{\uparrow} := \{x \in L : a \leq x\} \text{ and } a^{\downarrow} := \{x \in L : x \leq a\}$$
are, respectively, an filter and an ideal in L. Filters and ideals generated in this way by a single element are called principal filters (ideals).
2. *Let $f : L \to K$ be a lattice homomorphism. Then $f^{-1}(\{\perp\})$ is an ideal of L and $f^{-1}(\{\top\})$ is a filter of L.*
3. *Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$, which is a sublattice of $\mathcal{P}(\mathbb{R})$. Let $\mathcal{N}$ be the set of Borel sets of $\mathbb{R}$ with (Lebesgue) measure 0. Then $\mathcal{N}$ is an ideal of $\mathcal{B}$.*

Two important properties that ideals and filters may enjoy are *primeness* and *maximality*. The definitions are given here for filters and their duals apply equally to ideals.

Definition 1.1.2. Let L be a lattice and $I \subset F$ a proper filter. Then F is said to be *prime* if, whenever $a, b \in F$ and $a \vee b \in F$ then $a \in F$ or $b \in F$.

Example 1.1.2. *Let X be a nonempty set.*

1. *Let $x \in X$. The set of subsets of X containing x is a prime filter of X; if $U \cup V$ contains x then one of U and V must contain x.*
2. *Suppose $x, y \in X$ are distinct. Then $\{x, y\}^{\uparrow}$ is a filter but is not prime, since $\{x\} \cup \{y\} \in \{x, y\}^{\uparrow}$ but neither of the singletons contain both x and y.*

Strengthenings of the condition of primeness are of huge importance in later chapters.

Definition 1.1.3. Let L be a lattice and $F \subset L$. Then F is said to be *maximal* if F is a maximal element of the poset of filters on L, ordered by inclusion. That is, every proper filter F' containing F satisfies $F = F'$.

Remark. A maximal filter is called an *ultrafilter*, although a maximal ideal is **not** called an *ultraideal.*

Example 1.1.3. *The filter* $\{x\}^{\uparrow}$ *of example 1.1.2 (1) is an ultrafilter, but* $\{x, y\}^{\uparrow}$ *is not (*$\{x\}^{\uparrow}$ *is strictly larger). The ideal* $\mathcal{N}$ *of example 1.1.1 (3) is maximal.*

1.2 Completeness

Definition 1.2.1. Let L be a lattice. Then L is said to be *complete* if each set $S \subset L$ has both a join and a meet in L.

Example 1.2.1. *Any finite lattice is complete, as is any power set or any topology. The chain* $\{-\infty\} \cup \mathbb{Q} \cup \{\infty\}$ *is not complete: the set* $\{x \in \mathbb{Q} : x^2 < 2\}$ *has no supremum. The Borel* σ*-algebra on* $\mathbb{R}$ *is not complete if and only if the axiom of choice holds.*

Remark. Sometimes, for convenience, "S has a join/meet in L" will be shortened to "S meets/joins in L".

Proposition 1.2.1. *Let* L *be a lattice. Then* L *is complete if and only if every subset* $S \subset L$ *has a join in* L*.*

Proof. The forward direction is trivial. For the converse, let $S \subset L$. Then the set $S^{\downarrow}$ of lower bounds of S has a join $\vee(S^{\downarrow})$, which is clearly the meet $\wedge S$. □

One also has notions of completeness for homomorphisms.

Definition 1.2.2. Let L and T be complete lattices and $f : L \to T$ a homomorphism. Then f is said to be *complete* if, for all $S \subset L$, $f(\vee S) = \vee f(S)$ and $f(\wedge S) = \wedge f(S)$.

In cases where the lattices involved are not necessarily complete, there is still a notion of completeness, but it requires that the sets in consideration actually do meet or join.

Definition 1.2.3. Let L and T be lattices and $f : L \to T$ a homomorphism. Then f is said to be *conditionally complete* if, whenever $S \subset L$ joins in L then $f(S)$ joins in T and $f(\vee S) = \vee f(S)$ and whenever $S \subset L$ meets in L then $f(S)$ meets in T and $f(\wedge S) = \wedge f(S)$.

Note that if f is conditionally complete and L is complete then f is complete.

Proposition 1.2.2. *Let* L *and* T *be lattices and* $f : L \to T$ *an order isomorphism. Then* f *is a conditionally complete lattice homomorphism.*

Proof. Let $S \subset L$ and suppose that S has a meet $\vee S$. Then, since f is isotone, $f(\vee S)$ is an upper bound for $f(S)$. Let y be an upper bound for $f(S)$. Then, since f^{-1} is isotone, $f^{-1}(y) \geq \vee S$. Therefore

$$y = f(f^{-1}(y)) \geq f(\vee S)$$

so that $f(\vee S)$ is the least upper bound for $f(S)$, i.e., $f(\vee S) = \vee f(S)$. The argument that $f(\wedge S) = \wedge f(S)$ is analogous. □

Corollary 1.2.0.1. *Let L and T be lattices and $f : L \to T$ an order isomorphism. Then f is conditionally complete, and L is complete if and only if T is.*

1.3 Boolean Algebras

Definition 1.3.1. A *Boolean algebra* is a distributive, complemented lattice.

Since a Boolean algebra is distributive, its complements are unique. There do, however, exist uniquely complemented lattices which are not distributive lattices, hence not Boolean algebras. Our prime examples of Boolean algebras are the power sets of sets. Now, every sublattice of a distributive lattice is distributive, but in general a sublattice of a Boolean algebra is not a Boolean algebra. For instance, a topology on a set X is often not a Boolean algebra, though it is always a sublattice.

1.3.1 Basic Facts of Boolean Algebras

Here we give some of the basic theorems about Boolean algebras, all of which will come in handy later.

Theorem 1.3.1. *Let B be a Boolean algebra. Then B satisfies the De Morgan laws*

$$(x \wedge y)^c = x^c \vee y^c$$
$$(x \vee y)^c = x^c \wedge y^c$$

for all $x, y \in B$.

Proof. By duality, (since the dual of a Boolean algebra is again a Boolean algebra), it suffices to prove only one of the equations. Let $x, y \in B$. Then

$$\begin{aligned}(x \wedge y) \wedge (x^c \vee y^c) &= x \wedge (y \wedge (x^c \vee y^c)) \\ &= x \wedge x^c \\ &= \perp\end{aligned}$$

and

$$\begin{aligned}(x \wedge y) \vee (x^c \vee y^c) &= (x \vee (x^c \vee y^c)) \wedge (y \vee (x^c \vee y^c)) \\ &= (\top \vee y^c) \wedge (\top \vee x^c) \\ &= \top\end{aligned}$$

hence $(x \wedge y)^c = x^c \vee y^c$. ☐

Corollary 1.3.1.1. *The map $\cdot^c : B \to B$ is an antihomomorphism, i.e., a lattice homomorphism $B \to B^{\mathrm{op}}$, and hence is order reversing.*

1.3.2 Boolean Rings

Somewhat surprisingly, Boolean algebras can be represented equivalently as lattices *or rings*. Define a *Boolean ring* to be a ring R in which each element is idempotent, i.e., $x^2 = x$ for all $x \in R$. If B is a Boolean algebra, then the operations

$$x + y := (x \wedge y^c) \vee (y \wedge x^c), \text{ and}$$

$$xy := x \wedge y$$

yield a Boolean ring structure on B, and if B is a Boolean ring, then the order given by $x \leq y$ iff $xy = x$ gives B the structure of a Boolean algebra. The verification is quite standard so will not be replicated here (see [3]. The other direction is more interesting.

Proposition 1.3.1. *Let B be a Boolean ring. Then the relation given by $x \leq y$ iff $xy = x$ is the unique partial order $\leq$ on B such that $(B, \leq)$ is a lattice and*

1. $\perp = 0$,
2. $\top = 1$,
3. $x + y = (x \wedge y^c) \vee (y \wedge x^c)$, *and*
4. $xy = x \wedge y$.

Again, the proof is omitted, as it is very standard. Henceforth, Boolean algebras will be considered to be both lattices and rings. Now, in a Boolean algebra, one has

$$x + y = (x \wedge y^c) \vee (y \wedge x^c) \quad \text{and} \quad x \vee y = x + y + xy \quad \text{and} \quad xy = x \wedge y$$

so that a function of Boolean algebras is a ring homomorphism if and only if it is a lattice homomorphism. Moreover, ideals of Boolean rings and Boolean algebras coincide:

Proposition 1.3.2. *Let B be a Boolean algebra. Then a subset $I \subset B$ is an ideal of the lattice B if and only if it is an ideal of the ring B.*

Proof. Suppose I is an ideal of the lattice B. Let $x \in I$, $y \in B$. Then $x \geq x \wedge y = xy = yx \in I$. Now let $x, y \in B$. Then, since $x \in I$, $x \wedge y^c \in I$. Similarly, $y \wedge x^c \in I$. Hence $x + y = (x \wedge y^c) \vee (y \wedge x^c) \in I$. The reverse implication follows by reversing the argument. □

Corollary 1.3.1.2. *Let $I \subset B$ be a proper ideal. Then B is a prime ideal of the lattice B if and only if it is a prime ideal of the ring B.*

Proof. Immediate since the product and meet coincide. □

Corollary 1.3.1.3. *Let $I \subset B$ be a proper ideal. Then B is a maximal ideal of the lattice B if and only if it is a maximal ideal of the ring B.*

Corollary 1.3.1.4. *Prime ideals of Boolean algebras are maximal.*

Proof. Let $I \subset B$ be a prime ideal. Then the quotient ring B/I is an integral domain where one has $b^2 = b$ for all b, or, equivalently, $b(1-b) = 0$ for all b. Hence $B/I = \mathbb{Z}/2\mathbb{Z}$, which is a field, thus I is maximal. □

Corollary 1.3.1.5. *Dually, prime filters of Boolean algebras are ultrafilters.*

Hence one has the following characterisation of ultrafilters in Boolean algebras.

Proposition 1.3.1. *Let B be a Boolean algebra and $F \subset B$ a proper filter. Then the following are equivalent.*

1. *F is a prime filter,*
2. *F is an ultrafilter,*
3. *$b \in F$ or $b^c \in F$ for all $b \in B$.*

Proof. See [3]. □

The following result, the Ultrafilter lemma, is independent of ZF and uses Zorn's lemma for its proof, although it is strictly weaker than choice. For details on this, see [4]. It is needed in the proof of Stone's representation theorem for Boolean algebras.

Theorem* 1.3.1. ***Ultrafilter Lemma***
Let B be a Boolean algebra and $F \subset B$ a proper filter. Then there exists an ultrafilter F' containing F.

Proof. Let P be the set of proper filters containing F, ordered by inclusion. P is nonempty since $F \in P$. Claim: every chain in P has an upper bound in P. Let $\{P_i\}_{i \in I}$ be a chain in P. Let $P' := \bigcup_{i \in I} P_i$. That P' is a filter is easy. That P' is an upper bound for the chain $\{P_{i \in I}\}$ is also easy. Hence, by Zorn's lemma, P has a maximal element F'. By definition, F' is a filter. If any proper filter G strictly contains F', then it contains F and hence is in P, contradicting maximality of F'. Hence F' is an ultrafilter. □

Corollary 1.3.1.6. *Let B be a Boolean algebra and $S \subset B$ be such that the filter F generated by S is proper. Then there is an ultrafilter containing S.*

Corollary 1.3.1.7. *Let B be a Boolean algebra and $\perp \neq x \in B$. Then x lies in some ultrafilter of B.*

Having now seen the essentials of lattice theory, we turn to an application of lattice theory, namely the study of *frames and locales* as a model of *pointless topology.*

Chapter 2

Pointless Topology

First, recall the definition of a topology.

Definition 2.0.1. Let X be a set. Then a *topology on* X is a set $T \subset \mathcal{P}(X)$ such that

1. $\varnothing, X \in T$,
2. $\bigcup S \in T$ whenever $S \subset T$, and
3. $\bigcap S \in T$ whenever $S \subset T$ is finite.

Hence, in particular, a topology is a sublattice of $\mathcal{P}(X)$. To generalise, we drop the restriction that the elements of our lattice are subsets of some fixed set, and impose a distributivity condition.

Definition 2.0.2. Let F be a lattice. Then F is said to be a *frame* if

1. Every set $S \subset F$ joins in F,
2. Every finite set $S \subset F$ meets in F, and
3. For all $x \in F$ and $S \subset F$, the distributivity law

$$x \wedge \left(\bigvee_{s \in S} s \right) = \bigvee_{s \in S} (x \wedge s)$$

holds.

Note that, by Proposition 1.2.1, a frame is a complete lattice.

Example 2.0.1. *Examples of frames include, of course, all topologies. In fact, to describe a frame which is not a topology is nontrivial, and is deferred until section 2.3.*

A homomorphism of frames is a function $f : X \to Y$ between frames X and Y which respects all meets and all finite joins. The inverse of a bijective frame homomorphism is again a frame homomorphism, hence a bijective frame homomorphism is an isomorphism.

2.1 Frames and Locales

Let (X, T) and (Y, S) be topological spaces and $f : X \to Y$ a function. Then f induces a function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ given by taking inverse images. The function f is then continuous precisely when f^{-1} restricts to a function $f^{-1}|_S : S \to T$. Note that, since inverse images respect all unions and intersections, $f^{-1}|_S$ is a frame homomorphism. We will denote $f^{-1}|_S$ by f^* when no confusion is possible. Hence one obtains, for any topological space (X, T) a frame T and for any continuous map $f : (X, T) \to (Y, S)$ a frame homomorphism $f^* : S \to T$. Thus one has a contrvariant functor

$$\begin{aligned} \Omega : \mathbf{Top} &\to \mathbf{Frm} \\ \Omega((X, T)) &= T \\ \Omega(f) &= f^* \end{aligned}$$

In order to work with a category that behaves more like **Top**, one defines the category **Loc** of *locales* to be the opposite category to the category of frames. That is, a locale is a frame, but morphisms of locales run in the opposite direction and are called *continuous maps*. Hence Ω is a *co*variant functor $\mathbf{Top} \to \mathbf{Loc}$.

2.2 Points

With the exclusion of points as a *primitive* notion in our conception of spaces, we wish to recover them as a *structural* notion. The notion of points developed has a number of ways to arrive to it, revealing it as a very natural construction.

2.2.1 Points as Completely Prime Filters

Let (X, T) be a topological space and let $x \in X$. The point x makes its mark on the topology T by being contained in (some of) its sets, hence the set $N(x) := \{U \in T : x \in U\}$ is a natural object to consider. Now $N(x)$ can readily be observed to be not only a filter in T but a *completely prime* filter in T, i.e,. a filter such that, whenever $S \subset T$ and $\bigcup S \in N(x)$ then $S \cap N(x) \neq \varnothing$. Formally,

Definition 2.2.1. Let L be a complete lattice and $F \subset L$ a proper filter. Then F is said to be a *completely prime filter* if

$$\bigvee S \in F \implies S \cap F \neq \varnothing$$

for all $S \subset L$.

Hence every point $x \in X$ induces a completely prime filter in T. Define a *point* in T to be a completely prime filter and write $\operatorname{pt} T := \{F \subset T : F \text{ is a completely prime filter}\}$.

Completely prime filters, however, don't reflect the point set X exactly; for instance, (X, T) is not T_0 precisely when $N(x) = N(y)$ for some pair $x, y \in X$ of distinct points, so completely prime filters cannot distinguish points which are topologically indistinguishable. This flaw may be regarded, though, as a virtue, since it removes some of the pathology of general topological space; two points which behave the same *are* the same. As an example, two indiscrete topological spaces of differing cardinalities would have both precisely one point.

A slightly more bizarre problem with the notion of completely prime filters as points goes in the other direction: consider $\mathbb{N}$ equipped with the cofinite topology T. Set $F := \{U \in T : \mathbb{N} \setminus U \text{ is finite}\} = T \setminus \{\varnothing\}$. Then F is a completely prime filter. Clearly F is upper closed and closed under intersections: since $\mathbb{N}$ is infinite, no open set can be contained in the complement of another. If $S \subset T$ and $\bigcup S \in F$ then $S \neq \{\varnothing\}$ hence $S \cap F \neq \varnothing$. But for each $n \in \mathbb{N}$, one has $\mathbb{N} \setminus \{n\} \in F$ so that $F \neq N(x)$. So $(\mathbb{N}, T)$ has a completely prime filter that doesn't correspond to any points of $\mathbb{N}$. The pathology here lies in the fact that, under the cofinite topology, the set $\mathbb{N}$ is an *irreducible closed set* in the space $\mathbb{N}$. As we have seen, irredicuble closed sets that are "too large" induce completely prime filters that don't correspond to points. Luckily, this is the only way this problem can occur.

Definition 2.2.2. Let (X, T) be a topological space. Then X is said to be a *sober space* if either (and hence both) of the following equivalent conditions hold.

1. Every irreducible closed set $C \subset X$ takes the form $\overline{\{x\}}$ for a unique $x \in X$,

2. Every open set W for which

$$U \cap V \subset W \Rightarrow U \subset W \text{ or } V \subset W \quad \forall U, V \in T \tag{2.1}$$

   takes the form $X \setminus \overline{\{x\}}$ for a unique $x \in X$.

Proposition 2.2.1. *Let (X, T) be a sober topological space. Then (X, T) is T_0.*

Proof. Suppose $x, y \in X$ are distinct and suppose that $N(x) = N(y)$. Every closed set which contains x contains y and vice versa, hence $\overline{\{x, y\}} = \overline{\{x\}} = \overline{\{y\}}$ violating sobriety. □

Theorem 2.2.1. *Let (X, T) be a topological space. Then the function $\varphi : X \to \operatorname{pt} T$ given by $\varphi(x) = N(x)$ is a bijection if and only if (X, T) is sober.*

Proof. Suppose (X, T) is sober. Then, since (X, T) is T_0, φ is injective. Let $F \in \operatorname{pt} X$. Set

$$W := \bigcup (T \setminus F)$$

Since F is completely prime, $X \notin F$ and, in particular, $W \neq X$. Now suppose that $U, V \in T$ and $U \cap V \subset W$. Since $W \notin F$ it follows that, without loss of generality, $U \notin F$. But then $U \subset W$. Hence, by sobriety, $W = X \setminus \overline{\{x\}}$ for some $x \in X$. But then, for any $U \in T$,

$$U \notin F \iff U \subset W \iff x \notin U$$

hence $F = N(x)$.

Suppose that φ is bijective and let $W \in T$ be a set satisfying 2.1. Define

$$F := \{U \in T : U \not\subset W\}$$

Claim: F is a completely prime filter. First, $X \not\subset W$. Whenever $U, V \not\subset W$ then $U \cap V \not\subset W$ by (2.1). If $U \in F$ and $V \supset U$ then clearly $V \in F$, and finally, if $S \subset T$ and $\cup S \in F$, at least one $U \in S$ must contain an element not in W, hence $U \in F$. It follows that $F = N(x)$ for some $x \in X$, but then

$$x \in U \iff U \not\subset W$$

for all $U \in T$, thus $W = X \setminus \overline{\{x\}}$. Hence (X, T) is sober. □

2.2.2 Points as Global Elements

Let X be a set. Then any element $x \in X$ induces a unique function $\hat{x}$ from the one-element set $\mathbf{1} = \{0\}$ to X such that $\hat{x}(0) = x$ and, conversely, every function $f : \mathbf{1} \to X$ "picks" a single element of X. Hence one has a bijection

$$X \to \operatorname{Hom}_{\mathbf{Set}}(\mathbf{1}, X)$$

Similarly, in the category **Top**, if one supplies $\mathbf{1}$ with its unique topology (namely, $\{\varnothing, \mathbf{1}\}$), then for a topological space (X, T), elements $x \in X$ are in bijection to continuous maps $\mathbf{1} \to X$. The defining property of $\mathbf{1}$ in this context is that it is a terminal object in the categories **Set** and **Top**. Hence define

Definition 2.2.3. Let $\mathscr{C}$ be a category with a terminal object T, and let X be an object of $\mathscr{C}$. Then a *global element* (or simply an *element*) of the object X is a morphism $x : T \to X$.

Now the terminal object of the category **Loc** is the two-element frame **2**, and hence global elements of a locale X are continuous functions $\mathbf{2} \to X$, i.e., frame homomorphisms $X \to \mathbf{2}$.

Proposition 2.2.1. *Let X be a frame and $x : X \to \mathbf{2}$ a function. Then x is a frame homommorphism if and only if $x^{-1}(\{\top\})$ is a completely prime filter.*

Proof. Suppose that x is a frame homomorphism. To see that $x^{-1}(\{\top\})$ is a completely prime filter, let $S = \{a, b, \ldots\} \subset X$. Then

1. $x(\top) = \top$ hence $\top \in x^{-1}(\{\top\})$.
2. If $a \leq b$, then $\top = x(a) \leq x(b)$ hence $c \in x^{-1}(\{\top\})$.
3. If $a, b \in x^{-1}(\{\top\})$ then $x(a \wedge b) = x(a) \wedge x(b) = \top$ hence $a \wedge b \in x^{-1}(\{\top\})$.
4. If $\vee S \in x^{-1}(\{\top\})$ then $\top = x(\vee S) = \vee x(S)$ hence $S \cap x^{-1}(\{\top\})$ since, otherwise, $x(S) = \{\perp\}$, which has join $\perp$.

The converse is essentially the reverse of the above argument, and is left to the reader. $\square$

Since a function $X \to \mathbf{2}$ takes at most two values, it is completely determined by its inverse image of $\top$ (or, equivalently, of $\perp$). Hence Proposition 2.2.1 establishes bijection

$$\mathrm{Hom}_{\mathbf{Loc}}(\mathbf{2}, X) \to \mathrm{pt}\, X$$

Given by $x \mapsto x^{-1}(\{\top\})$. One may thus define points equivalently to be global elements. The choice of thinking of points as completely prime filters or globel elements is then a matter of convenience and different contexts can call for one or the other. The strength of the definition of points as global elements is illustrated in the slickness of the following.

Definition 2.2.4. Let X be a locale, $U \in X$ and $x : \mathbf{2} \to X$ a point. Then x is said to *inhabit* U if $x(U) = \top$. Denote by $e(U)$ the set of points inhabiting U.

Proposition 2.2.2. *Let X be a locale. Then the map $e : X \to \mathcal{P}(\mathrm{pt}\, X)$ is a frame homomorphism.*

Proof. Let $S \subset X$. Then

$$\begin{aligned} e(\vee S) &= \{x \in \mathrm{pt}\, X : x(\vee S) = \top\} \\ &= \{x \in \mathrm{pt}\, X : \vee x(S) = \top\} \\ &= \{x \in \mathrm{pt}\, X : \exists s \in S\; x(s) = \top\} \\ &= \bigcup e(S) \end{aligned}$$

Similarly, if S is finite then

$$e(\wedge S) = \{x \in \mathrm{pt}\, X : x(s) = \top : \forall s \in S\} = \bigcap e(S)$$

as required. $\square$

Corollary 2.2.0.1. *The image* $e(X)$ *is a frame, hence a topology on* $\mathrm{pt}\, X$.

Thus each locale can be transformed into a topological space in a natural way. And more: if X and Y are locales and $f : X \to Y$ a continuous map, then one has a map of points $\mathrm{pt}\, f : \mathrm{pt}\, X \to \mathrm{pt}\, Y$ given by

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ 2 \arrow[u, "x"] \arrow[ur, "(\mathrm{pt}\, f)(x)"'] & \end{tikzcd}$$

i.e., $(\mathrm{pt}\, f)(x) = f \hat{\circ} x \in \mathrm{pt}\, Y$ for all $x \in \mathrm{pt}\, X$.[1] The function $\mathrm{pt}\, f$ is continuous. The proof is deferred, and will be proved in much greater generality in a following chapter. The takeaway here is that the map pt, defined on both locales and localic maps, yields a functor

$$\mathrm{pt} : \mathbf{Loc} \to \mathbf{Top}$$

If one defines a locale to be *spatial* if it arises as the topology of a topological space, and writes **SpLoc** for the full subcategory (of **Loc**) consisting of the spatial locales, then the restriction $\mathrm{pt}_{\mathbf{SpLoc}} \to \mathbf{SobTop}$ of pt is an equivalence, having right adjoint the forgetful functor Ω. This is proved, in much greater generality, later.

2.3 A Non-Spatial Locale

Having seen spatiality and sobriety, it would now be prudent to provide an example of a locale which is not spatial. This will, however, require some work. The example given provides a solution to *the problem of random sequences* and was explored in a paper by Simpson [5].

Consider the set $\mathbf{2}^\omega$ of binary sequences. Equipped with the product topology when $\mathbf{2}$ is equipped with the discrete topology, $\mathbf{2}^\omega$ is a cantor set. Moreover, $\mathbf{2}^\omega$ comes with a canonical probability measure λ.

A sequence is intuitively said to be *random* if it satisfies all generic properties, i.e., if one could expect property P to hold for *almost all* sequences then it holds for all random sequences. This suggests that a random sequence should lie in the intersection

[1] Note that this only agrees with intuition in the opposite category **Loc**!

of all full measure sets. The *problem of random sequences*, then, is that this definition is vacuous: if $x \in \mathbf{2}^\omega$, then x does not occupy the measure 1 set $\mathbf{2}^\omega \setminus \{x\}$, thus

$$\bigcup_{\substack{U \subset \mathbf{2}^\omega \\ \lambda(U)=1}} U = \varnothing \tag{2.2}$$

The solution lies in locale theory. Instead of considering the *subspace* of (2.2), one considers the appropriate *sublocale* of the frame Ω of opens of $\mathbf{2}^\omega$.

2.3.1 Sublocales

The notion of a subframe is the usual algebraic notion: a subset of a frame which is also, itself, a frame, when the operations of the superframe are restricted to it. Now every subframe of a frame carries an inclusion morphism, which is of course injective, and every injective morphism has image a subframe of its codomain, isomorphic to its domain. Hence subframes correspond to monomorphisms. So for locales, sublocales should again correspond to monomorphisms. But monomorphisms in **Loc** are *epi*morphisms in the category **Frm** - which correspond to *quotients*. So how does one quotient a lattice? For the quotient of a lattice by an equivalence relation to be again a lattice, the equivalence relation simply needs to respect the operations (of join and meet). For our purposes we will neglect a formal definition and proceed with the rough notion that sublocales should be quotients induced by sufficiently well behaves equivalence relations. In general, for a locale X, each subspace of $\operatorname{pt} X$ gives a sublocale, but X has *many more* sublocales than $\operatorname{pt} X$ has subspaces[2].

Now to provide a sublocale analogue of (2.2), observe that, moving from $\mathbf{2}^\omega$, the top element of the frame of opens switches from the set $\mathbf{2}^\omega$ to an object that should simultaneously represent all full-measure subsets. Hence the right equivalence relation should identify all such subsets. In order for this equivalence relation to have any hope of respecting the operations, it cannot apply to just measure 1 subsets. It turns out that the appropriate equivalence relation is given by: for each $U, V \in \Omega$

$$U \sim V \iff \lambda\big((U \cap V^c) \cup (V \cap U^c)\big) = 0$$

It is clear to see that all full measure sets are identified under this equivalence relation. Hence let R be the sublocale $\Omega/\sim$ of Ω.

The locale R has the striking (and famous) property that $\operatorname{pt} R = \varnothing$. This agrees with the *problem of random sequences*: a point of R should correspond, in some way, to a random sequence - but there are none! To see that this is the case, suppose $F \subset R$ is a completely prime filter. Then the set $F' := \{U \in \Omega : [U] \in F\}$ is clearly a completely

[2]These extra sublocales also provide a conceptual solution to the Banach-Tarski paradox.

prime filter in Ω (where of course $[U]$ denotes the equivalence class of U under $\sim$). Since $\mathbf{2}^\omega$ is Hausdorff, $F' = N(x)$ for some $x \in \mathbf{2}^\omega$. But $\mathbf{2}^\omega \setminus \{x\} \sim \mathbf{2}^\omega$ and $\mathbf{2}^\omega \setminus \{x\} \notin F'$, which is clearly nonsense. This also implies the R is not spatial: $\Omega(\operatorname{pt} R) = \{\varnothing\}$, but R contains at least two elements[3], hence the extent map cannot be injective.

It can be shown that many of the expected properties of a "space of random sequences" can be rephrased for R (and locales formed by analagous constructions) and indeed hold, thus R provides a satisfactory notion of randomness. Much attention is paid to this by Simpson [5].

2.4 Properties of Locales

Aside from providing solutions to seeming paradoxes, the theory of locales is generally thought to provide a more well-behaved notion of space. The reasons for this are many; the pathology of indistinguishable points is removed, and spaces can have structure in the absence of points. In particular, this last point indicates that sublocales of a space can be related while having no points in common. This viewpoint leads to a conceptual change that some regard as a solution to the Banach-Tarski paradox, as mentioned. Some more examples of the good behavior of locales include the following theorems, of which proofs and relevant definitions can be found in [1].

Theorem 2.4.1. *Let X be a locale. Then X has a unique minimal dense sublocale.*

Corollary 2.4.1.1. *The intersection of any collection of dense sublocales of a locale is dense.*

Remark. This is certainly not true for topological spaces; recall Baire's theorem! In particular, the sublocales of $\Omega(\mathbb{R})$ determined by the subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ have a dense intersection (with no points!).

Theorem 2.4.2. *A product of paracompact locales is paracompact.*

Remark. This is not true for topological spaces!

Theorem 2.4.3. *The analogue, for locales, of Tychonoff's theorem can be proved constructively.*

Remark. Whereas Tychonoff's theorem is equivalent to the axiom of choice.

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[3] $[\varnothing] \neq [\mathbf{2}^\omega]$.

Chapter 3

Frames Of Type (p, q)

In this chapter we cover a new framework to give an appropriate language and context for new Stone duality theorems, and finally then prove the major theorem of this dissertation. To extend Stone duality to new contexts, it is instructive to look to the difference between the contexts. The difference between topologies and σ-algebras, for example, are

1. The *closure axioms* are different; topologies contain all unions and finite intersections, whereas σ-algebras contain countable unions and countable intersections.

2. σ-algebras are also required to contain complements.

To account for the first difference, we give a new framework for indexing categories of (distributive) lattices via their closure axioms.

3.1 Frames Of Type $(\mathbf{p}, \mathbf{q})$

Let $\mathscr{C}$ be the class containing the number 2, the infinite cardinals, and the symbol ∞. Let $p, q \in \mathscr{C}$. Then a *frame of type* (p, q) is a poset K satisfying

- All sets $S \subset K$ with $|S| \leq p$ join in K. (By convention, we write that $|T| \leq \infty$ for all sets T).

- All sets $S \subset K$ with $|S| \leq q$ meet in K.

- For all doubly-indexed sets $\{x_{j,k} : j \in J, k \in K_j\}$ in K where $|J| \leq q$ and $|K_j| \leq p$ for all $j \in J$, one has
$$\bigwedge_{j \in J} \bigvee_{k \in K_j} x_{j,k} = \bigvee_{f \in F} \bigwedge_{j \in J} x_{j,f(j)}$$
where F is the set of choice functions $f \in \prod_{i \in J} K_i$.

So frames of type $(2,2)$ are distributive lattice; frames of type $(\infty,2)$ are the frames of chapter 2. If $p=\infty$ then frames of type (p,q) are complete lattices, however their distributivity condition is different for that of (∞,∞)-frames, and so are their homomorphisms:

Definition 3.1.1. Let K and L be (p,q)-frames. Then a function $f: K \to L$ is a (p,q)-*homomorphism* if, for all $S \subset K$,

- If $|S| \leq p$ then $|f(S)| \leq p$ and $f(\bigvee S) = \bigvee f(S)$, and
- If $|S| \leq q$ then $|f(S)| \leq q$ and $f(\bigwedge S) = \bigwedge f(S)$.

Remark. Any (p,q)-homomorphism of (p,q)-frames is a lattice homomorphism, so the basic results of lattice homomorphisms such as monotonicity and their respecting complements holds. Moreover, it is clear that any composition of (p,q)-homomorphisms is again a (p,q)-homomorphism.

Now to adress the second difference between topologies and σ-algebras, we remark that any lattice homomorphism respects any complements which exist. Hence the requirement that σ-algebras are complemented doesn't affect the resulting notion of morphism. That is, one is essentially picking a full subcategory of the category of lattices contain countable unions and countable intersections. The only restriction on the choice of category is that it needs to contain the lattice $\mathbf{2}$ in order to have a notion of global element, and the category needs to be closed under homomorphic images.

3.2 $\mathscr{K}$-Locales and $\mathscr{K}$-spaces

Fix $p,q \in \mathscr{C}$ and let $\mathscr{K}$ be any full subcategory of the category of (p,q)-frames such that $\mathbf{2} \in \mathscr{K}$ and that, for each $\mathscr{K}$-homomorphism $f: K \to L$, $f(K)$ is an object of $\mathscr{K}$.

A $\mathscr{K}$-*space* is a pair (X,K) where $K \subset \mathcal{P}(X)$ is an object of $\mathscr{K}$ and that $\varnothing, X \in K$. For any $\mathscr{K}$-space (X,K), write $\Omega(X,K) := K$, and for any point $x \in X$, write

$$N_{(X,K)}(x) := \{U \in K : x \in U\}$$

If there is no risk of ambiguity, write $N(x)$ for $N_{(X,K)}$.

Definition 3.2.1. Let (X,K) and (Y,L) be $\mathscr{K}$-spaces and let $f: X \to Y$ be a function. Then f is said to be $\mathscr{K}$-continuous if and only if $f^{-1}(U) \in K$ for all $U \in L$.

Define $\mathbf{Sp}(\mathscr{K})$ to be the category of $\mathscr{K}$-spaces and $\mathscr{K}$-continuous maps. One has a functor Ω from $\mathbf{Sp}(\mathscr{K})$ to the opposite category $\mathscr{K}^{\mathrm{op}}$. Define also the category of $\mathscr{K}$-*locales*

$$\mathbf{Loc}(\mathscr{K}) := \mathscr{K}^{\mathrm{op}}.$$

and call a morphism in $\mathbf{Loc}(\mathscr{K})$ a *$\mathscr{K}$-localic map.* Familiar special cases arise when one takes $\mathscr{K}$ to be the category of frames, in which case $\mathbf{Loc}(\mathscr{K})$ is the usual category of locales, and $\mathbf{Sp}(\mathscr{K})$ is the category of topological spaces. Similarly, if one takes $\mathscr{K}$ to be the category of σ-complete Boolean algebras then $\mathbf{Sp}(\mathscr{K})$ is the category of measurable spaces.

We can now proceed to transfer the concepts of chapter 2 to the new context of $\mathscr{K}$-locales and $\mathscr{K}$-spaces.

Definition 3.2.2. Let $K \in \mathbf{Loc}(\mathscr{K})$. Then a *point in* K is a $\mathscr{K}$-localic map $x : \mathbf{2} \to K$. Denote by $\operatorname{pt} K$ the set of points in K.

Definition 3.2.3. Let $K \in \mathbf{Loc}(\mathscr{K})$, $x \in \operatorname{pt} K$ and $U \in K$. Then we say that the point x *inhabits* U if $x(U) = \top$. Define the *extent* $e(U)$ of U to be the set of points inhabiting U. Then e gives a map $e : K \to \mathcal{P}(\operatorname{pt} K)$.

Let $K \in \mathbf{Loc}(\mathscr{K})$. Then $e : K \to \mathcal{P}(\operatorname{pt} K)$ is a $\mathscr{K}$-morphism, and hence the image $e(K)$ is a $\mathscr{K}$-object. In particular, $(\operatorname{pt} K, e(K))$ is a $\mathscr{K}$-space. We will often write $\Omega(\operatorname{pt} K)$ for $e(K)$, i.e., $(\operatorname{pt} K, \Omega(\operatorname{pt} K))$ for $(\operatorname{pt} K, e(K))$. Let $K, L \in \mathbf{Loc}(\mathscr{K})$ and let $f : K \to L$ be a $\mathscr{K}$-localic map. Then, for each point $x \in \operatorname{pt} K$, one has a point $(\operatorname{pt} f)(x) \in \operatorname{pt} L$ given by $(\operatorname{pt} f)(x) = f \circ x$:

$$\begin{tikzcd} K \arrow[r, "f"] & L \\ \mathbf{2} \arrow[u, "x"] \arrow[ur, "(\operatorname{pt} f)(x)"'] & \end{tikzcd}$$

hence one has a function $\operatorname{pt} f : \operatorname{pt} K \to \operatorname{pt} L$ given by $x \mapsto (\operatorname{pt} f)(x)$.

Proposition 3.2.1. *Let $K, L \in \mathbf{Loc}(\mathscr{K})$ and let $f : K \to L$ be a $\mathscr{K}$-localic map. Then $\operatorname{pt} f$ is a $\mathscr{K}$-continuous map.*

Proof. Let $U \in \Omega(\operatorname{pt} L)$. Then $U = e(V)$ for some $V \in L$. Now

$$\begin{aligned} (\operatorname{pt} f)^{-1}(U) &= \{x \in \operatorname{pt} K : (\operatorname{pt} f)(x) \in U\} \\ &= \{x \in \operatorname{pt} K : (\operatorname{pt} f)(x)(V) = \top\} \\ &= \{x \in \operatorname{pt} K : (f \circ x)(V) = \top\} \\ &= \{x \in \operatorname{pt} K : x \in e(f(V))\} \\ &= e(f(V)) \in \Omega(\operatorname{pt} K) \end{aligned}$$

hence $\operatorname{pt} f$ is continuous. □

Thus one has a functor $\operatorname{pt} : \mathbf{Loc}(\mathscr{K}) \to \mathbf{Sp}(\mathscr{K})$ taking a $\mathscr{K}$-locale to the $\mathscr{K}$-space of its points and $\mathscr{K}$-localic maps to their corresponding continuous functions of points.

3.3 Spatiality

Those objects of $\mathscr{K}$ which take the form $\Omega(X, L)$ for some $\mathscr{K}$-space (X, L) are of particular importance. Luckily enough, there is a simple characterisation of which $\mathscr{K}$-objects K have this property, which makes reference only to the intrinsic properties of K.

Definition 3.3.1. Let $K \in \mathbf{Loc}(\mathscr{K})$. Then K is said to be *spatial* if $K \cong \Omega(X, L)$ for some $\mathscr{K}$-space (X, L). Denote by $\mathbf{SpLoc}(\mathscr{K})$ the category of spatial $\mathscr{K}$-locales.

Definition 3.3.2. Let $K \in \mathbf{Loc}(\mathscr{K})$. Then K is said to *have enough points* if the extent map $e : K \to \mathcal{P}(\operatorname{pt} K)$ is injective.

That is, K is said to *have enough points* precisely when

$$\textit{if } x \in e(U) \textit{ precisely when } x \in e(V) \textit{ then } U = V$$

In particular, this means that two elements U and V differ only when they can be distinguished by which points inhabit them, i.e., K has enough points to distinguish its elements.

Now, we can turn elements of the point-set of a $\mathscr{K}$ space (X, K) into points of K in a natural way.

Proposition 3.3.1. *Let (X, K) be a $\mathscr{K}$-space. For all $x \in X$, define the map $P_x : K \to \mathbf{2}$ by*

$$P_x(U) = \begin{cases} \top & \textit{if } x \in U \\ \bot & \textit{otherwise} \end{cases}$$

Then, for all $x \in X$, P_x is $\mathscr{K}$-continuous, i.e., a point in K. Moreover, the map $P : X \to \operatorname{pt} K$ given by $x \mapsto P_x$ is continuous and $P^* = e^{-1}$.

Remark. Recall that, if (X, K) and (Y, L) are $\mathscr{K}$-spaces and $f : X \to Y$ a $\mathscr{K}$-continuous map, then $f^* : L \to K$ is the map $U \mapsto f^{-1}(U)$.

Proof. That any P_x is continuous is clear. Let $V \in K$. Then

$$\begin{aligned} P^{-1}(e(V)) &= \{x \in X : P_x \in e(V)\} \\ &= \{x \in X : P_x(V) = \top\} \\ &= \{x \in X : x \in V\} = V \end{aligned}$$

hence $P^* \circ e = \mathrm{Id}_K$. Since every $U \in \Omega(\operatorname{pt} K)$ is of the form $e(V)$ for some $V \in K$, P is $\mathscr{K}$-continuous. Finally, since K is spatial, e is bijective, so that $P^* = e^{-1}$. $\square$

Finally, we may characterise spatiality.

Theorem 3.3.1. *Let $K \in \mathbf{Loc}(\mathscr{K})$. Then K is spatial if and only if K has enough points.*

Proof. Suppose K has enough points. Then $e : K \to e(K)$ is injective and surjective, hence $K \cong \Omega(\mathrm{pt}\, K, e(K))$. Suppose K is spatial. Then we may assume, without loss of generality, that $K = \Omega(X, L) = L \subset \mathcal{P}(X)$ for some $\mathscr{K}$-space (X, L), i.e., that K contains subsets of some point-set X. Let $U, V \in K$. If U and V are distinct, there is an element $x \in X$ such that, without loss of generality, $x \in U \setminus V$. But then the point $x^* \in \mathrm{pt}\, K$ given by x has $x^*(U) = \top$ and $x^*(V) = \bot$ hence $x^* \in e(U) \setminus e(V)$, i.e., $e(U) \neq e(V)$. Thus e is injective. □

Corollary 3.3.1.1. *K is spatial if and only if $K \cong \Omega(\mathrm{pt}\, K)$.*

3.4 Sobriety

We may use the characterisation of sobriety for frames (chapter 2) to define a notion of sobriety in this more general setting. Recall the map P

Definition 3.4.1. Let (X, K) be a $\mathscr{K}$-space. Then (X, K) is said to be *sober* if the map $P : X \to \mathrm{pt}\, K$ of Proposition 3.3.1 is a $\mathscr{K}$-homeomorphism.

Denote by $\mathbf{SobSp}(\mathscr{K})$ the category of sober $\mathscr{K}$-spaces. Since P is continuous and $P^* = e^{-1}$, it is immediate that P is a homeomorphism if and only if it is bijective. Hence sobriety is, in effect, two requirements: that P is injective, and that P is surjective. The proof that a particular space or type of space is sober will generally have two steps;

1. One to show that P is injective, which will often be phrased as "K separates the points of X", since P being injective is the same as no two points occupying the same sets in K, and

2. One to show that P is surjective. This is generally the hard part.

In chapter 2 we saw that points of frames could be viewed as completely prime filters or global elements. There is an analogue for this in the context of $\mathscr{K}$-spaces, and it will come in handy. In particular completely prime filters in a locale are precisely the inverse images of $\{\top\}$ under global elements. Recall that a choice of p and q was fixed at the start of this chapter. Finally p and q begin to appear in our definitions.

Definition 3.4.2. Let $K \in \mathscr{K}$. A *(p, q)-filter* is a proper filter $F \subset K$ such that,

1. Whenever $S \subset K$, $|S| \leq p$ and $\vee S \in F$ then $S \cap F \neq \varnothing$, and

2. Whenever $S \subset F$ and $|S| \leq q$ then $\wedge S \in F$.

Condition 1 will be referred to as the *primeness condition*; if $p = 2$ then F is said to be *prime*; if $p = \omega$ then F is *σ-prime*; if $p = \infty$ then F is completely prime. The following is immediate.

Proposition 3.4.1. *Let (X, K) be a $\mathscr{K}$-space and let $x \in X$. Then $N(x)$ is a (p, q)-filter.*

Now we have our analogue of Theorem 2.2.1 for $\mathscr{K}$-spaces.

Theorem 3.4.1. *Let $K \in \mathscr{K}$. Then, for every point $x \in \operatorname{pt} K$, $x^{-1}(\{\top\})$ is a (p, q)-filter in K and for every (p, q)-filter F in K, the function $f : K \to \mathbf{2}$ given by*

$$U \mapsto \begin{cases} \top & \text{if } U \in F \\ \bot & \text{otherwise} \end{cases}$$

is a $\mathscr{K}$-homomorphism, i.e., $f \in \operatorname{pt} K$.

Proof. First, let $x \in \operatorname{pt} K$. Then x is a $\mathscr{K}$-homomorphism $K \to 2$. Now x is surjective since $x(\top) = \top$ and $x(\bot) = \bot$ so that $x^{-1}(\{\top\})$ is proper. If $S \in x^{-1}(\{\top\})$ and $|S| \leq q$ then

$$x(\wedge S) = \wedge x(S) = \wedge\{\top\} = \top$$

Suppose that $V \in K$ and $U \in x^{-1}(\{\top\})$ and $U \leq V$. Then, by monotonicity, $x(V) = \top$ hence $V \in x^{-1}(\{\top\})$. This establishes that $x^{-1}(\{\top\})$ is a proper filter. Now, to see that it is a (p, q)-filter, let $S \subset K$ and suppose that $|S| \leq p$ and that $\vee S \in x^{-1}(\{\top\})$. Now, since x is a $\mathscr{K}$-homomorphism, it follows that

$$\top = x(\vee S) = \vee x(S)$$

hence $\top \in x(S)$, i.e., $S \cap x^{-1}(\{\top\}) \neq \varnothing$.

For the converse, let $F \subset K$ be a (p, q)-filter, and let $S \subset K$ and suppose $|S| \leq p$. If $\vee S \in F$ then $S \cap F \neq \varnothing$ hence $\vee f(S) = \top = f(\wedge S)$. If $\vee S \notin F$ then $S \cap F = \varnothing$, since otherwise $U \leq \vee S$ for any $U \in S \cap F$ but then $\vee S \in F$ since F is a filter. Thus

$$\bot = f(\vee S) = \vee f(S).$$

If now we suppose that $|S| \leq q$ then one has two cases

1. There is some $U \in S$ with $U \notin F$, in which case $\wedge S < V$ for all $V \in F$. Hence

$$\bot = f(\wedge S) = \wedge f(S)$$

2. $U \subset S$, in which case $\wedge S \in F$ and

$$\top = f(\wedge S) = \wedge f(S)$$

hence f is a $\mathscr{K}$-homomorphism, thus $\mathscr{K}$-localic map $\mathbf{2} \to K$, i.e., a point. □

Hence, one may equally regard points as being localic maps $2 \to K$ or (p, q)-filters. In particular, we see that a $\mathscr{K}$-space (X, K) is sober if and only if the (p, q)-filters (points) are precisely those of the form $N(x)$ and are in bijection to the elements $x \in X$.

3.5 Generalised Stone Duality

We are now ready to prove the main result of this dissertation.

Theorem 3.5.1. *Let (X, K) be a $\mathscr{K}$-space. Then $(\operatorname{pt} K, \Omega(\operatorname{pt} K))$ is sober.*

Proof. First, since K is spatial, $e : K \to \Omega(\operatorname{pt} K)$ is an isomorphism. Hence, a set $F \subset \Omega(\operatorname{pt} K)$ is a (p, q)-filter in $\Omega(\operatorname{pt} K)$ if and only $e^{-1}(F)$ is a (p, q)-filter in K. But $\operatorname{pt} K$ consists of the points on K, which are precisely the (p, q)-filters of K. Hence $(\operatorname{pt} K, \Omega(\operatorname{pt} K))$ has, for its point set, precisely its (p, q)-filters. □

Corollary 3.5.1.1. *Every spatial $\mathscr{K}$-locale K satisfies $K \cong \Omega(X, L)$ where (X, L) is some sober $\mathscr{K}$-space.*

Proposition 3.5.1. *Let (X, K) and (Y, L) be sober $\mathscr{K}$-spaces and $f : X \to Y$ a $\mathscr{K}$-continuous map. Then, for each $x \in X$,*

$$P^{-1}((\operatorname{pt} f^*)(P_x)) = f(x)$$

Proof. Let $x \in X$ and $U \in L$. Then

$$(\operatorname{pt} f^*)(P_x)(U) = (P_x \circ f^*)(U) = P_x(f^{-1}(U))$$

so that $(\operatorname{pt} f^*)(P_x)(U) = \top$ if and only if $f(x) \in U$, from which the result follows. □

Theorem 3.5.2. *The categories $\mathbf{SpLoc}(\mathscr{K})$ and $\mathbf{SobSp}(\mathscr{K})$ are equivalent.*

Proof. By Corollary 3.5.1.1, the functor Ω is essentially surjective on objects in $\mathbf{SpLoc}(\mathscr{K})$. Let (X, K) and (Y, L) be sober $\mathscr{K}$-spaces. If remains only to see that the map

$$\Omega : \operatorname{Hom}_{\mathbf{SobSp}(\mathscr{K})}((X, K), (Y, L)) \to \operatorname{Hom}_{\mathbf{SpLoc}(\mathscr{K})}(K, L)$$

induced by Ω, which is given by $f \mapsto f^*$, is bijective. By sobriety, one has $\mathscr{K}$-homeomorphisms $P_X : X \to \operatorname{pt} K$ and $P_Y : Y \to \operatorname{pt} L$. Let φ be the map $\varphi(f) = P_Y^{-1} \circ f \circ P_X$. Then, by Proposition 3.5.1, one has, for all $\mathscr{K}$-continuous $f : X \to Y$ and all $x \in X$,

$$(\varphi \circ \operatorname{pt} \circ \Omega)(f)(x) = P_Y^{-1}(\operatorname{pt} f^*)(P_x) = f(x)$$

hence $(\varphi \circ \mathrm{pt}) \circ \Omega = \mathrm{Id}$. Conversely, if $f : K \to L$ is a $\mathscr{K}$-localic map and $U \in K$,

$$\begin{aligned}(\Omega \circ \varphi \circ \mathrm{pt})(f)(U) &= \Omega(P_Y^{-1}(\mathrm{pt}\, f)(P_X))(U) \\ &= (P_Y^{-1}(\mathrm{pt}\, f)(P_X))^{-1}(U) \\ &= P_X^{-1}((\mathrm{pt}\, f)^{-1}(P_Y(U)))\end{aligned}$$

Now $P_X^{-1}((\mathrm{pt}\, f)^{-1}(P_Y(U)))$ is the set of those points in X whose corresponding locale points in K are mapped, under $\mathrm{pt}\, f$, into $P_Y(U)$. Let $x \in X$. Then

$$\begin{aligned}x \in P_X^{-1}((\mathrm{pt}\, f)^{-1}(P_Y(U))) &\Leftrightarrow P_x \in (\mathrm{pt}\, f)^{-1}(P_Y(U)) \\ &\Leftrightarrow (\mathrm{pt}\, f)(P_x) \in P_Y(U) \\ &\Leftrightarrow (P_x \circ f) \in P_Y(U) \\ &\Leftrightarrow (P_x \circ f)(U) = \top \\ &\Leftrightarrow P_x(f(U)) = \top \\ &\Leftrightarrow x \in f(U)\end{aligned}$$

Hence $P_X^{-1}((\mathrm{pt}\, f)^{-1}(P_Y(U))) = f(U)$. Thus $\Omega \circ \varphi \circ \mathrm{pt} = \mathrm{Id}$, hence Ω is full, faithful and essentially surjective on objects. $\square$

Of course, a special case of Theorem 3.5.2 is the theorem

$$\mathbf{SpLoc} \simeq \mathbf{SobTop}$$

i.e., frames are dual to sober topological spaces. The rest of this document will focus on special cases and consequences of Theorem 3.5.2.

Chapter 4

Special Cases and Consequences of Theorem 3.5.2

4.1 Stone's Representation Theorem for Boolean Algebras

In this section we will give a proof of Stone's famous representation theorem for Boolean algebras. We are working in the case $\mathscr{K} = \mathbf{Bool}$, with $p = q = 2$, where (p, q)-filters are equivalently prime filters and ultrafilters.

Recall that a topological space is said to be *totally disconnected* if the only connected sets are singletons, and *totally separated* if, for each pair x, y of distinct points, one has a clopen set U such that $x \in U$ and $y \notin U$.

Theorem 4.1.1. ***Characterisation of Stone Spaces***
Let (X, T) be a topological space. Then the following are equivalent.

1. *(X, T) is compact, Hausdorff and totally disconnected,*
2. *(X, T) is compact and totally separated,*
3. *(X, T) is compact, T_0 and the clopen sets of (X, T) form a basis for T.*

For a proof, see [1], page 69. A topological space satisfying any (and hence every) of the conditions of Theorem 4.1.1 is called a *Stone space*. Denote the category of Stone spaces with continuous maps by **Stone**.

Theorem* 4.1.1. *Let B be a Boolean algebra. Then B is spatial.*

Proof. It suffices to show that the extent map $e : B \to \Omega(\mathrm{pt}\, B)$ is injective. Let $\perp \neq a \in B$. Then, by the Ultrafilter lemma, $\{a\}$ extends to an ultrafilter F. But F induces a point $x : \mathbf{2} \to B$ and $x(a) = \top$ hence $e(a) \neq \varnothing$. □

We recall (without proof) a standard result from topology.

Lemma 4.1.2. *Let X be a set and let $B \subset \mathcal{P}(X)$. Then B is a basis for the topology generated by B if and only if*

1. *$X = \bigcup B$, and*

2. *For each $U, V \in B$ and $x \in U \cap V$, there exists $W \in B$ such that $x \in W \subset U \cap V$*

Corollary 4.1.2.1. *If B is a lattice then B is a basis for the topology it generates.*

We will also need the following theorem.

Theorem 4.1.3. *Let (X, T) be a topological space. Then, if for every ultrafilter F in $\mathcal{P}(X)$ there exists a point $x \in X$ such that $N(x) \subset F$, then (X, T) is compact.*

Proof. See [6]. □

Theorem* 4.1.2. *Let (X, B) be a sober* **Bool***-space and let T be the topology on X generated by B. Then (X, T) is a Stone space.*

Proof. We will use condition 3 of Theorem 4.1.1. Clearly every set $U \in B$ is clopen in (X, T). Since B is a lattice, generates T and is included in the clopen sets of (X, T), the clopen sets of (X, T) form a basis for T.

Now since (X, B) is sober, for any two points $x, y \in B$, one has a clopen set U of (X, T) such that $|U \cap \{x, y\}| = 1$, hence (X, T) is T_0.

Finally, let $F \subset \mathcal{P}(X)$ be an ultrafilter in $\mathcal{P}(X)$. Let $G := F \cap B$. Then G is clearly a filter, and, since F is an ultrafilter, if $U \in B$, either $U \in F$ or $U^c \in F$ so that either $U \in G$ or $U^c \in G$, hence G is an ultrafilter in B. By sobriety, $G = N_B(x)$ for some $x \in X$. Now suppose that $U \in N(x)$. Then, since B is a basis for T,

$$U = \bigcup_{i \in I} U_i$$

for some family $\{U_i\}_{i \in I}$ in B. But $x \in U$ so $x \in U_i$ for some $i \in I$. Thus $U_i \in N_B(x) \subset F$ hence, since F is upper closed, $U \in F$. Thus $N_T(x) \subset F$. Therefore (X, T) is compact. □

Denote the Stone space generated by (X, B) in this way by $\mathrm{St}(X, B)$.

Proposition 4.1.1. *Let (X, B) be a* **Bool***-space. Then every clopen set of* $\mathrm{St}(X, B)$ *lies in B.*

Proof. Let $U \subset X$ be clopen. Then, in particular, U is a closed, and thus a closed subset of a compact space, hence compact. Since U is open and B generates the topology, there exists an open cover $\{U_i\}_{i\in I}$ of U such that $U = \bigcup_{i\in I} U_i$ and $U_i \in B$ for all $i \in I$. Now since U is compact one has a finite subcover $\{U_i\}_{i\in J}$. But then $U = \bigcup_{i\in J} U_i$, which is a finite join of elements of B. Hence $U \in B$. □

Lemma 4.1.4. *Let (X, T) be a topological space. Then the set B of clopen sets of X is a Boolean algebra.*

Proof. Clear. □

Proposition* 4.1.1. *Let (X, T) be a Stone space and let B be the set of clopen sets in X. Then (X, B) is a sober* **Bool***-space.*

Proof. Since (X, T) is Hausdorff and B is a basis for T, B separates points of X. Now let $F \subset B$ be an ultrafilter. By the Ultrafilter lemma (Theorem 1.3.1), F extends an to an ultrafilter G in $\mathcal{P}(X)$. Since (X, T) is compact, there exists $x \in X$ such that $N_T(x) \subset G$. But clearly $N_B(x) = N_T(x) \cap B$ and $F = G \cap B$ hence

$$N_B(x) = N_T(x) \cap B \subset G \cap B = F$$

since $N_B(x)$ is maximal in B, $N_B(x) = F$. Thus (X, B) is sober. □

So we see that each Stone space has the form $\mathrm{St}(X, B)$ for some **Bool**-space (X, B).

Theorem 4.1.5. *Let (X, B) and (Y, C) be* **Bool***-spaces and $f : X \to Y$ a function. Then f is a continuous map of* **Bool***-spaces if and only if f is a continuous map of the Stone spaces $\mathrm{St}(X, B) \to \mathrm{St}(Y, C)$.*

Proof. Suppose that f is **Bool**-continuous. The the inverse image under f of every open set in the basis C is clopen and hence open, f is continuous. For the converse, suppose that f is continuous. Then for any clopen set U in Y, $U \in C$, and, by continuity, $f^{-1}(U)$ is clopen, hence $f^{-1}(U) \in B$. □

By Theorem 3.5.2 and 4.1.1, the functor $\mathrm{pt} : \mathbf{Loc}(\mathbf{Bool}) \to \mathbf{SobSp}(\mathbf{Bool})$ is an equivalence. By the preceding theorems, the map $\mathrm{St} : \mathbf{Sp}(\mathbf{Bool}) \to \mathbf{Stone}$ (extended to do nothing on morphisms) is an equivalence. Hence one has an equivalence $\mathrm{pt} \circ \mathrm{St} : \mathbf{Bool}^{\mathrm{op}} \to \mathbf{Stone}$.

4.2 Pointless Measure Theory

A notable special case is when $\mathscr{K} = \sigma\mathbf{Bool}$, i.e., the full subcategory of the category of (ω, ω)-frames consisting of those (ω, ω)-frames which are also Boolean algebras. These are called *σ-complete Boolean algebras*. In this case, $\mathbf{Sp}(\mathscr{K})$ is precisely the category of measurable spaces with measurable functions.

Recalling Stone's representation theorem, one might suspect that all σ-complete Boolean algebras are spatial. This is, unfortunately, not the case.

Theorem 4.2.1. *Let B be a σ-complete Boolean algebra. Then there exists a set X and a Boolean algebra $C \subset \mathcal{P}(X)$ and a Boolean algebra isomorphism $i : B \to C$. Moreover, this implies that C is σ-complete and the map i is a σ-isomohprism*[1]

Proof. By Stone's theorem, one has X, C and i. Since i is an invertible lattice homomorphism, is it an order isomorphism, hence preserves all joins and meets. Thus C has countable joins and meets and is isomorphic to B as a σ-complete Boolean algebra. $\square$

The problem is, however, that *the join and meet in C may not coincide with the union and intersection in $\mathcal{P}(X)$.*

Definition 4.2.1. Let B be a σ-complete Boolean algebra. Then an ideal $N \subset B$ is said to be a σ-ideal if it is closed under countable joins[2].

If N is a σ-ideal of B, then the quotient ring B/N is a Boolean ring, and the closure properties of N ensure the existence of countable joins and meets, hence B/N is a σ-complete Boolean algebra.

Proposition 4.2.1. *Let $\mathcal{B}$ be the Borel σ-algebra on $[0, 1]$ and let $\mathcal{N} \subset \mathcal{B}$ be the set of sets whose Lebesgue measure is zero. Then $\mathcal{B}/\mathcal{N}$ is a non-spatial σ-complete Boolean algebra.*

Proof. Let (X, H) be a $\sigma\mathbf{Bool}$-space and suppose $\varphi : \mathcal{B}/\mathcal{N} \to H$ is an isomorphism of σ-complete Boolean algebras. Then one may lift φ to a map $\overline{\varphi} : \mathcal{B} \to H$ via setting

$$\overline{\varphi}(X) = \varphi(X + \mathcal{N})$$

then $\overline{\varphi}$ is an homomorphism, and moreover $\overline{\varphi}(N) = \varnothing$ for all sets N with measure 0. Let $x \in X$ and let $X_1 = [0, 1]$. Then $x \in \overline{\varphi}(X)$. However a Borel set on $[0, 1]$ may be partitioned into two sets U_1 and U_2 of equal measure. Suppose, without loss of

[1] i.e., an isomorphism in the category $\sigma\mathbf{Bool}$.

[2] Closure under countable meets is guaranteed by downward closure

generality that $x \in \overline{\varphi}(U_1)$. Set $X_2 = U_1$. Continuing this process one forms a sequence X_i of Borel sets for which $\mu(X_i) = 1/2^{i-1}$ and $x \in \overline{\varphi}(X_i)$ for all $i \in \mathbb{N}$. Set

$$N = \bigcap_{i=1}^{\infty} X_i$$

Now $x \in \bigcap_{i=1}^{\infty} \overline{\varphi}(X_i) = \overline{\varphi}(N)$ however $\mu(N) \leq \mu(X_i) = 1/2^{i-1}$ for all $i \in \mathbb{N}$. Hence $\mu(N) = 0$ thus $\overline{\varphi}(N) = \varnothing$, a contradiction. □

Remark. Let $x : \mathcal{B}/\mathcal{N} \to \mathbf{2}$ be a homomorphism of σ-complete Boolean algebras. Then, as above, one has a sequence of sets X_i such that $x(X_i + N) = \top$ and $\mu(X_i) = 1/2^{i-1}$ for all $i \in \mathbb{N}$. But then $\wedge_{i=1}^{\infty} X_i \in \mathcal{N}$ hence

$$\top = \bigwedge_{i=1}^{\infty} x(X_i + \mathcal{N}) = x\left(\bigwedge_{i=1}^{\infty} X_i + \mathcal{N}\right) = x(\mathcal{N}) = \bot$$

a contradiction. Hence $\mathcal{B}/\mathcal{N}$ *has no points.*

So considering σ-algebras and measure spaces from a point free perspective does increase the scope of the theory, but not by much. The following proof uses the argument of Loomis [7].

Theorem 4.2.2. *(Loomis-Sikorski)*
*Let B be a σ-complete Boolean algebra. Then there exists a σ**Bool**-space (X, A) and a σ-ideal $N \subset A$ such that $B \cong A/N$.*

Remark. In this proof we will heavily use the fact that De Morgan's laws extend to countable joins and meets: one has

$$\bigvee_{n=1}^{\infty} x_n^c = \left(\bigwedge_{n=1}^{\infty} x_n\right)^c$$

and

$$\bigwedge_{n=1}^{\infty} x_n^c = \left(\bigvee_{n=1}^{\infty} x_n\right)^c$$

for all x_n in any σ-complete Boolean algebra. This follows from the fact that complementation is a bijective lattice homomorphism $B^{\mathrm{op}} \to B$ hence is conditionally complete[3].

[3]Thus also respects arbitrary joins and meets.

Proof. Denote by $\mathrm{pt}_{\mathbf{Bool}}$ the point-space functor for $\mathbf{Loc}(\mathbf{Bool})$. Denote by e the extent map $e : B \to \Omega(\mathrm{pt}_{\mathbf{Bool}}(B))$. Let A be the σ-algebra generated by $e(B)$. Now e is a Boolean algebra homomorphism but does not, in general, respect countable meets and joins. If a set $S \subset \mathrm{pt}_{\mathbf{Bool}}(B)$ takes the form

$$S = \bigcap_{i=1}^{\infty} e(B_i)$$

where $B_1, B_2, \ldots \in B$ and $\wedge\{B_i : i \in \mathbb{N}\} = \perp$, we say that S is a *basic null set.* Let N be the set of those sets in $e(B)$ which can be covered by countably many basic null sets. It is clear that N is a σ-ideal in $e(B)$.

Let $\varphi : e(B) \to e(B)/N$ be the projection, and write $\hat{e} = \varphi \circ e : B \to e(B)/N$. The claim is that $\hat{e}$ is an isomorphism of σ-complete Boolean algebras. Now $\hat{e}$ is a lattice homomorphism[4], so it suffices to show that e respects countable joins[5]. Let $a_1, a_2, \ldots \in B$ and set $a := \bigvee_{n=1}^{\infty} a_n$. Then, by De Morgan, $e(a) \cap e(a_n)^c = \perp$ for all $i \in \mathbb{N}$ and $e(a) \cap \wedge_{n=1}^{\infty} e(a_n) = \cap_{n=1}^{\infty}(e(a) \wedge e(a_n)^c)$ so that $e(a) \cap \wedge_{n=1}^{\infty} e(a_n)^c$ can be covered by countably many basic null sets. Similarly, $e(a)^c \cup \wedge_{n=1}^{\infty} e(a_n) = \mathrm{pt}_{\mathbf{Bool}}(B)$. Hence

$$\hat{e}\left(\vee_{n=1}^{\infty} a_n\right) = \left(\bigwedge_{n=1}^{\infty} \hat{e}(a_n)^c\right)^c = \bigvee_{n=1}^{\infty} \hat{e}(a_n)$$

By De Morgan. Hence $\hat{e}$ is a σ-homomorphism. Thus, since $\hat{e}(B)$ generates the σ-algebra A/N, it follows that $\hat{e}(B) = A/N$ so that $\hat{e}$ is surjective. Finally, to see that $\hat{e}$ is injective, simply note that N is the kernel of the Boolean ring homomorphism e. $\square$

Now it will come in handy to have some results of atomic Boolean algebras.

Definition 4.2.2. Let P be a poset with lower bound $\perp$. Then an *atom* $\perp \neq a \in P$ is an element covering $\perp$. That is, an element such that, if $\perp \leq y \leq a$ then $y = \perp$ or $y = a$. A poset P if said to be *atomic* if each element of P not equal to $\perp$ is preceded by an atom.

Remark. If L is a lattice and $a \in L$ then a is an atom if and only if, for each $x \in L$, either $a \wedge x = a$ or $a \wedge x = \perp$ (but not both).

Now all power sets are atomic, and since all finite posets are atomic, all finite Boolean algebras are. The Boolean algebra $\mathcal{B}/\mathcal{N}$ of Proposition 4.2.1 is non-atomic. In fact, it contains no atoms at all.

[4] Hence respects complements.

[5] Then e respects countable meets by De Morgan.

Theorem 4.2.3. *Let B be an atomic Boolean algebra and let $X := \{x \in B : x \text{ is an atom}\}$. Define $f : B \to \mathcal{P}(X)$ by $f(x) = \{a \in X : a \leq x\}$. Then f is an injective homomorphism.*

Proof. Let $x, y \in B$. Then

$$\begin{aligned} f(x \vee y) &= \{a \in X : a \leq x \vee y\} \\ &= \{a \in X : a \leq x \text{ or } a \leq y\} \\ &= f(x) \cup f(y) \end{aligned}$$

and similarly, $f(x \wedge y) = f(x) \cap f(y)$. Now if $f(x) = \varnothing$ then x is preceded by no atoms, i.e., $x = \perp$. Hence f is injective. □

Proposition 4.2.2. *The homomorphism $f : B \to \mathcal{P}(X)$ is conditionally complete.*

Proof. Let $S \subset B$ and suppose that S has a join. Then

$$\begin{aligned} f(\vee S) &= \{a \in X : a \leq \vee S\} \\ &= \{a \in X : \exists s \in S : a \leq s\} \\ &= \bigcup f(S) \end{aligned}$$

And similarly for meets. □

Corollary 4.2.3.1. *If B is complete, f is surjective.*

Proof. Let $S \in \mathcal{P}(X)$. Note that $S \subset B$. Then

$$S = \bigcup_{a \in S} \{a\} = \bigcup_{a \in S} f(a) = \bigcup f(S) = f(\vee S)$$

so that $S \in f(B)$. □

Corollary 4.2.3.2. *If B is complete, f is a complete isomorphism.*

Corollary 4.2.3.3. *A complete atomic Boolean algebra is isomorphic (as a complete Boolean algebra) to the power set of some set.*

Corollary 4.2.3.4. *A finite Boolean algebra is isomorphic to the power set of a finite set.*

Corollary 4.2.3.5. *A finite Boolean algebra has order 2^n for some $n \in \mathbb{N}_0$.*

Corollary 4.2.3.6. *Every atomic σ-complete Boolean algebra is spatial.*

Having some rather comprehensive results on spatiality of σ-complete Boolean algebras, we look to sobriety of measurable spaces. It is a pleasant result that, if one considers a topological space equipped with its Borel σ-algebra, it is enough to ask for complete metrizability and separability to ensure that the resulting measurable space is sober. To prove this, we will first need a couple of lemmas.

Definition 4.2.3. Let (X, d) be a metric space and $S \subset X$. We say that S is *σ-bounded* if, for each $\varepsilon > 0$ there exists a sequence (x_n) in X such that

$$S \subset \bigcup_{n=1}^{\infty} B_\varepsilon(x_n)$$

Clearly a subset of a σ-bounded set is σ-bounded, hence X is σ-bounded if and only if each of its subsets is.

Lemma 4.2.4. *Let (X, d) be a metric space. Then X is separable if and only if X is σ-bounded.*

Proof. The forward direction is easy: suppose that X is separable and let $\{x_n : n \in \mathbb{N}\}$ be dense. Then, if $\varepsilon > 0$, $X \subset \bigcup_{n=1}^{\infty} B_\varepsilon(x_n)$. For the converse, suppose that X is σ-bounded. Let, for each $i \in \mathbb{N}$, $(x_n^i)_{n\in\mathbb{N}}$ be a sequence in X such that $X \subset \bigcup_{n=1}^{\infty} B_{1/n}(x_n^i)$. Set $S := \{x_n^i : i, n \in \mathbb{N}\}$. Clearly, S is countable. To see that S is dense, let $\varepsilon > 0$ and $x \in X$. Then if $i \in \mathbb{N}$ is such that $1/i < \varepsilon$, there is some $n \in \mathbb{N}$ such that $d(x, x_n^i) < 1/i < \varepsilon$. $\square$

Lemma 4.2.5. *Let B be a Boolean algebra, x an atom of B, F a proper filter in B and suppose that $x \in F$. Then $F = x^\uparrow$.*

Proof. Since F is upwards closed, $x^\uparrow \subset F$. If $y \in F \setminus x^\uparrow$ then $y \vee x = \perp$ but then $F = B$, a contradiction. $\square$

Finally, we see that a large class of measurable spaces are sober. The following is an original result.

Theorem 4.2.6. *Let (X, d) be a complete separable metric space and let B be the Borel σ-algebra on X. Then (X, B) is a sober measurable space.*

Proof. Since metric spaces are Hausdorff, B separates points. Let F be an (ω, ω)-filter in B. Then $X \in B$ but $X = \bigcup_{n=1}^{\infty} \overline{B_1(x_n)}$ for some sequence (x_n) in X, hence, since F is σ-prime, $\overline{B_1(x_n)} \in F$ for some $n \in \mathbb{N}$. Set $X_0 = \overline{B_1(x_n)}$. Now suppose that X_i is a closed ball of radius $1/2^i$ in F. Then, similarly (i.e., since X_i is σ-bounded), there is a closed ball $X_{i+1} \subset X_i$ of radius $1/2^{i+1}$ with $X_{i+1} \in F$. Hence one inductively obtains a sequence X_i of closed sets in F with radius going to zero, hence by completeness, $\bigcap_{i=1}^{\infty} X_i = \{x\}$ for some $x \in X$, and $\{x\} \in F$ by σ-primeness. But $\{x\}$ is an atom of B hence $F = N_B(x)$. $\square$

One can translate the usual ideas of measure theory to apply to σ-complete Boolean algebras. For instance, a measure may be defined as follows.

Definition 4.2.4. Let B be a σ-complete Boolean algebra. Then a function $\mu : B \to [0, \infty]$ is said to be a *measure* on B if

1. $\mu(\perp) = 0$, and
2. For each family $\{X_i\}_{i \in \mathbb{N}}$ such that $X_i \wedge X_j = \perp$ whenever $i \neq j$, one has

$$\mu\left(\bigvee_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mu(X_i)$$

A development of these ideas, including a construction of the Lebesgue measure and Lebesgue integral for σ-complete Boolean algebras can be found in [8].

Conclusion

In this dissertation we have covered the basics of lattice theory and the basic theory of Boolean algebras and seen its application to the study of topological spaces. We have seen how topological spaces can be viewed purely as algebraic/order theoretic objects via Stone's duality, with large overlap in scope. We have seen how a natural generalisation of this leads to duality theorems in a much more general context, which includes Stone duality for lattices and topological spaces, Stone's representation theorem for Boolean algebras and a new duality for σ-complete Boolean algebras.

Future avenues of research could include a study of the products and coproducts in various choices of $\mathscr{K}$; the products in **Loc** and **Top** differ, and this difference is important in pointless topology. Does the product in $\mathbf{Loc}(\sigma\mathbf{Bool})$ differ to that for measurable spaces? What does the product in $\mathbf{Loc}(\sigma\mathbf{Bool})$ tell us about product measures? Another difference between **Loc** and **Top** are the subobjects, as seen in chapter 2. How do subobjects in $\mathbf{Loc}(\mathscr{K})$ compare to those in $\mathbf{Sp}(\mathscr{K})$? Finally, it is possible that deep analogies unite $\mathscr{K}$-spaces and $\mathscr{K}$-locales for different values of $\mathscr{K}$, beyond the superficial similarities. What can the study of $\mathscr{K}$-spaces tell us about what topological spaces and measurable spaces and Boolean algebras and so on have in common?

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