Khovanov Homology of Knots and Links

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1. Career Development

As an aspiring pure mathematician, the University of Auckland's summer research programme has provided me with an invaluable insight into the world of academia. My supervisor has exposed me to many fascinating and highly advanced areas of mathematics and, recognising my ambitions, he has expertly guided me to the forefront of research in knot invariants. I am very grateful for all of the opportunities I have had to interact with and learn from various faculty members, PhD students and other academics from institutions around the world whilst at the University of Auckland; in particular, for the opportunity to attend the conference, Character Varieties and Topological Quantum Field Theory, which introduced me to a number of active research areas related to knot theory and allowed me the chance to briefly discuss my research topic with New Zealand's only Fields medalist and the creator of the Jones polynomial, Vaughan Jones.

2. Research Summary and Significance

Classical knot theory is the study of embeddings of the one-sphere S^1 into Euclidean three-space $\mathbb{R}^3$ or the three-sphere S^3. Knot theorists aim to completely classify links into disjoint equivalence classes and, failing that, to develop tools which can distinguish two distinct links. Due to the difficulty of studying links directly in $\mathbb{R}^3$, knot theorists tend to study projections of the link onto a plane with crossing information included. Such a projection of a link is known as a link diagram. A link invariant is a function from the set of all link diagrams into another set whose value remains constant over all possible link diagrams of the same link. The celebrated Jones polynomial is an example of a link invariant whose output is a Laurent polynomial in t. While the Jones polynomial is an important link invariant in the history of knot theory, much of its significance lies in its links to other fields such as Chern-Simons theory and its generalisations to stronger link invariants such as Khovanov homology and the HOMFLY polynomial. Khovanov homology categorifies the Jones polynomial in that Khovanov homology assigns to every link diagram a bi-graded cochain complex $(C^{*,*}, d^*)$ whose graded Euler characteristic returns the unnormalised Jones polynomial of the link. Given a link's cochain complex, we can then find the Khovanov homology of the link by taking the cohomology of the complex. Khovanov homology is a much stronger link invariant than the Jones polynomial. For example, the Jones polynomial fails to distinguish an infinite class of links from the n-component unlink whereas Khovanov homology can always distinguish the n component unlink from a non-trivial link of n components. Furthermore, Khovanov homology reveals richer information about a link, such as torsion, and is a topological quantum field theory (a monoidal functor $Cob_{1+1} \to Mod_R$).

3. Aim

The research scholarship has aimed to understand the construction of Khovanov homology and compute the Khovanov homology of some simple knots.

4. Method

I began my research by reading through Peter R. Cromwell's book, *Knots and Links*, to acquaint myself with the theory of knots whilst concurrently reading through George McCarty's book, *Topology: An Introduction with Applications to Topological Groups.* From McCarty's book, I gained a sufficient background in topology and algebraic topology to study knot theory at a more formal level. Specifically, McCarty's book introduced me to the fundamental group of a topological space and provided me with the necessary background to continue my education in algebraic topology by skimming through sections on homological algebra in Alan Hatcher's book, *Algebraic Topology.* Having acquired an understanding of knot theory and the basics of homological algebra, I began reading through the papers listed in the references of this report which introduced the construction of Khovanov homology. I found it helpful to read through multiple papers at once to gain a better understanding of each step in the construction of Khovanov homology. I then turned my attention to computing the Khovanov homology of some simple knots. I found Dror Bar-Natan's paper, *On Khovanov's Categorification of the Jones Polynomial*, most instructive in computing a link's Khovanov homology.

5. Discussion

5.1 Knots and Links

Definition 1. *A knot is an oriented embedding of the circle (S^1) into three dimensional Euclidean space ($\mathbb{R}^3$), or the 3-sphere (S^3).*

Definition 2. *Two knots K and K' are defined to be equivalent if there exists a map*

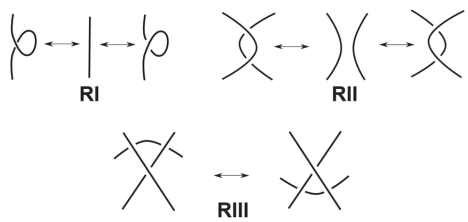
$$h : S^3 \times [0,1] \to S^3$$

such that for all $t \in [0,1]$, the map $h_t : S^3 \to S^3$ is a homeomorphism, h_0 is the identity map and $h_1(K) = K'$. Such a map is called an ambient isotopy and the knots K and K' are said to be ambient isotopic.

Essentially, a knot is a closed, nonintersecting space curve and two knots are called equivalent if one knot can be continuously deformed into the other without two strands passing through each other in space. I will also require that our knots are representable as finite closed polygonal chains. Hence, when I refer to a knot I implicitly assume that the knot is tame. A link is simply a collection of knots in $\mathbb{R}^3$ or S^3 which do not intersect, but which may be linked together. A knot is a one-component link. Often, one may wish to visualise a knot diagrammatically. One can always project a knot onto a plane $\mathbb{R}^2$ such that the projection is injective everywhere except for at a finite number of points called crossings in which it intersects itself in a standard way. Such a projection which also indicates the lower and upper strands at every crossing is called a knot diagram.

Theorem 1. Reidemeister's Theorem *Two links L and L' can be continuously deformed into each other if and only if any diagram of L can be transformed into a diagram of L' by a sequence of Reidemeister moves and planar isotopies.*

The Reidemeister moves are defined as follows:



Definition 3. *A link invariant is a function from the set of all links to another set such that the function outputs the same value for any two ambient isotopic links.*

Hence, a function is a link invariant if it is invariant under the three Reidemeister moves. An almost trivial example of a link invariant is a link's crossing number (the minimal number of crossings on any possible diagram of the link). Another link invariant, as we shall soon see, is the Jones polynomial.

5.2 The Jones Polynomial

Definition 4. The Unnormalised Jones Polynomial *Let L be an oriented link and D a diagram of L with n crossings. Further, suppose that of these n crossings, n_+ are positive (like this $\times$) and n_- are negative (like this $\times$). The unnormalized Jones polynomial of the diagram D is defined as:*

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

Where $\langle D \rangle$ is the Kaufmann bracket of D recursively defined by:

1. $$\langle \emptyset \rangle = 1$$
2. $$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$$
3. $$\langle O \sqcup L \rangle = (q + q^{-1}) \langle L \rangle$$

As the formula for the unnormalised Jones polynomial above is defined recursively, we would like to find a general formula for a given link diagram. To do this, first note that each time the Kauffman bracket resolves a crossing in our diagram, the number of terms on the right hand side of our equation doubles. Hence, for a diagram D with n crossings, we apply the Kauffman bracket n times to end up with 2^n pictures on the right hand side, each consisting of a number of circles in the plane which we can then evaluate using the third equation in the definition of the Kauffman bracket. Now arbitrarily label the crossings of D by $1, 2, ..., n$ and form a word consisting of n zeros and ones by assigning to each crossing either a zero or a one representing a zero-smoothing (like this $\asymp$) or a one-smoothing (like this $)($) respectively (i.e. each word will be an element of $\{0,1\}^n$). We, thus, have 2^n possible words and a little thought reveals that each element of $\{0,1\}^n$ corresponds to exactly one of our 2^n

pictures. Hence, I shall define each element of $\{0,1\}^n$ to be a smoothing of D and by connecting edges between smoothings which differ in exactly one place, one can see that the space of all smoothings of D lies on the vertices of a hyper-cube.

Let $\alpha \in \{0,1\}^n$ and denote Γ_α to be the smoothing associated to α. Now we shall define

$$\gamma_\alpha = \text{number of 1's in } \alpha$$

$$\kappa_\alpha = \text{number of circles in } \Gamma_\alpha.$$

Lemma 1. *The state-sum formula for the unnormalized Jones polynomial is:*

$$\hat{J}(D) = \sum_{\alpha \in \{0,1\}^n} (-1)^{\gamma_\alpha + n_-} q^{\gamma_\alpha + n_+ - 2n_-} (q+q^{-1})^{\kappa_\alpha}$$

Proof. Note that

$$\sum_{\alpha \in \{0,1\}^n} (-1)^{\gamma_\alpha + n_-} q^{\gamma_\alpha + n_+ - 2n_-} (q+q^{-1})^{\kappa_\alpha} = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{\alpha \in \{0,1\}^n} (-1)^{\gamma_\alpha} q^{\gamma_\alpha} (q+q^{-1})^{\kappa_\alpha}$$

Where the factor in front of the sum is just the normalization for the Jones polynomial. Hence, to complete the proof it will suffice to show that

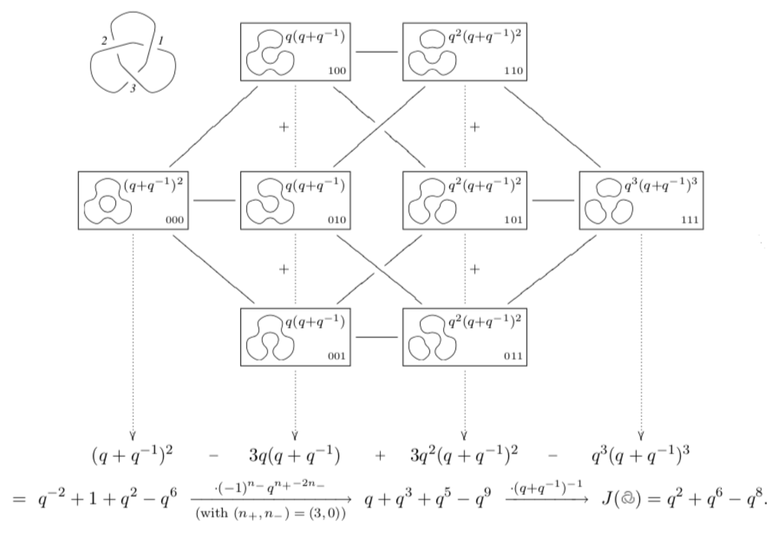
$$\sum_{\alpha \in \{0,1\}^n} (-1)^{\gamma_\alpha} q^{\gamma_\alpha} (q+q^{-1})^{\kappa_\alpha} = \langle D \rangle$$

Verifying this equation is simply a matter of checking the definition of the Kauffman bracket. For each Γ_α, we have exactly γ_α one-smoothings and κ_α planar circles. Thus, by the second and third equations in the definition of the Kauffman bracket, it follows that

$$\langle \Gamma_\alpha \rangle = (-1)^{\gamma_\alpha} q^{\gamma_\alpha} (q+q^{-1})^{\kappa_\alpha}$$

and the equation follows simply from the fact that the elements of $\{0,1\}^n$ represent all possible smoothings of D. ☐

Example 1. *Computing the unnormalized Jones polynomial of the Trefoil Knot*



5.3 Graded Vector Spaces and Cohomology

Definition 5. Graded Vector Space *We call a vector space V $\mathbb{Z}$-graded if it can be decomposed into a direct sum $V = \bigoplus_{n\in\mathbb{Z}} V_n$ where each V_n is a vector space. The elements of V_n are called homogenous components of q-degree n.*

Given two graded vector spaces V and V', we define their direct sum with grading by:

$$(V \bigoplus V')_i = V_i \bigoplus V'_i$$

We define the tensor product of V with V' by:

$$(V \bigotimes V')_i = \bigoplus_{k+l=i} (V_k \bigoplus V'_l)$$

The graded (or quantum) dimension of a graded vector space is defined to be the power series:

$$qdimV := \sum_{n\in\mathbb{Z}} q^n dimV_n$$

In this report, we will only concern ourselves with graded vector spaces with finitely many non-zero homogenous components.

Definition 6. Degree Shift Operator *Let V be a graded vector space. The degree shift operator $\cdot\{l\}$ is defined such that $V_n\{l\} = V_{n-l}$.*

Properties of Graded Vector Spaces Let V and V' be two graded vector spaces, then

1. $$qdimV\{l\} = q^l qdimV$$

2. $$qdim(V \bigotimes V') = qdim(V)qdim(V')$$

3. $$qdim(V \bigoplus V') = qdim(V) + qdim(V')$$

Definition 7. Chain Complex *A cochain complex $(C^\bullet, d^\bullet)$ is a sequence of (graded) vector spaces $. \ . \ . \ , C^{n-1}, C^n, C^{n+1}, \ . \ . \ .$ connected by a family of homomorphisms (usually referred to as differentials) $\{d^n : C^n \to C^{n+1}\}$ such that for every $n \in \mathbb{Z}$, $d^{n+1} \circ d^n = 0$, normally written out as*

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

The cohomology of a chain complex $(C^\bullet, d^\bullet)$ is defined to be the sequence of quotients

$$H^\bullet C := \left\{ \frac{Ker(d^n)}{Im(d^{n-1})} \right\}_{n\in\mathbb{Z}}$$

where the elements in $ker(d^n)$ are called degree n cocyles and elements in $Im(d^{n-1})$ are called degree n coboundaries.

5.4 Khovanov Homology

Khovanov's idea is to turn the set of smoothings Γ_α of a link diagram D into a bi-graded cochain complex $(C^{*,*}(D), d)$ whose homotopy type is a link invariant and whose graded Euler characteristic returns the unnormalised Jones polynomial of D.

Definition 8. Khovanov Complex *Let $V = \mathbb{Q}\{1, x\}$ be a two dimensional graded vector space over $\mathbb{Q}$ with basis elements 1 and x and grading, $deg(1) = 1$ and $deg(x) = -1$. For each $\alpha \in \{0,1\}^n$ associate the graded vector space*

$$V_\alpha = V^{\bigotimes \kappa_\alpha}\{\gamma_\alpha + n_+ - 2n_-\}$$

and for each $\alpha \in \{0,1\}^n$, define

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ r_\alpha = i+n_-}} V_\alpha$$

An element $v \in V_\alpha \subset C^{i,j}(D)$ is said to have homological grading i and q-grading j where

$$i = \gamma_\alpha - n_+$$

$$j = deg(v) + i + n_+ - n_-$$

Hence, we replace each smoothing Γ_α in the definition of the unnormalised Jones polynomial above by the graded vector space V_α and $C^{i,*}(D)$ is then the direct sum of vector spaces in the $i + n_-$ column. To complete our definition of the Khovanov complex, we need to define the differentials between the cochain components of $C^{*,*}(D)$. We turn the edges between elements of $\{0,1\}^n$ into arrows $\zeta \in \{0, 1, \diamondsuit\}$ so that the tail of ζ is found by setting $\diamondsuit$ to 0 and the head is found by setting $\diamondsuit$ to 1. Now we replace each arrow $\alpha \xrightarrow{\zeta} \alpha'$ by the linear map $d_\zeta : V_\alpha \longrightarrow V_{\alpha'}$ which is defined by the two linear maps

$$d_\zeta = \begin{cases} m : V \bigotimes V \longrightarrow V, & \text{if } \kappa_\alpha > \kappa_{\alpha'} \\ \Delta : V \longrightarrow V \bigotimes V, & \text{if } \kappa_\alpha < \kappa_{\alpha'} \end{cases}$$

Where $m = \begin{cases} 1 \bigotimes 1 \mapsto 1, & 1 \bigotimes x \mapsto x \\ x \bigotimes 1 \mapsto x, & x \bigotimes x \mapsto 0 \end{cases}$ and $\Delta = \begin{cases} x \mapsto x \bigotimes x \\ 1 \mapsto 1 \bigotimes x + x \bigotimes 1 \end{cases}$

Finally, define $d^i : C^{i,*}(D) \longrightarrow C^{i+1,*}(D)$ by the rule

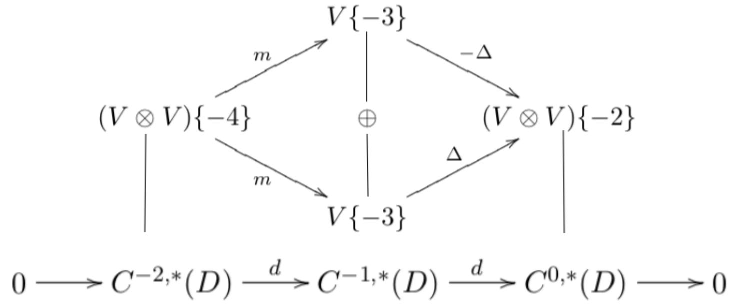
$$v \in V_\alpha \subset C^{i,*}(D) \quad \mapsto \sum_{\text{tail}(\zeta) = \alpha} \text{sign}(\zeta) d_\zeta(v)$$

Where $\text{sign}(\zeta) = (-1)^{\text{number of 1's to the left of } \diamondsuit \text{ in } \zeta}$.

To keep my final report within the specified limitations, I will neither prove that $d^{i+1} \circ d^i = 0$, thus leaving the verification that $(C^{*,*}(D), d)$ actually defines a cochain complex incomplete, nor will I prove that the Khovanov homology of D, $KH^{*,*}(D) = H(C^{*,*}(D), d)$, is a link invariant. For proofs of these two important propositions, refer to Christoffer Söderberg's paper, *Khovanov homology of Knots.* I will, however, include a computation of the Khovanov homology of the Hopf link.

Example 2. *Computing the Khovanov homology of the Hopf link,* 

By replacing each smoothing Γ_α in the resolution of the Hopf link  by its corresponding graded vector space, we obtain $C^{*,*}($$)$:



We then compute the cycles, boundaries and homology of the chain components:

Homological degree	*-2*	*-1*	*0*
Cycles	$\{1\otimes x - x\otimes 1, x\otimes x\}$	$\{(1,1),(x,x)\}$	$\{1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\}$
Boundaries	-	$\{(1,1),(x,x)\}$	$\{1\otimes x + x\otimes 1, x\otimes x\}$
Homology	$\{1\otimes x - x\otimes 1, x\otimes x\}$	-	$\{1\otimes 1, 1\otimes x\}$
q-degrees	*-4, -6*		*0, -2*

And finally we obtain the following table summarising $KH^{*,*}($$)$

j \ *i*	*-2*	*-1*	*0*
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

References

[1] Paul Turner, Five Lectures on Khovanov Homology, *arXiv:0606464v1*, (2006).

[2] Paul Turner, A Hitchhiker's Guide to Khovanov Homology, *arXiv:1409.6442*, (2014).

[3] Dror Bar-Natan, On Khovanov's Categorification of the Jones Polynomial, *Algebraic and Geometric Topology. 2, (2002)*, 337-370.

[4] Christoffer Söderberg, Khovanov Homology of Knots, *diva2:1117052*, (2017).

[5] George McCarty, Topology: An Introduction with Applications, (2010).

[6] Peter R. Cromwell, Knots and Links, (2004).